Appendix for “An Efficient Ascending-Bid Auction for Multiple Objects: Comment”
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The following counterexample shows that sincere bidding by all bidders is not always an ex post perfect equilibrium under all rationing rules that satisfy the monotonicity property.

Example 3

Consider a case where there are two bidders, A and B, and three quantities of an object. Let \( u_A, u_B \) be the marginal value functions of the two bidders such that

\[
  u_A(q) = u_B(q) = \begin{cases} 
    5 & \text{if } q \in [0, 1) \\
    1 & \text{if } q \in [1, 3]. 
  \end{cases}
\]

Consider the history \( h^4 = (x_A^t, x_B^t)_{t=0,1,2,3} = ((3, 3), (3, 3), (3, 3), (3, 3)). \) After \( h^4 \), sincere bidding of each bidder is 1.

The result under sincere bidding \( x_A^4 = 1 \)

If the bidders report \( x_A^4 = x_B^4 = 1 \) after \( h^4 \), then the auction ends at \( z^5 = (h^4, (1, 1)) \), yielding an assignment \( (x_A^*, x_B^*) \) such that

\[
1 \leq x_A^* \leq 3, \\
1 \leq x_B^* \leq 3, \\
x_A^* + x_B^* = 3.
\]

Without loss of generality, suppose that \( x_A^* \geq \frac{3}{2} \). Since bidder A did not clinch at \( t \leq 3 \), A’s payment is \( y_A^* = 4x_A^* \). Therefore, A’s utility is \( U_A(x_A^*) - 4x_A^* \) at \( z^5 \).
The result under misreporting $\hat{x}_A^4 = 0$

If bidder $A$ reports $\hat{x}_A^4 = 0$, and bidder $B$ reports $x_B^4 = 1$ after $h^4$, then the auction ends at $\hat{z}_5 = (h^4, (0, 1))$, yielding an assignment $(\hat{x}_A, \hat{x}_B)$. Since $0 = \hat{x}_A^4 < x_A^4 = 1$ and any other condition of $\hat{z}_5$ is the same as $z_5$, by the monotonicity property, $\hat{x}_A$ must be strictly less than $x_A^*$. Similarly to case with sincere bidding, $A$’s utility at $\hat{z}_5$ is $U_A(\hat{x}_A) - 4\hat{x}_A$.

We calculate the difference between $A$’s utilities at $z_5$ and $\hat{z}_5$,

$$\begin{align*}
(U_A(x_A^*) - 4x_A^*) - (U_A(\hat{x}_A) - 4\hat{x}_A) \\
= & \left( \int_0^{x_A^*} u_A(q)dq - 4x_A^* \right) - \left( \int_{\hat{x}_A}^{x_A^*} u_A(q)dq - 4\hat{x}_A \right) \\
= & \left( \int_0^{x_A^*} u_A(q)dq - \int_0^{\hat{x}_A} u_A(q)dq \right) - 4 \cdot (x_A^* - \hat{x}_A) \\
= & \int_{\hat{x}_A}^{x_A^*} u_A(q)dq - 4 \cdot (x_A^* - \hat{x}_A). \quad (1)
\end{align*}$$

**Case 1: $\hat{x}_A \geq 1$.** We calculate (1) such that

$$x_A^* - \hat{x}_A - 4 \cdot (x_A^* - \hat{x}_A) = -3 \cdot (x_A^* - \hat{x}_A) < 0.$$

**Case 2: $\hat{x}_A < 1$.** We calculate (1) such that

$$\left(x_A^* - 1\right) + 5 \cdot \left(1 - \hat{x}_A\right) - 4 \cdot (x_A^* - \hat{x}_A)$$

$$= -3x_A^* - \hat{x}_A + 4 < 0 \quad (\therefore x_A^* \geq \frac{3}{2}).$$

Thus, $A$’s utility at $\hat{z}_5$ is strictly greater than that at $z_5$, and bidder $A$ has an incentive to misreport after $h^4$. Therefore, sincere bidding by all bidder is not an ex post perfect equilibrium.
Proof of Lemma 1

Since \( u_i \) is a weakly decreasing integer-valued function, there is a partition \( \{a_0, \ldots, a_m\} \subset X_i \) with \( 0 = a_0 < \cdots < a_m = \lambda_i \) and values \( \{b_1, \ldots, b_m\} \subset \{0, 1, \ldots, \overline{\mu}\} \) with \( b_1 > b_2 > \cdots > b_m \) such that for each \( k \) with \( 1 \leq k \leq m \),

\[
u_i(x_i) = b_k \text{ if } a_{k-1} < x_i < a_k.
\]

Note that \( m \leq T \). Consider any \( x_i' \in X_i \). Let

\[
k = \arg\min_{\ell} \{a_{\ell} : x_i' \leq a_{\ell}\}.
\]

By the definition of Riemann Integral,

\[
U_i(x_i') = \int_{0}^{x_i'} u_i(q) dq = \sum_{\ell=1}^{k-1} b_{\ell} (a_{\ell} - a_{\ell-1}) + b_k (x_i' - a_{k-1}).
\]

Take any \( p \in \{1, \ldots, T\} \). Define \( b_0 = T + 1 \). Let

\[
r = \arg\min_{\ell} \{b_{\ell} : p-1 < b_{\ell}\},
\]

\[
r' = \arg\min_{\ell} \{b_{\ell} : p \leq b_{\ell}\}.
\]

By equation (2), we can verify that

\[
a_r = \min \{ \arg\max_{x_i \in X_i} U_i(x_i) - (p-1)x_i \},
\]

\[
a_{r'} = \max \{ \arg\max_{x_i \in X_i} U_i(x_i) - px_i \}.
\]

Because \( b_{\ell} \in \mathbb{Z} \) for each \( \ell \),

\[
\{b_{\ell} : p-1 < b_{\ell}\} = \{b_{\ell} : p \leq b_{\ell}\}.
\]

Therefore \( a_r = a_{r'} \), that is,

\[
\min \{ \arg\max_{x_i \in X_i} U_i(x_i) - (p-1)x_i \} = \max \{ \arg\max_{x_i \in X_i} U_i(x_i) - px_i \}.
\]
To prove Lemma 2 and Proposition 1, we explain some notation and a property of the Ausubel auction.

**Notation**

- With full bid information, a *strategy* $\sigma_i$ is a function that maps each non-terminal history $h \in H \setminus Z$ to a quantity $x_i \in X_i$, that is, $\sigma_i : H \setminus Z \to X_i$.
- For each non-terminal history $h \in H \setminus Z$, the set of histories in the subgame that follows $h$ is given by
  $$H|_h = \{h' \in H : h' = (h, h'') \text{ for some sequence } h''\},$$
  and the set of terminal histories in the subgame is given by
  $$Z|_h = Z \cap H|_h.$$
- For each non-terminal history $h \in H \setminus Z$ and each strategy $\sigma_i$, we denote $\sigma_i|_h : H|_h \setminus Z|_h \to X_i$ the induced strategy in the subgame that follows $h$. For each $h' \in H_h \setminus Z_h$, $\sigma_i(h') = \sigma_i|_h(h')$.
- Let $\pi_i(\cdot)$ be the utility of bidder $i$ at an $n$-tuple of strategies.

**Property 1**

For each $t \geq 1$, if there exists a bidder $i \in N$ such that $x^t_i = C^t_{i-1}$ and $C^t_{i-1} > 0$, then the auction ends at $t$, i.e., $t = L$. Therefore, if the auction does not end at $t$, then for each bidder $i \in N$, $x^t_i \neq C^t_{i-1}$ or $C^t_{i-1} = 0$.

*Proof.* Suppose that $x^t_i = C^t_{i-1}$ and $C^t_{i-1} > 0$. Then, $x^t_i = M - \sum_{j \neq i} x^{t-1}_j$. By bidding constraint for each $j \in N$, $x^t_j \leq x^{t-1}_j$. Therefore $\sum_{j \in N} x^t_j \leq M$. □

Note that this property holds under all rationing rules. We use the property in proofs of Lemma 2 and Proposition 1.
Proof of Lemma 2

Consider any \( t \in \{0, 1, \ldots, T\}, \)

\[ h^t = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq t-1} \in H^t \setminus Z^t, \]

and \((u_j)_{j \in N}\). For each \( j \in N \), let \( \sigma_j^* \) be sincere bidding which is corresponding to \( u_j \), and \( \sigma_j^*|_{h^t} \) be induced sincere bidding in the subgame that follows \( h^t \).

Take any \( i \in N \) and \( \sigma_i \in \Sigma_i|_{h^t} \). Suppose that \( x_{i}^{t-1} \leq Q_i(p^{t-1}) \). We shall show that

\[ \pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i((\sigma_j^*|_{h^t})_{j \neq i}). \]

Let

\[ z^{L+1} = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq L} \]

be the terminal history which is reached by \((\sigma_j^*|_{h^t})_{j \in N}\), and

\[ w^{L'} = (\hat{x}_1^s, \hat{x}_2^s, \ldots, \hat{x}_n^s)_{s \leq L'} \]

be the terminal history which is reached by \((\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i})\). Denote \( \{(C_j^u)_{j \in N}\}_{L=0}^L \) the cumulative clinches of \( z^{L+1} \), and \( \{(\hat{C}_j^u)_{j \in N}\}_{L=0}^{L'} \) the cumulative clinches of \( w^{L'+1} \).

**Step 1.** \( x_{i}^{L-1} \leq Q_i(p^{L-1}) \).

If \( L - 1 = t - 1 \), \( x_{i}^{L-1} = x_{i}^{t-1} \leq Q_i(p^{t-1}) = Q_i(p^{L-1}) \). Then, let \( L - 1 \geq t \). By the definition of sincere bidding,

\[ x_{i}^{L-1} = \sigma_i^*|_{h^t}((x_1^s, \ldots, x_n^s)_{s \leq L-2}) = \min\{x_{i}^{L-2}, \max\{C_{i}^{L-2}, Q_i(p^{L-1})\}\}. \]

By Property 1, \( x_{i}^{L-1} \neq C_{i}^{L-2} \) or \( C_{i}^{L-2} = 0 \). Then, \( x_{i}^{L-1} = \min\{x_{i}^{L-2}, Q_i(p^{L-1})\} \).

Therefore, \( x_{i}^{L-1} \leq Q_i(p^{L-1}) \).

**Step 2.** For each \( j \neq i \) and \( s \leq \min\{L - 1, L' - 1\} \), \( x_{j}^{s} = \hat{x}_{j}^{s} \). This implies that for each \( s \leq \min\{L - 1, L' - 1\} \),

\[ C_{i}^{s} = M - \sum_{j \neq i} x_{j}^{s} = M - \sum_{j \neq i} \hat{x}_{j}^{s} = \hat{C}_{i}^{s}. \]
For each $s \leq t - 1$, obviously $x^s_j = \hat{x}^s_j$.

For the cases with $t \leq s \leq \min\{L - 1, L' - 1\}$, we shall show by induction. Let $s = t$. Because $x^t_j = \sigma^*_s|_{h^t}(h^t)$ and $\hat{x}^t_j = \sigma^*_s|_{h^t}(h^t)$, $x^t_j = \hat{x}^t_j$.

Let $s = k$ with $t + 1 \leq k \leq \min\{L - 1, L' - 1\}$. Suppose that $x^\ell_j = \hat{x}^\ell_j$ for all $\ell$ with $t + 1 \leq \ell \leq k - 1$. By the definition of sincere bidding,

$$
\begin{align*}
  x^k_j &= \sigma^*_s|_{h^t}(x^t_1, \ldots, x^t_n)_{t \leq k - 1}) = \min\{x^{k-1}_j, \max\{C_{j}^{k-1}, Q_j(p^k)\}\}, \\
  \hat{x}^k_j &= \sigma^*_s|_{h^t}(\hat{x}^t_1, \ldots, \hat{x}^t_n)_{t \leq k - 1}) = \min\{\hat{x}^{k-1}_j, \max\{\hat{C}_{j}^{k-1}, Q_j(p^k)\}\}.
\end{align*}
$$

Since $k \leq \min\{L - 1, L' - 1\}$, by Property 1, $x^k_j \neq C_{j}^{k-1}$ or $C_{j}^{k-1} = 0$. Thus, $x^k_j = \min\{x^{k-1}_j, Q_j(k)\}$. Similarly, we have $\hat{x}^k_j = \min\{\hat{x}^{k-1}_j, Q_j(k)\}$. Since $x^{k-1}_j = \hat{x}^{k-1}_j$, $x^k_j = \hat{x}^k_j$.

**Step 3.** $\pi_i((\sigma^*_s|_{h^t})_{j \in N}) \geq \pi_i((\sigma^*_s|_{h^t})_{j \neq i})$.

We consider three cases: $L = L'$, $L > L'$ and $L < L'$.

**Case 1.** $L = L'$.

By step 2, for all $s \leq L - 1 = L - 1$, $C^s_i = \hat{C}^s_i$. We calculate $C^{L}_i$ and $\hat{C}^{L}_i$ for two cases with $x^L_i \geq Q_i(p^L)$ and $x^L_i < Q_i(p^L)$.

**Case 1-1.** $x^L_i \geq Q_i(p^L)$.

By step 1, $x^{L-1}_i \leq Q^{L-1}_i$. Thus,

$$
Q_i(p^L) \leq x^L_i \leq C^L_i \leq x^{L-1}_i \leq Q_i(p^{L-1}).
$$

Therefore, by lemma 1,

$$
\min\{\arg\max_{x_i \in X_i}(U_i(x_i) - p^L x_i)\} \leq C^L_i \leq \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - p^L x_i)\}.
$$

Hence,

$$
\pi_i((\sigma^*_s|_{h^t})_{j \in N}) \geq \pi_i((\sigma^*_s|_{h^t})_{j \neq i}).
$$

**Case 1-2.** Let $x^L_i < Q_i(p^L)$.

We shall show that $x^L_i = x^{L-1}_i$. By the definition of sincere bidding,

$$
\begin{align*}
  x^L_i &= \sigma^*_s|_{h^t}(x^s_1, \ldots, x^s_n)_{s \leq L - 1}) = \min\{x^{L-1}_i, \max\{C^{L-1}_i, Q_i(p^L)\}\}.
\end{align*}
$$
Since \( x_i^L < Q_i(p^L) \), \( x_i^L = x_i^{L-1} \). If \( L - 1 = t - 1 \), \( x_i^{L-1} = x_i^{t-1} \). Then, we assume \( t - 1 \neq L - 1 \). By the definition of sincere bidding,

\[
x_i^{L-1} = \sigma_i^*|_{h'}((x_1^a, \ldots, x_n^a)_{s \leq L-2}) = \min\{x_i^{L-2}, \max\{C_{i}^{L-2}, Q_i(p^{L-1})\}\}.
\]

Since \( Q_i(p^L) \leq Q_i(p^{L-1}) \), \( x_i^{L-1} = x_i^L < Q_i(p^{L-1}) \). Hence, we have \( x_i^{L-1} = x_i^{L-2} \).

By repeating this procedure, \( x_i^L = x_i^{L-1} = \cdots = x_i^{t-1} \). Thus, \( C_i^L = x_i^{t-1} \).

Since bidder \( i \) cannot bid more quantity than \( x_i^{t-1} \) after \( h' \), \( \hat{C}_i^L \leq x_i^{t-1} \). Then,

\[
\hat{C}_i^L \leq C_i^L < Q_i(p^L).
\]

Hence,

\[
\pi_i((\sigma_j^*|_{h'}|_{j \in N}) \geq \pi_i((\sigma_j^*|_{h'}|_{j \neq i})).
\]

**Case 2.** \( L > L' \).

By step 2, for each \( s \leq L' - 1 \), \( C_i^s = \hat{C}_i^s \). Then, we calculate \( \{C_i^s\}_{s=L'} \) and \( \hat{C}_i^L \). Since the auction does not end at \( L' \) in the history \( z^{L+1} \), by Property 1 for each \( j \neq i \), \( x_j^{L'} \neq C_j^{L'-1} \) or \( C_j^{L'-1} = 0 \). Then, by the definition of sincere bidding, for each \( j \neq i \),

\[
x_j^{L'} = \min\{x_j^{L'-1}, Q_j(p^{L'})\}.
\]

On the other hand,

\[
\hat{x}_j^{L'} = \min\{\hat{x}_j^{L'-1}, \max\{\hat{C}_j^{L'-1}, Q_j(p^{L'})\}\}.
\]

Since \( x_j^{L'-1} = \hat{x}_j^{L'-1} \), \( x_j^{L'} \leq \hat{x}_j^{L'} \). Thus,

\[
\hat{C}_i^{L'} \leq M - \sum_{j \neq i} \hat{x}_j^{L'} \leq M - \sum_{j \neq i} x_j^{L'} = C_i^{L'}.
\]

By the definition of cumulative clinches, for each \( s \in \{L', \ldots, L - 1\} \), \( C_i^s \leq x_i^s \) and \( x_i^L \leq C_i^L \leq x_i^{L-1} \). For each \( s \in \{L', \ldots, L - 1\} \), because \( s \geq t \), \( x_i^s \) is sincere bidding. That is,

\[
x_i^s = \min\{x_i^{s-1}, \max\{C_i^{s-1}, Q_i(p^s)\}\}.
\]
Since the auction does not end at \( s \leq L - 1 \) in the history \( z^{L+1} \), by Property 1,

\[
x^* = \min \{ x^{s+1}_i, Q_i(p^s) \}.
\]

Therefore, for each \( s \in \{ L', \ldots, L - 1 \} \), \( x^*_i \leq Q_i(p^s) \). Thus,

\[
C^*_i \leq Q_i(p^s) \quad \forall s \in \{ L', \ldots, L - 1 \},
\]

\[
\hat{C}^L \leq C^L' \leq Q_i(p^L),
\]

\[
C^*_i \leq x^L_i \leq Q_i(p^L - 1) = \max \{ \arg \max_{x_i(X_i)} (U_i(x_i) - p^L x_i) \}.
\]

Hence,

\[
\pi_i((\sigma_j^*|_{H^s})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{H^s})_{j \neq i}).
\]

**Case 3.** \( L < L' \).

We first show that \( Q_i(p^L) \leq x^L_i \). Suppose that \( Q_i(p^L) > x^L_i \). Similarly to case 1-2, we have \( x^L_i = x^L_i - 1 \). By bidding constraint, \( \hat{x}^L_i \leq x^L_i - 1 \). Then, \( \hat{x}^L_i \leq x^L_i \).

Since \( L < L' \), the auction does not end at \( L \) in the history \( w^{L+1} \). Therefore, by Property 1, for each \( j \neq i \), \( \hat{x}^L_i \neq \hat{C}_i^L \) or \( \hat{C}_i^L = 0 \). By the definition of sincere bidding

\[
x^L_j = \min \{ x^L_j - 1, \max \{ C^L_j - 1, Q_j(p^L) \} \},
\]

\[
\hat{x}^L_j = \min \{ \hat{x}^L_j - 1, \max \{ C^L_j - 1, Q_j(p^L) \} \} \leq \min \{ \hat{x}^L_j - 1, Q_j(p^L) \}.
\]

For each \( j \neq i \), since by step 2, \( x^L_j - 1 = \hat{x}^L_j - 1 \), we have \( x^L_j \geq \hat{x}^L_j \). Hence for each \( j \in N \), \( x^L_j \geq \hat{x}^L_j \). Since the auction ends at \( L \) in the history \( z^{L+1} \), \( \sum_{j \in N} x^L_j \leq M \).

Therefore, \( \sum_{j \in N} \hat{x}^L_j \leq M \). This implies the auction ends at \( L \) in the history \( w^{L+1} \). This contradicts to \( L < L' \). Thus, \( Q_i(p^L) \leq x^L_i \).

By step 2, for each \( s \leq L - 1 \), \( C^*_i = \hat{C}^*_i \). Similarly to case 2, we have \( C^L_i \leq \hat{C}_i^L \).

Because \( x^L_i \leq \hat{C}^L_i \), \( Q_i(p^L) \leq \hat{C}^L_i \leq \hat{C}_i^L \). Moreover, for each \( s \geq L \), \( \hat{C}_i^L \leq \hat{C}^*_i \) and \( Q_i(p^s) \leq Q_i(p^L) \). Thus, for each \( s \geq L \), \( Q_i(p^s) \leq \hat{C}^*_i \). Hence,

\[
\pi_i((\sigma_j^*|_{H^s})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{H^s})_{j \neq i}).
\]
Proof of Proposition 1

Consider any \(t \in \{0, 1, \ldots, T\}\),
\[
h^t = (x^s_1, x^s_2, \ldots, x^s_n)_{s \leq t-1} \in H^t \setminus Z^t,
\]
and \((u_j)_{j \in N}\). For each \(j \in N\), let \(\sigma^*_j\) be sincere bidding which is corresponding to \(u_j\), and \(\sigma^*_j|_{h^t}\) be induced sincere bidding in the subgame that follows \(h^t\).

Take any \(i \in N\) and \(\sigma_i \in \Sigma_i|_{h^t}\). We shall show that
\[
\pi_i((\sigma^*_j|_{h^t})_{j \in N}) \geq \pi_i((\sigma^*_j|_{h^t})_{j \neq i}).
\]
If \(x^t_{i-1} \leq Q_i(p^t_{i-1})\), we can show by Lemma 2. Suppose that \(x^t_{i-1} > Q_i(p^t_{i-1})\).

Let
\[
z^{L+1} = (x^s_1, x^s_2, \ldots, x^s_n)_{s \leq L}
\]
be the terminal history which is reached by \((\sigma^*_j|_{h^t})_{j \in N}\), and
\[
w^{L'+1} = (\hat{x}^s_1, \hat{x}^s_2, \ldots, \hat{x}^s_n)_{s \leq L'}
\]
be the terminal history which is reached by \((\sigma^*_j|_{h^t})_{j \neq i}\). Denote \(\{(C^t_j)_{j \in N}\}_{t=0}^{L+1}\) the cumulative clinches of \(z^{L+1}\), and \(\{(\hat{C}^t_j)_{j \in N}\}_{t=0}^{L'+1}\) the cumulative clinches of \(w^{L'+1}\).

We consider three cases; \(L > t\), \(L' > L = t\) and \(L' = L = t\).

Case 1. \(L > t\).

Since \(L - 1 \geq t\), by the definition of sincere bidding,
\[
x^{L-1}_t = \sigma^*_i|_{h^t}((x^t_1, \ldots, x^t_n)_{\ell \leq L-2}) = \min\{x^{L-2}_i, \max\{C^{L-2}_i, Q_i(p^{L-1})\}\}.
\]
By Property 1, \(x^{L-1}_t \neq C^{L-2}_i\) or \(C^{L-2}_i = 0\). Then, \(x^{L-1}_t = \min\{x^{L-2}_i, Q_i(p^{L-1})\}\). Therefore, \(x^{L-1}_t \leq Q_i(p^{L-1})\), which is the same argument as step 1 of Lemma 2. Note that we only use the assumption \(x^{t-1}_i \leq Q_i(p^{t-1})\) in step 1 of Lemma 2. Thus, we can prove this case similarly to Lemma 2.

Case 2. \(L' > L = t\).

For each \(j \in N\) and each \(s \leq t - 1\), obviously \(x^s_j = \hat{x}^s_j\). Therefore, for each
s \leq t - 1 = L - 1, C_i^s = \hat{C}_i^s. We will calculate \(C_i^L\) and \{\hat{C}_i^s\}_{s=L}^{L'}.

We first show that \(Q_i(p^L) \leq C_i^L\). By the definition of sincere bidding,

\[x_i^L = \min\{x_i^{L-1}, \max\{C_i^{L-1}, Q_i(p^L)\}\}.

Since \(x_i^{L-1} \geq C_i^{L-1}\) and \(x_i^{L-1} > Q_i(p^{L-1}) \geq Q_i(p^L)\),

\[x_i^L = \max\{C_i^{L-1}, Q_i(p^L)\}.

Therefore, \(Q_i(p^L) \leq x_i^L\). Because \(x_i^L \leq C_i^L \leq x_i^{L-1}\), \(Q_i(p^L) \leq C_i^L\).

Next we show that \(C_i^L \leq \hat{C}_i^L\). For each \(j \neq i\), because \(t = L\), \(x_j^L = \sigma_j^s|_{h^L(h^t)}\) and \(\hat{x}_j^L = \sigma_j^s|_{h^t(h^t)}\). Therefore, for each \(j \neq i\), \(x_j^L = \hat{x}_j^L\). Since the auction does not end at \(L\) in the history \(w^{L+1}\),

\[\hat{C}_i^L = M - \sum_{j \neq i} \hat{x}_j^L = M - \sum_{j \neq i} x_j^L.

On the other hand, since the auction ends at \(L\) in the history \(z^{L+1}\),

\[C_i^L \leq M - \sum_{j \neq i} x_j^L.

Therefore, \(C_i^L \leq \hat{C}_i^L\).

Hence, \(Q_i(p^L) \leq C_i^L \leq \hat{C}_i^L\). Furthermore, for all \(s \geq L + 1\), \(Q_i(p^s) \leq \hat{C}_i^s\), because \(Q_i(p^s) \leq Q_i(p^L)\) and \(\hat{C}_i^s \leq \hat{C}_i^L\). Thus,

\[\pi_i((\sigma_j^s|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^s|_{h^t})_{j \neq i}).

Case 3. \(L' = L = t\).

For each \(j \in N\) and each \(s \leq t - 1\), obviously \(x_j^s = \hat{x}_j^s\). Furthermore, for each \(j \neq i\), \(x_j^L = \sigma_j^s|_{h^t(h^t)}\). Since for each \(s \leq L - 1\), \(C_i^s = \hat{C}_i^s\), we calculate \(C_i^L\) and \(\hat{C}_i^L\).

Case 3-1. \(C_i^L = x_i^L\).

By the definition of sincere bidding,

\[x_i^L = \min\{x_i^{L-1}, \max\{C_i^{L-1}, Q_i(p^L)\}\}.

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Since $x_i^{L-1} \geq C_i^{L-1}$ and $x_i^{L-1} > Q_i(p^{L-1}) \geq Q_i(p^L)$,

$$x_i^{L} = \max\{C_i^{L-1}, Q_i(p^L)\}.$$  

If $x_i^{L} = Q_i(p^L)$, then $C_i^{L} = Q_i(p^L)$ and we have

$$\pi_i((\sigma_j^{*}|_{h'})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^{*}|_{h'})_{j \neq i}).$$

Suppose that $x_i^{L} = C_i^{L-1}$. Then, $C_i^{L-1} \geq Q_i(p^L)$ and $C_i^{L} = C_i^{L-1}$. Since $\hat{C}_i^{L} \geq \hat{C}_i^{L-1}$ and $C_i^{L-1} = \hat{C}_i^{L-1}$, $\hat{C}_i^{L} \geq C_i^{L}$. Therefore, $\hat{C}_i^{L} \geq C_i^{L} \geq Q_i(p^L)$. Hence

$$\pi_i((\sigma_j^{*}|_{h'})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^{*}|_{h'})_{j \neq i}).$$

**Case 3-2.** $C_i^{L} > x_i^{L}$.  

First, we show that for each $j \in \{1, \ldots, i-1\}$, $C_j^{L} = x_j^{L-1}$. Suppose that there exists $j \in \{1, \ldots, i-1\}$ such that $C_j^{L} \neq x_j^{L-1}$. By the definition of our rationing rule,

$$C_j^{L} = \min\{x_j^{L-1}, x_j^{L} + M - \sum_{k=j}^{n} x_k^{L} - \sum_{k=1}^{j-1} C_k^{L}\} = x_j^{L} + M - \sum_{k=j}^{n} x_k^{L} - \sum_{k=1}^{j-1} C_k^{L}.\$$

Therefore,

$$M = \sum_{k=j+1}^{n} x_k^{L} - \sum_{k=1}^{j} C_k^{L}.\$$

Since for each $k \in N$, $x_k^{L} \leq C_k^{L}$, and $\sum_{k \in N} C_k^{L} = M$,

$$M = \sum_{k=j+1}^{n} x_k^{L} - \sum_{k=1}^{j} C_k^{L} \leq \sum_{k \in N} C_k^{L} = M.\$$

Therefore, for each $k \geq j + 1$, $x_k^{L} = C_k^{L}$. Because $i \geq j + 1$, this contradicts to $C_i^{L} > x_i^{L}$. 

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Next, we show that $C_L^i \leq \hat{C}_L^i$. By the definition of our rationing rule,

$$C_L^i = \min \{x_i^{L-1}, x_i^L + M - \sum_{j=i}^{n-1} x_j^L - \sum_{j=1}^{i-1} C_j^L \} = \min \{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} C_j^L \},$$

$$\hat{C}_L^i = \min \{x_i^{L-1}, \hat{x}_i^L + M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} \hat{C}_j^L \} = \min \{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} \hat{C}_j^L \}.$$

For each $j \leq i - 1$, since $x_j^{L-1} = C_j^L$ and $x_j^{L-1} \geq \hat{C}_j^L$,

$$C_j^L \geq \hat{C}_j^L.$$

Therefore,

$$\min \{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} C_j^L \} \leq \min \{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} \hat{C}_j^L \}.$$

Hence, $C_L^i \leq \hat{C}_L^i$.

Similarly to case 2, we can show that $Q_i(p^L) \leq C_L^i$. Therefore, $Q_i(p^L) \leq C_L^i \leq \hat{C}_L^i$. Thus,

$$\pi_i((\sigma_j^h |_{h \in N})_{j \in N}) \geq \pi_i(\sigma_i^L |_{h \notin i}) \geq \pi_i((\sigma_j^h |_{h \in N})_{j \notin i}).$$