Online Appendices to
“Cadet-branch Matching in a Kelso–Crawford Economy”

Ravi Jagadeesan*

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The online appendices present example that illustrate how DA-equivalence relates to weakened substitutability conditions from the matching literature. Online Appendix 1 presents examples omitted from Section VI.A. Online Appendix 2 discusses the relationship of Section VI.B with the results of [Hatfield and Kominers (2015)].

1 DA-equivalence and unilateral substitutability: Examples

The following example shows that the law of aggregate demand for $\hat{C}^b$ is necessary in Theorem 4(a). The law of aggregate demand is clearly necessary in Theorem 4(b).

Example 1 (Necessity of the law of aggregate demand in Theorem 4(a)). Let $I = \{i, j\}$, let $B = \{b\}$, and let $X = \{x, x’, y\}$ with $i(x) = i(x’) = d$ and $i(y) = e$. Let $C^b$ be the choice function associated to the priority order

$\{x’, y\} \succ_b \{x\} \succ_b \{x’\} \succ_b \{y\} \succ_b \emptyset,$

and let $\hat{C}^b$ be the choice function associated to the priority order

$\{x\} \hat{\succ}_b \{x’, y\} \hat{\succ}_b \{x’\} \hat{\succ}_b \{y\} \hat{\succ}_b \emptyset.$

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*Aygün and Sönmez (2012, 2013) showed that substitutability and the law of aggregate demand together imply the irrelevance of rejected contracts condition. Example 1 shows that, even to deduce only unilateral substitutability, the hypothesis that $\hat{C}^b$ satisfy the law of aggregate demand cannot be weakened to require $\hat{C}^b$ to only satisfy the irrelevance of rejected contracts condition.
It is straightforward to verify that $C^b$ and $\hat{C}^b$ are feasible and DA-equivalent, and that $\hat{C}^b$ is substitutable. However, $C^b$ is not unilaterally substitutable because $y \in C^b(\{x, x', y\})$ but $y \notin C^b(\{x, y\})$. Note that $\hat{C}^b$ does not satisfy the law of aggregate demand because $|C^b(\{x, x', y\})| = |\{x\}| = 1$ while $|C^b(\{x', y\})| = |\{x', y\}| = 2$.

The following two examples show that the feasibility of $\hat{C}^b$ is necessary in both parts of Theorem 4. In the language of Section VI.B, the examples show that DA-strategy-proofness and the irrelevance of rejected contracts condition do not imply unilateral substitutability or the law of aggregate demand.

Example 2 shows furthermore that DA-strategy-proofness and the irrelevance of rejected contracts condition do not imply that deferred acceptance mechanism is stable (see also Footnote 29). By the contrapositive of Theorem 4 in [Hatfield and Kojima (2010)], DA-strategy-proofness does not imply unilateral substitutability either.

**Example 2 (DA-strategy-proofness + irrelevance of rejected contracts does not imply that deferred acceptance is stable).** Let $X = \{x, x', y, y'\}$ with $B = \{b\}$ and $I = \{i, j\}$. Define $i(x) = i(x') = i$ and $i(y) = j$. Define $C^b$ to be the choice function induced by the priority order

$$\{x, y'\} \succ_b \{x', y'\} \succ_b \{y'\} \succ_b \{x'\} \succ_b \{y\} \succ_b \{x\} \succ_b \emptyset.$$ 

Note that if the preference of $i$ is $x \succ_i x'$ and the preference of $j$ is $y \succ_i y'$, then deferred acceptance with respect to $C^b$ returns the allocation $\{x', y\}$, which is blocked by $\{x\}$. By the contrapositive of Theorem 4 in [Hatfield and Kojima (2010)], $C^b$ is not unilaterally substitutable. More explicitly, we have that $x \in \{x, y'\} = C^b(\{x, y, y'\})$ but $x \notin \{y\} = C^b(\{x, y\})$, violating unilateral substitutability.

Let $\hat{C}^b$ be the choice function induced by the priority order

$$\{x', y'\} \sim_b \{y, y'\} \sim_b \{x, y'\} \sim_b \{y'\} \sim_b \{x'\} \sim_b \{y\} \sim_b \{x\} \sim_b \emptyset.$$ 

Clearly $\hat{C}^b$ and $C^b$ are DA-equivalent and $\hat{C}^b$ is substitutable and satisfies the law of aggregate demand. Hence, $C^b$ is DA-strategy-proof. However, $\hat{C}^b$ is not feasible.

**Example 3 (DA-strategy-proofness + irrelevance of rejected contracts does not imply the law of aggregate demand).** The choice function $C^b$ in this example is taken from

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2Example 1 in [Kominers and Sonmez (2016)] provides another example of the necessity of feasibility in Theorem 4(a), when, as in Example 4, $\hat{C}^b$ is the substitutable completion of $C^b$ defined in the proof of Theorem F.1 in [Hatfield and Kominers (2015)].
Example 2 in [Kominers and Sönmez (2016)]. Let \( X = \{x, x', y\} \) with \( B = \{b\} \) and \( I = \{i, j\} \). Define \( i(x) = i(x') = i \) and \( i(y) = j \). Define \( C^b \) to be the choice function induced by the priority order

\[
\{x\} \succ_b \{x', y\} \succ_b \{y\} \succ_b \{x'\} \succ_b \emptyset.
\]

As \( |C^b(\{x, x', y\})| = |\{x\}| < |\{x', y\}| = |C^b(\{x', y\})| \), the choice function \( C^b \) does not satisfy the law of aggregate demand.

Let \( \hat{C}^b \) be the choice function induced by the priority order

\[
\{x, x'\} \sim_b \{x', y\} \sim_b \{x\} \sim_b \{y\} \sim_b \{x'\} \sim_b \emptyset.
\]

Clearly \( \hat{C}^b \) and \( C^b \) are DA-equivalent and \( \hat{C}^b \) is substitutable and satisfies the law of aggregate demand. However, \( \hat{C}^b \) is not feasible.

The following example shows that one possible converse to Theorem 4 is not true. More precisely, the example shows that feasibility, unilateral substitutability, the law of aggregate demand, and the irrelevance of rejected contracts condition do not together imply DA-equivalence to a feasible, substitutable choice function. This provides a counterexample to a converse to Theorem 4.

**Example 4 (Unilateral substitutability + law of aggregate demand does not imply DA-equivalence to a feasible, substitutable choice function).** Let \( B = \{b\} \), let \( I = \{i, j, k\} \), and let \( X = \{x, x', y, z\} \) with \( i(x) = i(x') = i \), \( i(y) = j \), and \( i(z) = k \). Let \( C^b \) be the choice function induced by the priority order

\[
\{y, z\} \succ_b \{x', y\} \succ_b \{y\} \succ_b \{x, z\} \succ_b \{z\} \succ_b \{x\} \succ_b \{x'\} \succ_b \emptyset.
\]

It is straightforward to verify that \( C^b \) is unilaterally substitutable.

However, \( C^b \) is not DA-equivalent to a feasible, substitutable choice function that satisfies the irrelevance of rejected contracts condition. Suppose for the sake of deriving a contradiction that \( C^b \) is DA-equivalent to \( \hat{C}^b \), where \( \hat{C}^b \) is feasible, substitutable, and satisfies the irrelevance of rejected contracts condition. To obtain a contradiction, we divide into cases based on the value of \( \hat{C}^b(\{x, x'\}) \).

**Case 1:** \( \hat{C}^b(\{x, x'\}) = \{x\} \). Note that \( \hat{C}^b(\{x, y\}) = \{y\} \) because \( \hat{C}^b \) is DA-equivalent to \( C^b \). As \( \hat{C}^b \) is substitutable, it follows that \( \hat{C}^b(\{x, x', y\}) \subseteq \{y\} \).

\footnote{The choice function \( \hat{C}^b \) is the substitutable completion of \( C^b \) defined in the proof of Theorem F.1 in [Hatfield and Kominers (2015)].}
the irrelevance of rejected contracts condition, we have that \( \hat{C}_b(\{x', y\}) \subseteq \{y\} \), contradicting the assumption that \( \hat{C}_b \) is DA-equivalent to \( C_b \).

**Case 2:** \( \hat{C}_b(\{x, x'\}) = \{x'\} \). Note that \( \hat{C}_b(\{x', z\}) = \{z\} \) because \( \hat{C}_b \) is DA-equivalent to \( C_b \). As \( \hat{C}_b \) is substitutable, it follows that \( \hat{C}_b(\{x, x', z\}) \subseteq \{z\} \). By the irrelevance of rejected contracts condition, we have that \( \hat{C}_b(\{x, z\}) \subseteq \{z\} \), contradicting the assumption that \( \hat{C}_b \) is DA-equivalent to \( C_b \).

**Case 3:** \( \hat{C}_b(\{x, x'\}) = \emptyset \). By the irrelevance of rejected contracts condition, we have that \( \hat{C}_b(\{x\}) = \emptyset \), contradicting the assumption that \( \hat{C}_b \) is DA-equivalent to \( C_b \).

As \( \hat{C}_b \) was assumed to be feasible, the cases exhaust all possible values of \( \hat{C}_b(\{x, x'\}) \), and we have therefore produced the desired contradiction. Thus, we can conclude that \( C_b \) is not DA-equivalent to a feasible, substitutable choice function that satisfies the irrelevance of rejected contracts condition.

Example 4 and the main result of Kadam (2017) imply that substitutable completability (in the sense of Hatfield and Kominers, 2015) does not imply DA-equivalence to a feasible, substitutable choice function either.\(^4\)

### 2 DA-substitutability and substitutable completability:

**Examples**

Hatfield and Kominers (2015) introduced a notion of completing a (usually feasible) choice function to an unfeasible choice function to restore substitutability. Recall that a choice function \( \hat{C}_b \) completes \( C_b \) if \( \hat{C}_b(Y) \) is unfeasible whenever \( \hat{C}_b(Y) \neq C_b(Y) \). A choice function \( C_b \) is substitutably completable if \( C_b \) has a completion that is substitutable. The existence of a substitutable completion of \( C_b \) satisfying the law of aggregate demand for all \( b \in B \) implies that \( \mathcal{DA}_C \) is stable and strategy-proof (Hatfield and Kominers, 2015).

Clearly, a choice function \( \hat{C}_b \) is DA-equivalent to \( C_b \) if \( \hat{C}_b \) completes \( C_b \). Thus, substitutatable completability implies DA-substitutability. Similarly, DA-strategy-proofness is implied by the existence of a completion that is substitutable and satisfies the law of aggregate demand. The following example shows that DA-strategy-proofness does not imply substitutable completability, so that DA-strategy-proofness does not imply substitutable completability.

\(^4\)The main result of Kadam (2017) asserts that unilateral substitutability implies substitutable completability. See also Proposition 2 in Zhang (2016).
(and hence DA-substitutability) is a strictly weaker condition than requiring the existence of a completion that is substitutable and satisfies the law of aggregate demand.

**Example 5** (DA-strategy-proofness does not imply substitutable completability). This example is Example 2 in Hatfield et al. (2015). Let \( B = \{ b \} \), let \( I = \{ i, j, k \} \), and let \( X = \{ x, x', y, z, z' \} \) with \( i(x) = i(x') = i, i(y) = j \), and \( i(z) = i(z') = k \). Let \( C^b \) be the choice function induced by the priority order

\[
\{ x', z \} \succ_b \{ z', x \} \succ_b \{ z', y \} \succ_b \{ x', y \} \succ_b \{ x, y \} \succ_b \{ z, y \} \succ_b \{ x', z' \} \\
\succ_b \{ x, z \} \succ_b \{ y \} \succ_b \{ z' \} \succ_b \{ x' \} \succ_b \{ x \} \succ_b \{ z \} \succ_b \emptyset.
\]

Let \( \hat{C}^b \) be the choice function induced by the priority order

\[
\{ x, z' \} \succ_b \{ x', x \} \succ_b \{ x, y \} \succ_b \{ x, z' \} \succ_b \{ x \} \succ_b \{ z, z' \} \succ_b \{ y, z \} \succ_b \{ z \} \succ_b \{ x' \} \succ_b \{ x \} \succ_b \{ z \} \succ_b \emptyset.
\]

It is straightforward to verify that \( \hat{C}^b \) is DA-equivalent to \( C^b \), substitutable, and satisfies the law of aggregate demand. Thus, \( C^b \) is DA-strategy-proof.

However, as Hatfield et al. (2015) observed, the choice function \( C^b \) is not substitutably completable. I review their argument for the sake of completeness. Suppose for the sake of deriving a contradiction that \( \hat{C}^b \) is a substitutable completion of \( C^b \). Clearly \( \hat{C}^b \) is DA-equivalent to \( C^b \). Hence, we have that

\[
x' \notin C^b(\{x', y, z\}) \implies x' \notin \hat{C}^b(\{x', y', z'\}) \\
z \notin C^b(\{x, y, z\}) \implies z \notin \hat{C}^b(\{x, y, z\}) \\
y \notin C^b(\{x', y, z\}) \implies y \notin \hat{C}^b(\{x', y, z\}).
\]

As \( \hat{C}^b \) is substitutable, it follows that \( \hat{C}^b(\{x, z'\}) \subseteq \{x, z'\} \), contradicting the assumption that \( \hat{C}^b \) completes \( C^b \).

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5I could equivalently define \( \hat{C}^b \) by the following iterative process. Given a set of contracts \( Y \subseteq X \), apply the following two steps.

- Step 1: If one of \( x, z, y, x' \) is in \( Y \), accept the first one in the list that is available. Regardless, proceed to the next step.

- Step 2: If one of \( z', x', y, z \) is in \( Y \) and was not selected in the first step, accept the first one in the list that is available. Regardless, terminate the process.
References


