

Online Appendix for *Obviously Strategy-Proof Mechanisms*

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1 Proofs omitted from the main text

1.1 Proof of Theorem 1

Proof. First we prove the “if” direction. Fix agent 1 and preferences θ_1 . Suppose that S_1 is not obviously dominant in $G = \langle H, \prec, A, \mathcal{A}, P, \delta_c, (\mathcal{I}_i)_{i \in N}, g \rangle$. We need to demonstrate that there exists \tilde{G} that is i -indistinguishable from G , such that $\lambda_{G, \tilde{G}}(S_1)$ is not weakly dominant in \tilde{G} . We proceed by construction.

Let $(S'_1, I_1, h^{\text{sup}}, S_{-1}^{\text{sup}}, d_c^{\text{sup}}, h^{\text{inf}}, S_{-1}^{\text{inf}}, d_c^{\text{inf}})$ be such that $I_1 \in \alpha(S_1, S'_1)$, $h^{\text{inf}} \in I_1$, $h^{\text{sup}} \in I_1$, and

$$u_1^G(h^{\text{sup}}, S'_1, S_{-1}^{\text{sup}}, d_c^{\text{sup}}, \theta_1) > u_1^G(h^{\text{inf}}, S_1, S_{-1}^{\text{inf}}, d_c^{\text{inf}}, \theta_1) \quad (1)$$

Since G is a game of perfect recall, we can pick $(S_{-1}^{\text{inf}}, d_c^{\text{inf}})$ such that $h^{\text{inf}} \prec z^G(h_\emptyset, S_1, S_{-1}^{\text{inf}}, d_c^{\text{inf}})$, by specifying that $(S_{-1}^{\text{inf}}, d_c^{\text{inf}})$ plays in a way consistent with h^{inf} at any $h \prec h^{\text{inf}}$. Likewise for h^{sup} and $(S_{-1}^{\text{sup}}, d_c^{\text{sup}})$. Suppose we have so done.

We now define another game \tilde{G} is 1-indistinguishable from G . Intuitively, we construct this as follows:

1. We add a chance move at the start of the game; chance can play L or R .
2. Agent 1 does not at any history know whether chance played L or R .
3. If chance plays L , then the game proceeds as in G .
4. If chance plays R , then the game proceeds mechanically as though all players in $N \setminus 1$ and chance played according to $S_{-1}^{\text{sup}}, d_c^{\text{sup}}$ in G , with one exception:

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5. If chance played R , we reach the information set corresponding to I_1 , and agent 1 plays $S_1(I_1)$, then the game henceforth proceeds mechanically as though all players in $N \setminus 1$ and chance played according to $S_{-1}^{\text{inf}}, d_c^{\text{inf}}$ in G .

Formally, the construction proceeds as such: $\tilde{A} = A \cup \{L, R\}$, where $A \cap \{L, R\} = \emptyset$. There is a new starting history \tilde{h}_\emptyset , with two successors $\sigma(\tilde{h}_\emptyset) = \{\tilde{h}_L, \tilde{h}_R\}$, $\tilde{\mathcal{A}}(\tilde{h}_L) = L$, $\tilde{\mathcal{A}}(\tilde{h}_R) = R$, $\tilde{P}(\tilde{h}_\emptyset) = c$. The subtree $\tilde{H}_L \subset \tilde{H}$ starting from \tilde{h}_L ordered by $\tilde{\prec}$ is the same as the arborescence (H, \prec) . $(\tilde{\mathcal{A}}, \tilde{P}, \tilde{\delta}_c, \tilde{g})$ are defined on \tilde{H}_L exactly as $(\mathcal{A}, P, \delta_c, g)$ are on H . For $j \neq 1$, $\tilde{\mathcal{I}}_j$ is defined as on H .

We now construct the subtree starting from \tilde{h}_R . Let h^* be such that $h^* \in \sigma(h^{\text{sup}})$, $h^* \preceq z^G(h^{\text{sup}}, S_1, S_{-1}^{\text{sup}}, d_c^{\text{sup}})$.

$$\begin{aligned} H' \equiv & \{h \in H \mid \exists S_1'' : h \preceq z^G(h_\emptyset, S_1'', S_{-1}^{\text{sup}}, d_c^{\text{sup}})\} \\ & \cap [\{h \in H \mid P(h) = 1\} \cup \{h \in Z\}] \\ & \setminus \{h \in H \mid h^* \preceq h\} \end{aligned} \quad (2)$$

In words, these are the histories that can be reached by *some* S_1'' when facing $S_{-1}^{\text{sup}}, d_c^{\text{sup}}$, where either agent 1 is called to play or that history is terminal, and such that those histories are not h^* or its successors.

Let h^{**} be such that $h^{**} \in \sigma(h^{\text{inf}})$, $h^{**} \preceq z^G(h^{\text{inf}}, S_1, S_{-1}^{\text{inf}}, d_c^{\text{inf}})$.

$$\begin{aligned} H'' \equiv & \{h \in H \mid \exists S_1'' : h \preceq z^G(h_\emptyset, S_1'', S_{-1}^{\text{inf}}, d_c^{\text{inf}})\} \\ & \cap [\{h \in H \mid P(h) = 1\} \cup \{h \in Z\}] \\ & \cap \{h \in H \mid h^{**} \preceq h\} \end{aligned} \quad (3)$$

In words, these are the histories that can be reached by *some* S_1'' when facing $S_{-1}^{\text{inf}}, d_c^{\text{inf}}$, where either agent 1 is called to play or that history is terminal, and such that those histories are h^{**} or its successors.

We now paste these together. Let \tilde{H}_R be the rooted subtree ordered by $\tilde{\prec}$, for some bijection $\gamma : \tilde{H}_R \rightarrow H' \cup H''$, such that for all $\tilde{h}, \tilde{h}' \in \tilde{H}_R$, $\tilde{h} \tilde{\prec} \tilde{h}'$ if and only if

1. EITHER: $\gamma(\tilde{h}), \gamma(\tilde{h}') \in H'$ and $\gamma(\tilde{h}) \prec \gamma(\tilde{h}')$
2. OR: $\gamma(\tilde{h}), \gamma(\tilde{h}') \in H''$ and $\gamma(\tilde{h}) \prec \gamma(\tilde{h}')$
3. OR: $\gamma(\tilde{h}) \prec h^*$ and $h^{**} \preceq \gamma(\tilde{h}')$

The root of this subtree exists and is unique; it corresponds to $\gamma^{-1}(h)$, where h is the earliest history preceding $z^G(h_\emptyset, S_1, S_{-1}^{\text{sup}}, d_c^{\text{sup}})$ where 1 is called to play. Let \tilde{h}_R be the root of \tilde{H}_R . This completes the specification of \tilde{H} .

For all $\tilde{h} \in \tilde{H}_R$, we define:

1. $\tilde{g}(\tilde{h}) = g(\gamma(\tilde{h}))$ if \tilde{h} is a terminal history.
2. $\tilde{P}(\tilde{h}) = 1$ if \tilde{h} is not a terminal history.

For all $\tilde{h} \in \tilde{H}_R \setminus \tilde{h}_R$, we define $\tilde{A}(\tilde{h}) = A(h)$, for the unique (\tilde{h}', h) such that:

1. $\tilde{h} \in \sigma(\tilde{h}')$
2. $h \in \sigma(\gamma(\tilde{h}'))$
3. $h \preceq \gamma(\tilde{h})$

We now specify the information sets for agent 1. Every $\tilde{h} \in \tilde{H}_L$ corresponds to a unique history in H . We use γ_L to denote the bijection from \tilde{H}_L to H . Let $\hat{\gamma}$ be defined as γ_L on \tilde{H}_L and γ on \tilde{H}_R .

1's information partition $\tilde{\mathcal{I}}_1$ is defined as such: $\forall \tilde{h}, \tilde{h}' \in \tilde{H}$:

$$\exists \tilde{I}'_1 \in \tilde{\mathcal{I}}_1 : \tilde{h}, \tilde{h}' \in \tilde{I}'_1 \quad (4)$$

if and only if

$$\exists I'_1 \in \mathcal{I}_1 : \hat{\gamma}(\tilde{h}), \hat{\gamma}(\tilde{h}') \in I'_1 \quad (5)$$

All that remains is to define δ_c ; we need only specify that at \tilde{h}_\emptyset , c plays R with certainty.¹

$\tilde{G} = \langle \tilde{H}, \tilde{z}, \tilde{A}, \tilde{P}, \tilde{\delta}_c, (\tilde{\mathcal{I}}_i)_{i \in N}, \tilde{g} \rangle$ is 1-indistinguishable from G . Every experience at some history in \tilde{H}_L corresponds to some experience in G , and *vice versa*. Moreover, any experience at some history in \tilde{H}_R could also be produced by some history in \tilde{H}_L .

Let $\lambda_{G, \tilde{G}}$ be the appropriate bijection from 1's information sets and actions in G onto 1's information sets and actions in \tilde{G} . Take arbitrary \tilde{S}_{-1} . Observe that since $I_1 \in \alpha(S_1, S'_1)$, $\lambda_{G, \tilde{G}}(S_1)$ and $\lambda_{G, \tilde{G}}(S'_1)$ result in the same histories following \tilde{h}_R , until they reach information set $\lambda_{G, \tilde{G}}(I_1)$. Having reached that point, $\lambda_{G, \tilde{G}}(S_1)$ leads to outcome $g(z^G(h^{\text{inf}}, S_1, S_{-1}^{\text{inf}}, d_c^{\text{inf}}))$ and $\lambda_{G, \tilde{G}}(S'_1)$ leads to outcome $g(z^G(h^{\text{sup}}, S'_1, S_{-1}^{\text{sup}}, d_c^{\text{sup}}))$. Thus,

¹If one prefers to avoid δ_c without full support, an alternative proof puts ϵ probability on L , for ϵ close to 0, or for games with $|N| > 2$ assigns $\tilde{P}(\tilde{h}_\emptyset) = 2$.

$$\begin{aligned}
& \mathbb{E}_{\tilde{\delta}_c} [u_1^{\tilde{G}}(\tilde{h}_\emptyset, \lambda_{G, \tilde{G}}(S'_1), \tilde{S}_{-1}, \tilde{d}_c, \theta_1)] \\
& = u_1^G(h^{\text{sup}}, S'_1, S_{-1}^{\text{sup}}, d_c^{\text{sup}}, \theta_1) \\
& > u_1^G(h^{\text{inf}}, S_1, S_{-1}^{\text{inf}}, d_c^{\text{inf}}, \theta_1) \\
& = \mathbb{E}_{\tilde{\delta}_c} [u_1^{\tilde{G}}(\tilde{h}_\emptyset, \lambda_{G, \tilde{G}}(S_1), \tilde{S}_{-1}, \tilde{d}_c, \theta_1)]
\end{aligned} \tag{6}$$

So $\lambda_{G, \tilde{G}}(S_1)$ is not weakly dominant in \tilde{G} .

We now prove the “only if” direction. Take arbitrary \tilde{G} . Suppose $\lambda_{G, \tilde{G}}(S_1) \equiv \tilde{S}_1$ is not weakly dominant in \tilde{G} (given type θ_1). We want to show that S_1 is not obviously dominant in G .

There exist \tilde{S}'_1 and \tilde{S}'_{-1} such that:

$$\mathbb{E}_{\tilde{\delta}_c} [u_1^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}'_1, \tilde{S}'_{-1}, \tilde{d}_c, \theta_1)] > \mathbb{E}_{\tilde{\delta}_c} [u_1^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}_1, \tilde{S}'_{-1}, \tilde{d}_c, \theta_1)] \tag{7}$$

This inequality must hold for some realization of the chance function, so there exists \tilde{d}_c such that:

$$u_1^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}'_1, \tilde{S}'_{-1}, \tilde{d}_c, \theta_1) > u_1^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}_1, \tilde{S}'_{-1}, \tilde{d}_c, \theta_1) \tag{8}$$

Fix $(\tilde{S}'_1, \tilde{S}'_{-1}, \tilde{d}_c, \theta_1)$.

$$z^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}'_1, \tilde{S}'_{-1}, \tilde{d}_c) \neq z^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}_1, \tilde{S}'_{-1}, \tilde{d}_c) \tag{9}$$

Define:

$$\tilde{H}^* \equiv \{\tilde{h} \in \tilde{H} \mid \tilde{h} \prec z^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}'_1, \tilde{S}'_{-1}, \tilde{d}_c) \text{ and } \tilde{h} \prec z^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}_1, \tilde{S}'_{-1}, \tilde{d}_c)\} \tag{10}$$

$$\tilde{h}^* \equiv \tilde{h} \in \tilde{H}^* \text{ such that } \forall \tilde{h}' \in \tilde{H}^* : \tilde{h}' \preceq \tilde{h} \tag{11}$$

Since the opponent strategies and chance moves are held constant across both sides of Equation 9, $P(\tilde{h}^*) = 1$ and $\tilde{h}^* \in \tilde{I}_1$, where $\tilde{S}_1(\tilde{I}_1) \neq \tilde{S}'_1(\tilde{I}_1)$. Moreover, $\tilde{I}_1 \in \alpha(\tilde{S}_1, \tilde{S}'_1)$ and $\lambda_{\tilde{G}, G}(\tilde{I}_1) \in \alpha(S_1, S'_1)$, where we denote $S'_1 \equiv \lambda_{\tilde{G}, G}(\tilde{S}'_1)$.

Since G and \tilde{G} are 1-indistinguishable, consider the experiences $\lambda_{\tilde{G}, G}(\psi_1(z^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}_1, \tilde{S}'_{-1}, \tilde{d}_c)))$ and $\lambda_{\tilde{G}, G}(\psi_1(z^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}'_1, \tilde{S}'_{-1}, \tilde{d}_c)))$.

In G , $\lambda_{\tilde{G}, G}(\psi_1(z^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}_1, \tilde{S}'_{-1}, \tilde{d}_c)))$ could lead to outcome $\tilde{g}(z^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}_1, \tilde{S}'_{-1}, \tilde{d}_c))$. We use $(S_{-1}^{\text{inf}}, d_c^{\text{inf}})$ to denote the corresponding opponent strategies and chance realizations that lead to that outcome. We denote $h^{\text{inf}} \equiv h \in \lambda_{\tilde{G}, G}(\tilde{I}_1) : h \prec z^G(h_\emptyset, S_1, S_{-1}^{\text{inf}}, d_c^{\text{inf}})$.

In G , $\lambda_{\tilde{G},G}(\psi_1(z^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}'_1, \tilde{S}'_{-1}, \tilde{d}_c)))$ could lead to outcome $\tilde{g}(z^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}'_1, \tilde{S}'_{-1}, \tilde{d}_c))$. We use $(S_{-1}^{\text{sup}}, d_c^{\text{sup}})$ to denote the corresponding opponent strategies and chance realizations that lead to that outcome. We denote $h^{\text{sup}} \equiv h \in \lambda_{\tilde{G},G}(\tilde{I}_1) : h \prec z^G(h_\emptyset, S'_1, S_{-1}^{\text{sup}}, d_c^{\text{sup}})$.

$$\begin{aligned}
& u_1^G(h^{\text{sup}}, S'_1, S_{-1}^{\text{sup}}, d_c^{\text{sup}}, \theta_1) \\
&= u_1^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}'_1, \tilde{S}'_{-1}, \tilde{d}_c, \theta_1) \\
&> u_1^{\tilde{G}}(\tilde{h}_\emptyset, \tilde{S}_1, \tilde{S}'_{-1}, \tilde{d}_c, \theta_1) \\
&= u_1^G(h^{\text{inf}}, S_1, S_{-1}^{\text{inf}}, d_c^{\text{inf}}, \theta_1)
\end{aligned} \tag{12}$$

where $h^{\text{sup}}, h^{\text{inf}} \in \lambda_{\tilde{G},G}(\tilde{I}_1)$ and $\lambda_{\tilde{G},G}(\tilde{I}_1) \in \alpha(S_1, S'_1)$. Thus S_1 is not obviously dominant in G . □

1.2 Proof of Theorem 2

Proof. The key is to see that, for every $G \in \mathcal{G}$, there is a corresponding \tilde{S}_0^Δ , and *vice versa*. We use \bar{S}_0 to denote the support of \tilde{S}_0^Δ . In particular, observe the following isomorphism:

Table 1: Equivalence between extensive game forms and Planner mixed strategies

G	\tilde{S}_0^Δ
d_c	$\tilde{S}_0 \in \bar{S}_0$
δ_c	the probability measure specified by \tilde{S}_0^Δ
$g(z)$ for $z \in Z$	the Planner's choice of outcome when she ends the game
I_i	$((m_k, R_k, r_k)_{k=1}^{t-1}, m_t, R_t)$ consistent with some $\tilde{S}_0 \in \bar{S}_0$
$A(I_i)$	R_t
$\psi_i(z)$	o_i^c consistent with some $\tilde{S}_0 \in \bar{S}_0$ and \tilde{S}_N

Information sets in G are equivalent to sequences of past communication $((m_k, R_k, r_k)_{k=1}^{t-1}, m_t, R_t)$ under \tilde{S}_0^Δ . Available actions at some information set $A(I_i)$ are equivalent to acceptable responses R_t . Thus, for any strategy in some game G , we can construct an equivalent strategy given appropriate \tilde{S}_0^Δ , and vice versa.

Furthermore, fixing a chance realization d_c and agent strategies S_N uniquely results in some outcome. Similarly, fixing a realization of the planner's mixed strategy $\tilde{S}_0 \in \bar{S}_0$ and agent strategies \tilde{S}_N uniquely determines some outcome. Consequently, for any $G \in \mathcal{G}$, there exists \tilde{S}_0^Δ with the same strategies available for each agent and the same resulting

(probability measure over) outcomes, and *vice versa*.² Table 1 summarizes.

The next step is to see that a bilateral commitment $\hat{\mathbb{S}}_0^i$ is equivalent to the Planner promising to ‘run’ only games in some equivalence class that is i -indistinguishable.

Suppose that there is some G that OSP-implements f . Pick some equivalent \tilde{S}_0^Δ with support $\bar{\mathbb{S}}_0$. For each $i \in N$, specify the bilateral commitment $\hat{\mathbb{S}}_0^i \equiv \Phi_i^{-1}(\Phi_i(\bar{\mathbb{S}}_0))$. These bilateral commitments support f .

To see this, take any $\tilde{S}_0^{\Delta'} \in \Delta\hat{\mathbb{S}}_0^i$, with support $\bar{\mathbb{S}}_0'$. For any $\tilde{S}'_0 \in \bar{\mathbb{S}}_0'$, for any \tilde{S}'_N , there exists $\tilde{S}_0 \in \bar{\mathbb{S}}_0$ and \tilde{S}_N such that $\phi_i(\tilde{S}'_0, \tilde{S}'_N) = \phi_i(\tilde{S}_0, \tilde{S}_N)$. By construction, G is such that: There exists $z \in Z$ where $\psi_i(z)$ and $g(z)$ are equivalent to $\phi_i(\tilde{S}_0, \tilde{S}_N)$. Thus, for G' that is equivalent to $\tilde{S}_0^{\Delta'}$, every terminal history in G' results in the same experience for i and the same outcome as some terminal history in G . Consequently, G and G' are i -indistinguishable. Thus, by Theorem 1, the strategy assigned to agent i with type θ_i is weakly dominant in G' , which implies that it is a best response to $\tilde{S}_0^{\Delta'}$ and any $\tilde{S}_{N \setminus i}$ in the bilateral commitment game. Thus, if f is OSP-implementable, then f can be supported by bilateral commitments.

Suppose that f can be supported by bilateral commitments $(\hat{\mathbb{S}}_0^i)_{i \in N}$, with requisite \tilde{S}_0^Δ (with support $\bar{\mathbb{S}}_0$) and \tilde{S}_N . Without loss of generality, let us suppose these are ‘minimal’ bilateral commitments, *i.e.* $\hat{\mathbb{S}}_0^i = \Phi_i^{-1}(\Phi_i(\bar{\mathbb{S}}_0))$. Pick G that is equivalent to \tilde{S}_0^Δ . G OSP-implements f .

To see this, consider any G' such that G and G' are i -indistinguishable. Let $\tilde{S}_0^{\Delta'}$ denote the Planner strategy that corresponds to G' . At any terminal history z' in G' , the resulting experience $\psi_i(z')$ and outcome $g'(z')$ are equivalent to the experience $\psi_i(z)$ and outcome $g(z)$ for some terminal history z in G . These in turn correspond to some observation $o_i \in \Phi_i(\bar{\mathbb{S}}_0)$. Thus $\tilde{S}_0^{\Delta'} \in \Delta\hat{\mathbb{S}}_0^i$. Since f is supported by $(\hat{\mathbb{S}}_0^j)_{j \in N}$, $\tilde{\mathbf{S}}_i^{\theta_i}$ is a best response (for type θ_i) to $\tilde{S}_0^{\Delta'}$ and any $\tilde{S}_{N \setminus i}$. Thus, the equivalent strategy $\mathbf{S}_i(\theta_i)$ is weakly dominant in G' . Since this argument holds for all i -indistinguishable G' , by Theorem 1, $\mathbf{S}_i(\theta_i)$ is obviously dominant in G . Thus, if f can be supported by bilateral commitments, then f is OSP-implementable. □

1.3 Proof of Proposition 2

Proof. We prove the contrapositive. Suppose $(\tilde{G}, \tilde{\mathbf{S}}_N)$ does not OSP-implement f . Then there exists some $(i, \theta_i, \tilde{S}'_i, \tilde{I}_i)$ such that $\tilde{I}_i \in \alpha(\tilde{\mathbf{S}}_i(\theta_i), \tilde{S}'_i)$ and

²Implicitly, this relies on the requirement that both G and \tilde{S}_0^Δ have finite length. If one had finite length but the other could be infinitely long, the resulting outcome would not be well defined and the equivalence would not hold.

$$u_i^{\tilde{G}}(\tilde{h}, \tilde{\mathbf{S}}_i(\theta_i), \tilde{S}_{-i}, \tilde{d}_c, \theta_i) < u_i^{\tilde{G}}(\tilde{h}', \tilde{S}'_i, \tilde{S}'_{-i}, \tilde{d}'_c, \theta_i) \quad (13)$$

for some $(\tilde{h}, \tilde{S}_{-i}, \tilde{d}_c)$ and $(\tilde{h}', \tilde{S}'_{-i}, \tilde{d}'_c)$.

Notice that \tilde{h} and \tilde{h}' correspond to histories h and h' in G . Moreover, we can define $S'_i = \tilde{S}'_i$ at information sets containing histories that are shared by G and \tilde{G} , and specify S'_i arbitrarily elsewhere. We do the same for $(\tilde{S}_{-i}, \tilde{d}_c)$ and $(\tilde{S}'_{-i}, \tilde{d}'_c)$, to construct (S_{-i}, d_c) and (S'_{-i}, d'_c) . But, starting from h and h' respectively, these result in the same outcomes as their partners in \tilde{G} . Thus,

$$u_i^G(h, \mathbf{S}_i(\theta_i), S_{-i}, d_c, \theta_i) < u_i^G(h', S'_i, S'_{-i}, d'_c, \theta_i) \quad (14)$$

$h, h' \in I_i$, for $I_i \in \alpha(\mathbf{S}_i(\theta_i), S'_i)$. Thus (G, \mathbf{S}_N) does not OSP-implement f . □

1.4 Proof of Theorem 3

We split this proof into two parts.

Proposition 1. *If (G, \mathbf{S}_N) OSP-implements f_y , then $\mathcal{P}(G, \mathbf{S}_N)$ is a personal-clock auction.*

Proof. Throughout this proof, we use the following notation: Given some type-strategy \mathbf{S}_i , $S_i^{\theta_i}$ denotes the strategy assigned to θ_i by \mathbf{S}_i .

Take any (G, \mathbf{S}_N) that implements (f_y, f_t) . For any history h , we define

$$\Theta_h \equiv \{\theta_N \mid h \preceq z^G(h_\emptyset, (S_i^{\theta_i})_{i \in N})\} \quad (15)$$

$$\Theta_{h,i} \equiv \{\theta_i \mid \exists \theta_{-i} : (\theta_i, \theta_{-i}) \in \Theta_h\} \quad (16)$$

For information set I_i , we define

$$\Theta_{I_i} \equiv \cup_{h \in I_i} \Theta_h \quad (17)$$

$$\Theta_{I_i,i} \equiv \{\theta_i \mid \exists \theta_{-i} : (\theta_i, \theta_{-i}) \in \Theta_{I_i}\} \quad (18)$$

$$\Theta_{I_i,i}^1 \equiv \{\theta_i \mid \exists \theta_{-i} : (\theta_i, \theta_{-i}) \in \Theta_{I_i} \text{ and } i \in f_y(\theta_i, \theta_{-i})\} \quad (19)$$

$$\Theta_{I_i,i}^0 \equiv \{\theta_i \mid \exists \theta_{-i} : (\theta_i, \theta_{-i}) \in \Theta_{I_i} \text{ and } i \notin f_y(\theta_i, \theta_{-i})\} \quad (20)$$

Some observations about this construction:

1. Since player i 's type-strategy depends only on his own type, $\Theta_{I_i,i} = \Theta_{h,i}$ for all $h \in I_i$.
2. $\Theta_{I_i,i} = \Theta_{I_i,i}^1 \cup \Theta_{I_i,i}^0$
3. Since SP requires $1_{i \in f_y(\theta)}$ weakly increasing in θ_i , $\Theta_{I_i,i}^1$ dominates $\Theta_{I_i,i}^0$ in the strong set order.

Lemma 1. *Suppose (G, \mathbf{S}_N) OSP-implements (f_y, f_t) , where $G \equiv \langle H, \prec, A, \mathcal{A}, P, \delta_c, (\mathcal{I}_i)_{i \in N}, g \rangle$. For all i , for all $I_i \in \mathcal{I}_i$, if:*

1. $\theta_i < \theta'_i$
2. $\theta_i \in \Theta_{I_i,i}^1$
3. $\theta'_i \in \Theta_{I_i,i}^0$

then $S_i^{\theta_i}(I_i) = S_i^{\theta'_i}(I_i)$.

Equivalently, for any I_i , there exists $a_{I_i}^*$ such that for all $\theta_i \in \Theta_{I_i,i}^1 \cap \Theta_{I_i,i}^0$, $S_i^{\theta_i}(I_i) = a_{I_i}^*$.

Suppose not. Take $(i, I_i, \theta_i, \theta'_i)$ constituting a counterexample to Lemma 1. Since $\theta_i \in \Theta_{I_i,i}^1$, there exists $h \in I_i$ and S_{-i} such that $i \in g_y(z^G(h, S_i^{\theta_i}, S_{-i}))$. Fix $t_i \equiv g_{t,i}(z^G(h, S_i^{\theta_i}, S_{-i}))$. Since $\theta'_i \in \Theta_{I_i,i}^0$, there exists $h' \in I_i$ and S'_{-i} such that $i \notin g_y(z^G(h', S_i^{\theta'_i}, S'_{-i}))$. Fix $t'_i \equiv g_{t,i}(z^G(h', S_i^{\theta'_i}, S'_{-i}))$. Since $S_i^{\theta_i}(I_i) \neq S_i^{\theta'_i}(I_i)$ and $\theta_i \cup \theta'_i \subseteq \Theta_{I_i,i}$, $I_i \in \alpha(S_i^{\theta_i}, S_i^{\theta'_i})$. Thus, OSP requires that

$$u_i(\theta_i, h, S_i^{\theta_i}, S_{-i}) \geq u_i(\theta_i, h', S_i^{\theta'_i}, S'_{-i}) \quad (21)$$

which implies

$$\theta_i + t_i \geq t'_i \quad (22)$$

and

$$u_i(\theta'_i, h, S_i^{\theta_i}, S_{-i}) \leq u_i(\theta'_i, h', S_i^{\theta'_i}, S'_{-i}) \quad (23)$$

which implies

$$\theta'_i + t_i \leq t'_i \quad (24)$$

But $\theta'_i > \theta_i$, so

$$\theta'_i + t_i > t'_i \quad (25)$$

a contradiction. This proves Lemma 1.

Lemma 2. *Suppose (G, \mathbf{S}_N) OSP-implements (f_y, f_t) and $\mathcal{P}(G, \mathbf{S}_N) = G$. Take any I_i such that $\Theta_{I_i, i}^1 \cap \Theta_{I_i, i}^0 \neq \emptyset$, and associated $a_{I_i}^*$.*

1. *If there exists $\theta_i \in \Theta_{I_i, i}^0$ such that $S_i^{\theta_i}(I_i) \neq a_{I_i}^*$, then there exists t_i^0 such that:*

(a) *For all $\theta_i \in \Theta_{I_i, i}^0$ such that $S_i^{\theta_i}(I_i) \neq a_{I_i}^*$, for all $h \in I_i$, for all S_{-i} , $g_{t, i}(z^G(h, S_i^{\theta_i}, S_{-i})) = t_i^0$.*

(b) *For all $\theta_i \in \Theta_{I_i, i}^0$ such that $S_i^{\theta_i}(I_i) = a_{I_i}^*$, for all $h \in I_i$, for all S_{-i} , if $i \notin g_y(z^G(h, S_i^{\theta_i}, S_{-i}))$, then $g_{t, i}(z^G(h, S_i^{\theta_i}, S_{-i})) = t_i^0$.*

2. *If there exists $\theta_i \in \Theta_{I_i, i}^1$ such that $S_i^{\theta_i}(I_i) \neq a_{I_i}^*$, then there exists t_i^1 such that:*

(a) *For all $\theta_i \in \Theta_{I_i, i}^1$ such that $S_i^{\theta_i}(I_i) \neq a_{I_i}^*$, for all $h \in I_i$, for all S_{-i} , $g_{t, i}(z^G(h, S_i^{\theta_i}, S_{-i})) = t_i^1$.*

(b) *For all $\theta_i \in \Theta_{I_i, i}^1$ such that $S_i^{\theta_i}(I_i) = a_{I_i}^*$, for all $h \in I_i$, for all S_{-i} , if $i \in g_y(z^G(h, S_i^{\theta_i}, S_{-i}))$, then $g_{t, i}(z^G(h, S_i^{\theta_i}, S_{-i})) = t_i^1$.*

Take any type $\theta'_i \in \Theta_{I_i, i}^0$ such that $S_i^{\theta'_i}(I_i) \neq a_{I_i}^*$. Take any type $\theta''_i \in \Theta_{I_i, i}^0$ such that $S_i^{\theta''_i}(I_i) = a_{I_i}^*$. (By $\Theta_{I_i, i}^1 \cap \Theta_{I_i, i}^0 \neq \emptyset$ there exists at least one such type.) Notice that $I_i \in \alpha(S_i^{\theta'_i}, S_i^{\theta''_i})$.

By Lemma 1, $\theta'_i \notin \Theta_{I_i, i}^1$, and the game is pruned. Thus,

$$\forall h \in I_i : \forall S_{-i} : i \notin g_y(z^G(h, S_i^{\theta'_i}, S_{-i})) \quad (26)$$

Since $\theta''_i \in \Theta_{I_i, i}^0$,

$$\exists h \in I_i : \exists S_{-i} : i \notin g_y(z^G(h, S_i^{\theta''_i}, S_{-i})) \quad (27)$$

OSP requires that type θ'_i does not want to (inf-sup) deviate. Thus,

$$\begin{aligned} & \inf_{h \in I_i, S_{-i}} g_{t,i}(z^G(h, S_i^{\theta'_i}, S_{-i})) \geq \\ & \sup_{h \in I_i, S_{-i}} \{g_{t,i}(z^G(h, S_i^{\theta''_i}, S_{-i})) : i \notin g_y(z^G(h, S_i^{\theta''_i}, S_{-i}))\} \end{aligned} \quad (28)$$

OSP also requires that type θ''_i does not want to (inf-sup) deviate. This implies

$$\begin{aligned} & \inf_{h \in I_i, S_{-i}} \{g_{t,i}(z^G(h, S_i^{\theta''_i}, S_{-i})) : i \notin g_y(z^G(h, S_i^{\theta''_i}, S_{-i}))\} \\ & \geq \sup_{h \in I_i, S_{-i}} g_{t,i}(z^G(h, S_i^{\theta'_i}, S_{-i})) \end{aligned} \quad (29)$$

The RHS of Equation 28 is weakly greater than the LHS of Equation 29. The RHS of Equation 29 is weakly greater than the LHS of Equation 28. Consequently all four terms are equal. Moreover, this argument applies to every $\theta'_i \in \Theta_{I_i, i}^0$ such that $S_i^{\theta'_i}(I_i) \neq a_{I_i}^*$, and every $\theta''_i \in \Theta_{I_i, i}^0$ such that $S_i^{\theta''_i}(I_i) = a_{I_i}^*$. Since the game is pruned, θ''_i satisfies (1b) iff $\theta''_i \in \Theta_{I_i, i}^0$ and $S_i^{\theta''_i}(I_i) = a_{I_i}^*$. This proves part 1 of Lemma 2. Part 2 follows by symmetry; we omit the details since they involve only small changes to the above argument.

Lemma 3. *Suppose (G, \mathbf{S}_N) OSP-implements (f_y, f_t) and $\mathcal{P}(G, \mathbf{S}_N) = G$. Take any I_i such that $\Theta_{I_i, i}^1 \cap \Theta_{I_i, i}^0 \neq \emptyset$, and associated $a_{I_i}^*$. Let t_i^1 and t_i^0 be defined as before.*

1. *If there exists $\theta_i \in \Theta_{I_i, i}^0$ such that $S_i^{\theta_i}(I_i) \neq a_{I_i}^*$, then for all $(h \in I_i, S_i, S_{-i})$, if $i \in g_y(z^G(h, S_i, S_{-i}))$, then $g_{t,i}(z^G(h, S_i, S_{-i})) \leq t_i^0 - \sup\{\theta_i \in \Theta_{I_i, i}^0 : S_i^{\theta_i}(I_i) \neq a_{I_i}^*\}$.*
2. *If there exists $\theta_i \in \Theta_{I_i, i}^1$ such that $S_i^{\theta_i}(I_i) \neq a_{I_i}^*$, then for all $(h \in I_i, S_i, S_{-i})$, if $i \notin g_y(z^G(h, S_i, S_{-i}))$, then $g_{t,i}(z^G(h, S_i, S_{-i})) \leq \inf\{\theta_i \in \Theta_{I_i, i}^1 : S_i^{\theta_i}(I_i) \neq a_{I_i}^*\} + t_i^1$.*

Suppose that part 1 of Lemma 3 does not hold. Fix $(h \in I_i, S_i, S_{-i})$ such that $i \in g_y(z^G(h, S_i, S_{-i}))$ and $g_{t,i}(z^G(h, S_i, S_{-i})) > t_i^0 - \sup\{\theta_i \in \Theta_{I_i, i}^0 : S_i^{\theta_i}(I_i) \neq a_{I_i}^*\}$. Since G is pruned, we can find some $\theta'_i \in \Theta_{I_i, i}$ such that for every $\tilde{I}_i \in \{I'_i \in \mathcal{I}_i : I_i \in \psi(I'_i)\}$, $S_i^{\theta'_i}(\tilde{I}_i) = S_i(\tilde{I}_i)$. Fix that θ'_i .

Fix $\theta''_i \in \Theta_{I_i, i}^0$ such that $S_i^{\theta''_i}(I_i) \neq a_{I_i}^*$ and $\theta''_i \geq \sup\{\theta_i \in \Theta_{I_i, i}^0 : S_i^{\theta_i}(I_i) \neq a_{I_i}^*\} - \epsilon$. Since G is pruned and $\theta''_i \notin \Theta_{I_i, i}^1$ (by Lemma 1), it must be that $S_i^{\theta''_i}(I_i) \neq S_i^{\theta'_i}(I_i)$.

By construction, $I_i \in \alpha(S_i^{\theta'_i}, S_i^{\theta''_i})$.

OSP requires that, for all $h'' \in I_i, S_{-i}''$:

$$u_i(\theta_i'', h'', S_i^{\theta_i''}, S_{-i}'') \geq u_i(\theta_i'', h, S_i^{\theta_i''}, S_{-i}) \quad (30)$$

which entails

$$t_i^0 \geq \theta_i'' + g_{t,i}(z^G(h, S_i, S_{-i})) \quad (31)$$

which entails

$$t_i^0 - \sup\{\theta_i \in \Theta_{I_i,i}^0 : S_i^{\theta_i}(I_i) \neq a_{I_i}^*\} + \epsilon \geq g_{t,i}(z^G(h, S_i, S_{-i})) \quad (32)$$

But, by hypothesis,

$$t_i^0 - \sup\{\theta_i \in \Theta_{I_i,i}^0 : S_i^{\theta_i}(I_i) \neq a_{I_i}^*\} < g_{t,i}(z^G(h, S_i, S_{-i})) \quad (33)$$

Since this argument holds for all $\epsilon > 0$, we can pick ϵ small enough to create a contradiction. This proves part 1 of Lemma 3. Part 2 follows by symmetry.

Lemma 4. *Suppose (G, \mathbf{S}_N) OSP-implements (f_y, f_t) and $\mathcal{P}(G, \mathbf{S}_N) = G$. Take any I_i such that $|\Theta_{I_i,i}^1 \cap \Theta_{I_i,i}^0| > 1$ and associated $a_{I_i}^*$.*

1. *If there exists $\theta_i \in \Theta_{I_i,i}^0$ such that $S_i^{\theta_i}(I_i) \neq a_{I_i}^*$, then for all $\theta_i' \in \Theta_{I_i,i}^1$, $S_i^{\theta_i'}(I_i) = a_{I_i}^*$.*
2. *(Equivalently) If there exists $\theta_i \in \Theta_{I_i,i}^1$ such that $S_i^{\theta_i}(I_i) \neq a_{I_i}^*$, then for all $\theta_i' \in \Theta_{I_i,i}^0$, $S_i^{\theta_i'}(I_i) = a_{I_i}^*$.*

Suppose Part 1 of Lemma 4 does not hold. Fix I_i , and choose $\theta_i' < \theta_i''$ such that $\{\theta_i'\} \cup \{\theta_i''\} \subseteq \Theta_{I_i,i}^1 \cap \Theta_{I_i,i}^0$. Fix $\theta_i''' \in \Theta_{I_i,i}^1$ such that $S_i^{\theta_i'''}(I_i) \neq a_{I_i}^*$. By Lemma 1, if $\theta_i''' \in \Theta_{I_i,i}^0$, then $S_i^{\theta_i'''}(I_i) = a_{I_i}^*$, a contradiction. Thus, $\theta_i''' \in \Theta_{I_i,i}^1 \setminus \Theta_{I_i,i}^0$, and since $\Theta_{I_i,i}^1$ dominates $\Theta_{I_i,i}^0$ in the strong set order, $\theta_i'' < \theta_i'''$.

Since $\theta_i' \in \Theta_{I_i,i}^1$, there exists $h' \in I_i$ and θ_{-i}' such that $(\theta_i', \theta_{-i}') \in \Theta_{I_i}$ and $i \in g_y(z^G(h', S_i^{\theta_i'}, S_{-i}^{\theta_{-i}'}))$. By Lemma 2, there exists $a_i \in A_i(I_i)$ such that $a_i \neq a_{I_i}^*$ and choosing a_i ensures $i \notin y$ and $t_i = t_i^0$. Thus, by G SP

$$\theta_i' + g_{t,i}(z^G(h', S_i^{\theta_i'}, S_{-i}^{\theta_{-i}'})) \geq t_i^0 \quad (34)$$

By $\theta_i'' \in \Theta_{I_i,i}^0$, there exists $h'' \in I_i$ and θ_{-i}'' such that $i \notin g_y(z^G(h'', S_i^{\theta_i''}, S_{-i}^{\theta_{-i}''}))$. By Lemma 2

$$g_{t,i}(z^G(h'', S_i^{\theta_i''}, S_{-i}^{\theta_{-i}''})) = t_i^0 \quad (35)$$

By G SP, $i \in g_y(z^G(h', S_i^{\theta''''}, S_{-i}^{\theta'_{-i}}))$ and $g_{t,i}(z^G(h', S_i^{\theta''''}, S_{-i}^{\theta'_{-i}})) = g_{t,i}(z^G(h', S_i^{\theta'_i}, S_{-i}^{\theta'_{-i}}))$.

Notice that $I_i \in \alpha(S_i^{\theta''}, S_i^{\theta''''})$. Thus, OSP requires that θ''_i does not want to (inf-sup) deviate to θ''''_i 's strategy, which entails:

$$g_{t,i}(z^G(h'', S_i^{\theta''}, S_{-i}^{\theta''_{-i}})) \geq \theta''_i + g_{t,i}(z^G(h', S_i^{\theta''''}, S_{-i}^{\theta'_{-i}})) \quad (36)$$

$$\begin{aligned} t_i^0 &\geq \theta''_i + g_{t,i}(z^G(h', S_i^{\theta''''}, S_{-i}^{\theta'_{-i}})) \\ &> \theta'_i + g_{t,i}(z^G(h', S_i^{\theta'_i}, S_{-i}^{\theta'_{-i}})) \end{aligned} \quad (37)$$

which contradicts Equation 34.

Part 2 is the contrapositive of Part 1. This proves Lemma 4.

Lemma 5. *Suppose (G, \mathbf{S}_N) OSP-implements (f_y, f_t) and $\mathcal{P}(G, \mathbf{S}_N) = G$.*

For all I_i , if $|\Theta_{I_i,i}^1 \cap \Theta_{I_i,i}^0| \leq 1$ and $|A(I_i)| \geq 2$, then there exists t_i^1 and t_i^0 such that:

1. *For all $\theta_i \in \Theta_{I_i,i}$, $h \in I_i$, S_{-i} :*

- (a) *If $i \notin g_y(z^G(h, S_i^{\theta_i}, S_{-i}))$ then $g_{t,i}(z^G(h, S_i^{\theta_i}, S_{-i})) = t_i^0$*
- (b) *If $i \in g_y(z^G(h, S_i^{\theta_i}, S_{-i}))$ then $g_{t,i}(z^G(h, S_i^{\theta_i}, S_{-i})) = t_i^1$*

2. *If $|\Theta_{I_i,i}^1| > 0$ and $|\Theta_{I_i,i}^0| > 0$, then $t_i^1 = -\inf\{\theta_i \in \Theta_{I_i,i}^1\} + t_i^0$*

By G pruned, $\Theta_{I_i,i} \neq \emptyset$. By the Green-Laffont-Holmström Theorem,

$$f_{t,i}(\theta_i, \theta_{-i}) = -1_{i \in f_y(\theta)} \inf\{\theta'_i \mid i \in f_y(\theta'_i, \theta_{-i})\} + r_i(\theta_{-i}) \quad (38)$$

Consider the case where $\Theta_{I_i,i}^1 = \emptyset$. Pick some $\theta'_i \in \Theta_{I_i,i}^0$ and some $h' \in I_i, S'_{-i}$. Fix $t_i^0 \equiv g_{t,i}(z^G(h', S_i^{\theta'_i}, S'_{-i}))$. Suppose there exists some $(\theta''_i, \theta''_{-i}) \in \Theta_{I_i}$ such that $f_{t,i}(\theta''_i, \theta''_{-i}) = t_i^{0''} \neq t_i^0$. Pick $h'' \in I_i$ such that $h'' \preceq z^G(\emptyset, S_i^{\theta''_i}, S_{-i}^{\theta''_{-i}})$. By Equation 38, for all $\theta_i \in \Theta_{I_i,i}$, $f_{t,i}(\theta_i, \theta''_{-i}) = t_i^{0''}$. By G pruned and $|A(I_i)| \geq 2$, we can pick $\theta''''_i \in \Theta_{I_i,i}^0$ such that $S_i^{\theta''''_i}(I_i) \neq S_i^{\theta''_i}(I_i)$. Notice that $I_i \in \alpha(S_i^{\theta''''_i}, S_i^{\theta''_i})$. If $t_i^{0''} > t_i^0$, then

$$u_i(\theta'_i, h', S_i^{\theta'_i}, S'_{-i}) = t_i^0 < t_i^{0''} = u_i(\theta'_i, h'', S_i^{\theta''''_i}, S_{-i}^{\theta''_{-i}}) \quad (39)$$

so $S_i^{\theta'_i}$ is not obviously dominant for (i, θ'_i) . If $t_i^{0''} < t_i^0$, then

$$u_i(\theta''''_i, h', S_i^{\theta''_i}, S'_{-i}) = t_i^0 > t_i^{0''} = u_i(\theta''''_i, h'', S_i^{\theta''''_i}, S_{-i}^{\theta''_{-i}}) \quad (40)$$

so $S_i^{\theta_i''}$ is not obviously dominant for (i, θ_i''') . By contradiction, this proves Lemma 5 for this case. A symmetric argument proves Lemma 5 for the case where $\Theta_{I_i, i}^0 = \emptyset$.

Note that, if Lemma 5 holds at some information set I_i , it holds at all information sets I_i' that follow I_i . Thus, we need only consider some earliest information set I_i^* at which $|\Theta_{I_i^*, i}^1 \cap \Theta_{I_i^*, i}^0| \leq 1$ and $|A(I_i^*)| \geq 2$.

Now we consider the case where $\Theta_{I_i^*, i}^1 \neq \emptyset$ and $\Theta_{I_i^*, i}^0 \neq \emptyset$.

At every prior information set I_i prior to I_i^* , $|\Theta_{I_i, i}^1 \cap \Theta_{I_i, i}^0| > 1$. Since $\Theta_{I_i^*, i}^1 \neq \emptyset$ and $\Theta_{I_i^*, i}^0 \neq \emptyset$, by Lemma 4, I_i^* is reached by some interval of types all taking the same action. Thus $\sup\{\theta_i \in \Theta_{I_i^*, i}^0\} = \inf\{\theta_i \in \Theta_{I_i^*, i}^1\}$.

Fix $\hat{\theta}_i \in \Theta_{I_i^*, i}^0$ such that $\hat{\theta}_i \geq \sup\{\theta_i \in \Theta_{I_i^*, i}^0\} - \epsilon$. Choose corresponding $\hat{h} \in I_i^*$ and $\hat{\theta}_{-i} \in \Theta_{I_i^*, -i}$ such that $i \notin g_y(z^G(\hat{h}, S_i^{\hat{\theta}_i}, S_{-i}^{\hat{\theta}_{-i}}))$. Define $t_i^0 \equiv g_{t, i}(z^G(\hat{h}, S_i^{\hat{\theta}_i}, S_{-i}^{\hat{\theta}_{-i}}))$.

Fix $\check{\theta}_i \in \Theta_{I_i^*, i}^1$ such that $\check{\theta}_i \leq \inf\{\theta_i \in \Theta_{I_i^*, i}^1\} + \epsilon$. Choose corresponding $\check{h} \in I_i^*$ and $\check{\theta}_{-i} \in \Theta_{I_i^*, -i}$ such that $i \in g_y(z^G(\check{h}, S_i^{\check{\theta}_i}, S_{-i}^{\check{\theta}_{-i}}))$. Define $t_i^1 \equiv g_{t, i}(z^G(\check{h}, S_i^{\check{\theta}_i}, S_{-i}^{\check{\theta}_{-i}}))$.

Suppose there exists some $(\theta'_i, \theta'_{-i}) \in \Theta_{I_i^*}$ such that $i \notin f_y(\theta'_i, \theta'_{-i})$ and $f_{t, i}(\theta'_i, \theta'_{-i}) = t_i^{0'} \neq t_i^0$. Since $\sup\{\theta_i \in \Theta_{I_i^*, i}^0\} = \inf\{\theta_i \in \Theta_{I_i^*, i}^1\}$, it follows that for all $\theta_{-i} \in \Theta_{I_i^*, -i}$, $\inf\{\theta_i : i \in f_y(\theta_i, \theta_{-i})\} = \inf\{\theta_i \in \Theta_{I_i^*, i}^1\}$. Thus, by Equation 38, for all $\theta_i \in \Theta_{I_i^*, i}$: $f_{t, i}(\theta_i, \theta_{-i}) = -1_{i \in f_y(\theta_i, \theta_{-i})} \inf\{\theta_i \in \Theta_{I_i^*, i}^1\} + t_i^{0'}$. Fix $h' \in I_i^*$ such that $h' \preceq z^G(\emptyset, S_i^{\theta'_i}, S_{-i}^{\theta'_{-i}})$.

By $A(I_i) \geq 2$, we can pick some $\theta_i'' \in \Theta_{I_i^*, i}$ such that $S_i^{\theta_i''}(I_i) \neq S_i^{\hat{\theta}_i}(I_i)$. Notice that $I_i^* \in \alpha(S_i^{\theta_i''}, S_i^{\hat{\theta}_i})$. Either $\theta_i'' \in \Theta_{I_i^*}^0$ or $\theta_i'' \in \Theta_{I_i^*}^1 \setminus \Theta_{I_i^*}^0$. Suppose $\theta_i'' \in \Theta_{I_i^*}^0$. Suppose $t_i^{0'} > t_i^0$. By OSP,

$$u_i(\hat{\theta}_i, \hat{h}, S_i^{\hat{\theta}_i}, S_{-i}^{\hat{\theta}_{-i}}) \geq u_i(\hat{\theta}_i, h', S_i^{\theta_i''}, S_{-i}^{\theta'_{-i}}) \quad (41)$$

which entails

$$\begin{aligned} t_i^0 &\geq t_i^{0'} + 1_{i \in f_y(\theta_i'', \theta'_{-i})}(\hat{\theta}_i - \inf\{\theta_i \in \Theta_{I_i^*, i}^1\}) \\ &\geq t_i^{0'} + 1_{i \in f_y(\theta_i'', \theta'_{-i})}(-\epsilon) \\ &\geq t_i^{0'} - \epsilon \end{aligned} \quad (42)$$

and we can pick ϵ small enough to constitute a contradiction. Suppose $t_i^{0'} < t_i^0$. By OSP

$$u_i(\theta_i'', \hat{h}, S_i^{\hat{\theta}_i}, S_{-i}^{\hat{\theta}_{-i}}) \leq u_i(\theta_i'', h', S_i^{\theta_i''}, S_{-i}^{\theta'_{-i}}) \quad (43)$$

which entails

$$\begin{aligned} t_i^0 &\leq t_i^{0'} + 1_{i \in f_y(\theta_i'', \theta'_{-i})}(\theta_i'' - \inf\{\theta_i \in \Theta_{I_i^*, i}^1\}) \\ &= t_i^{0'} + 1_{i \in f_y(\theta_i'', \theta'_{-i})}(\theta_i'' - \sup\{\theta_i \in \Theta_{I_i^*, i}^0\}) \leq t_i^{0'} \end{aligned} \quad (44)$$

which is a contradiction.

The case that remains is $\theta_i'' \in \Theta_{I_i^*}^1 \setminus \Theta_{I_i^*}^0$. Then $i \in f_y(\theta_i'', \theta'_{-i})$ and $f_{t,i}(\theta_i'', \theta'_{-i}) = -\inf\{\theta_i \in \Theta_{I_i^*, i}^1\} + t_i^{0'}$. Suppose $t_i^{0'} > t_i^0$. OSP requires:

$$u_i(\hat{\theta}_i, \hat{h}, S_i^{\hat{\theta}_i}, S_{-i}^{\hat{\theta}_i}) \geq u_i(\hat{\theta}_i, h', S_i^{\theta_i''}, S_{-i}^{\theta'_{-i}}) \quad (45)$$

which entails

$$\begin{aligned} t_i^0 &\geq \hat{\theta}_i - \inf\{\theta_i \in \Theta_{I_i^*, i}^1\} + t_i^{0'} \\ &\geq t_i^{0'} - \epsilon \end{aligned} \quad (46)$$

and we can pick ϵ small enough to constitute a contradiction.

Suppose $t_i^{0'} < t_i^0$. Since $S_i^{\theta_i''}(I_i) \neq S_i^{\hat{\theta}_i}(I_i)$, either $S_i^{\hat{\theta}_i}(I_i) \neq S_i^{\check{\theta}_i}(I_i)$ or $S_i^{\check{\theta}_i}(I_i) \neq S_i^{\theta_i''}(I_i)$. Moreover, $f_{t,i}(\check{\theta}_i, \theta'_{-i}) = -1_{i \in f_y(\check{\theta}_i, \theta'_{-i})} \inf\{\theta_i \in \Theta_{I_i^*, i}^1\} + t_i^{0'}$.

Suppose $S_i^{\check{\theta}_i}(I_i) \neq S_i^{\hat{\theta}_i}(I_i)$. OSP requires:

$$u_i(\check{\theta}_i, h', S_i^{\check{\theta}_i}, S_{-i}^{\theta'_{-i}}) \geq u_i(\check{\theta}_i, \hat{h}, S_i^{\hat{\theta}_i}, S_{-i}^{\hat{\theta}_i}) \quad (47)$$

which entails

$$1_{i \in f_y(\check{\theta}_i, \theta'_{-i})}(\check{\theta}_i - \inf\{\theta_i \in \Theta_{I_i^*, i}^1\}) + t_i^{0'} \geq t_i^0 \quad (48)$$

which entails

$$1_{i \in f_y(\check{\theta}_i, \theta'_{-i})} \epsilon + t_i^{0'} \geq t_i^0 \quad (49)$$

and we can pick ϵ small enough to yield a contradiction. Suppose $S_i^{\check{\theta}_i}(I_i) \neq S_i^{\theta_i''}(I_i)$. By Equation 38, $f_{t,i}(\check{\theta}_i, \theta'_{-i}) = -1_{i \in f_y(\check{\theta}_i, \theta'_{-i})} \inf\{\theta_i \in \Theta_{I_i^*, i}^1\} + t_i^{0'}$, and $f_{t,i}(\theta_i'', \hat{\theta}_{-i}) = -\inf\{\theta_i \in \Theta_{I_i^*, i}^1\} + t_i^0$. OSP requires

$$u_i(\check{\theta}_i, h', S_i^{\check{\theta}_i}, S_{-i}^{\theta'_{-i}}) \geq u_i(\check{\theta}_i, \hat{h}, S_i^{\theta_i''}, S_{-i}^{\hat{\theta}_{-i}}) \quad (50)$$

which entails

$$1_{i \in f_y(\check{\theta}_i, \theta'_{-i})}(\check{\theta}_i - \inf\{\theta_i \in \Theta_{I_i^*, i}^1\}) + t_i^{0'} \geq (\check{\theta}_i - \inf\{\theta_i \in \Theta_{I_i^*, i}^1\}) + t_i^0 \quad (51)$$

which entails

$$1_{i \in f_y(\check{\theta}_i, \theta'_{-i})}\epsilon + t_i^{0'} \geq \epsilon + t_i^0 \quad (52)$$

which entails

$$t_i^{0'} \geq t_i^0 \quad (53)$$

a contradiction. By the above argument, for all I_i satisfying the assumptions of Lemma 5, there is a unique transfer t_i^0 for all terminal histories z passing through I_i such that $i \notin g_y(z)$. Equation 38 thus implies that there is a unique transfer t_i^1 for all terminal histories z passing through I_i such that $i \in g_y(z)$. Moreover, $t_i^1 = -\inf\{\theta_i \in \Theta_{I_i, i}^1\} + t_i^0$. This proves Lemma 5.

Now to bring this all together. We leave showing parts (1.d.iv) and (2.d.iv) of Definition 15 to the last. Take any $(\hat{G}, \hat{\mathbf{S}}_N)$ that OSP-implements (f_y, f_t) . Define $G \equiv \mathcal{P}(\hat{G}, \hat{\mathbf{S}}_N)$ and \mathbf{S}_N as $\hat{\mathbf{S}}_N$ restricted to G . By Proposition 2, (G, \mathbf{S}_N) OSP-implements (f_y, f_t) .

We now characterize (G, \mathbf{S}_N) . For any player i , consider any information set I_i^* such that $|A(I_i^*)| \geq 2$, and for all prior information sets $I_i' \prec I_i^*$, $|A(I_i')| = 1$. By Lemma 1, there is a unique action $a_{I_i^*}^*$ taken by all types in $\Theta_{I_i^*, i}^1 \cap \Theta_{I_i^*, i}^0$.

Either $|\Theta_{I_i^*, i}^1 \cap \Theta_{I_i^*, i}^0| > 1$ or $|\Theta_{I_i^*, i}^1 \cap \Theta_{I_i^*, i}^0| \leq 1$.

If $|\Theta_{I_i^*, i}^1 \cap \Theta_{I_i^*, i}^0| > 1$, then by Lemma 4, \tilde{G} pruned and $|A(I_i)| \geq 2$,

1. EITHER: There exists $\theta_i \in \Theta_{I_i^*, i}^0$ such that $S_i^{\theta_i}(I_i^*) \neq a_{I_i^*}^*$, and for all $\theta_i' \in \Theta_{I_i^*, i}^1$, $S_i^{\theta_i'}(I_i^*) = a_{I_i^*}^*$.
2. OR: There exists $\theta_i \in \Theta_{I_i^*, i}^1$ such that $S_i^{\theta_i}(I_i^*) \neq a_{I_i^*}^*$, and for all $\theta_i' \in \Theta_{I_i^*, i}^0$, $S_i^{\theta_i'}(I_i^*) = a_{I_i^*}^*$.

In the first case, then by Lemma 2, there is some \bar{t}_i such that, for all (S_i, S_{-i}) , for all $h \in I_i$, if $i \notin g_y(z^G(h, S_i, S_{-i}))$, then $g_{t, i}(z^G(h, S_i, S_{-i})) = \bar{t}_i$. Moreover, we can define a ‘going transfer’ at all information sets I_i' such that $I_i^* \preceq I_i'$:

$$\tilde{t}_i(I_i') \equiv \min_{I_i'' : I_i^* \preceq I_i'' \preceq I_i'} [\bar{t}_i - \sup\{\theta_i \in \Theta_{I_i'', i}^0 : S_i^{\theta_i}(I_i'') \neq a_{I_i''}^*\}] \quad (54)$$

Notice that this function falls monotonically as we move along the game tree; for any I'_i, I''_i such that $I'_i \preceq I''_i$, $\tilde{t}_i(I'_i) \geq \tilde{t}_i(I''_i)$. Moreover, by construction, at any I'_i, I''_i such that $I'_i \prec I''_i$, $\tilde{t}_i(I'_i) > \tilde{t}_i(I''_i)$, and there does not exist I'''_i such that $I'_i \prec I'''_i \prec I''_i$, then there exists $a \in A(I''_i)$ that yields $i \notin y$, and by Lemma 2 this yields transfer \bar{t}_i . We define A^q to include all such quitting actions; i.e. A^q is the set of all actions such that:

1. $a \in I_i$ for some $I_i \in \mathcal{I}_i$
2. For all z such that $a \in \psi_i(z)$: $i \notin g_y(z)$ and $g_{t,i}(z) = \bar{t}_i$

Lemma 3 and SP together imply that, at any terminal history z , if $i \in g_y(z)$, then

$$g_{t,i}(z) = \inf_{I_i: I_i^* \preceq I_i \prec z} \tilde{t}_i(I_i) \quad (55)$$

This holds because, if $g_{t,i}(z) < \inf_{I_i: I_i^* \preceq I_i \prec z} \tilde{t}_i(I_i)$, then type θ_i such that $\bar{t}_i - \inf_{I_i: I_i^* \preceq I_i \prec z} \tilde{t}_i(I_i) < \theta_i < \bar{t}_i - g_{t,i}(z)$ could profitably deviate to play $a \in A^0$ at information set I_i^* .

In the second case, then by Lemma 2, there is some \bar{t}_i such that, for all (S_i, S_{-i}) , for all $h \in I_i^*$, if $i \in g_y(z^G(h, S_i, S_{-i}))$, then $g_{t,i}(z^G(h, S_i, S_{-i})) = \bar{t}_i$. Moreover, we can define a ‘going transfer’ at all information sets I'_i such that $I_i^* \preceq I'_i$:

$$\tilde{t}_i(I'_i) \equiv \min_{I''_i: I_i^* \preceq I''_i \preceq I'_i} [\bar{t}_i + \inf\{\theta_i \in \Theta_{I''_i, i}^0 : S_i^{\theta_i}(I''_i) \neq a_{I''_i}^*\}] \quad (56)$$

We define A^1 symmetrically for this second case.

Part (1.d.iii) and (and its analog in Clause 2) of Definition 15 follow from Lemma 4. The above constructions suffice to prove Theorem 3 for cases where $|\Theta_{I_i^*, i}^1 \cap \Theta_{I_i^*, i}^0| > 1$. Cases where $|\Theta_{I_i^*, i}^1 \cap \Theta_{I_i^*, i}^0| \leq 1$ are dealt with by Lemma 5.

Now for the last piece: We prove that parts (1.d.iv) and (and its analog in Clause 2) of Definition 15 hold. The proof of part (1.d.iv) is as follows: Suppose we are facing Clause 1 of Definition 15, and for some I_i , $|A(I'_i) \setminus A^0| > 1$. By part (1.d.iii), we know that the going transfer \tilde{t}_i can fall no further. Since G is pruned and $|A(I'_i) \setminus A^0| > 1$, there exist two distinct types of i , $\theta_i, \theta'_i \in \Theta_{I'_i, i}$, who do not quit at I'_i , and take different actions. Since neither quits at I'_i and the going transfer falls no further, there exist $\theta_{-i}, \theta'_{-i} \in \Theta_{I'_i, -i}$ such that $i \in f_y(\theta_i, \theta_{-i})$ and $i \in f_y(\theta'_i, \theta'_{-i})$. So there exist $(h \in I'_i, S_{-i})$ and $(h' \in I'_i, S'_{-i})$ such that

$$i \in g_y(h, S_i^{\theta_i}, S_{-i}) \quad (57)$$

$$i \in g_y(h', S_i^{\theta'_i}, S'_{-i}) \quad (58)$$

$$g_{t,i}(h, S_i^{\theta_i}, S_{-i}) = g_{t,i}(h', S_i^{\theta'_i}, S'_{-i}) = \tilde{t}_i(I'_i) \quad (59)$$

WLOG suppose $\theta_i < \theta'_i$. Suppose that there does not exist $a \in A(I'_i)$ such that, for all z such that $a \in \psi_i(z)$, $i \in g_y(z)$. Then there must exist $(h'' \in I'_i, S''_{-i})$ such that

$$i \notin g_y(h'', S_i^{\theta'_i}, S''_{-i}) \quad (60)$$

$$g_{t,i}(h'', S_i^{\theta'_i}, S''_{-i}) = \bar{t}_i \quad (61)$$

Note that $I'_i \in \alpha(S_i^{\theta_i}, S_i^{\theta'_i})$. But then $S_i^{\theta'_i}$ is not obviously dominant, a contradiction, since

$$\begin{aligned} u_i^{\tilde{G}}(h'', S_i^{\theta'_i}, S''_{-i}, \theta'_i) &= \bar{t}_i \leq \theta_i + \tilde{t}_i(I'_i) \\ &< \theta'_i + \tilde{t}_i(I'_i) = u_i^{\tilde{G}}(h, S_i^{\theta_i}, S_{-i}, \theta'_i) \end{aligned} \quad (62)$$

(The first inequality holds because of type θ_i 's incentive constraint.) This shows that part (1.d.iv) of Definition 15 holds. Its analog in Clause 2 is proved symmetrically. \square

Proposition 2. *If G is a personal-clock auction, then there exist \mathbf{S}_N and f_y such that (G, \mathbf{S}_N) OSP-implements f_y .*

Proof. Take any personal-clock auction G . For any i and any θ_i , the following strategy S_i is obviously dominant:

1. If i encounters an information set consistent with Clause 1 of Definition 15, then, from that point forward:
 - (a) If $\theta_i + \tilde{t}_i(I_i) > \bar{t}_i$ and there exists $a \in A(I_i) \setminus A^q$, play $a \in A(I_i) \setminus A^q$.
 - i. If $|A(I_i) \setminus A^q| > 1$, then play $a \in A(I'_i)$ such that: For all z such that $a \in \psi_i(z)$: $i \in g_y(z)$.
 - (b) Else play some $a \in A^q$.³
2. If i encounters an information set consistent with Clause 2 of Definition 15, then, from that point forward:
 - (a) If $\theta_i + \bar{t}_i < \tilde{t}_i(I_i)$ and there exists $a \in A(I_i) \setminus A^q$, play $a \in A(I_i) \setminus A^q$.

³If $|A^q \cap A(I_i)| > 1$, the agent chooses deterministically but arbitrarily.

- i. If $|A(I_i) \setminus A^q| > 1$, then play $a \in A(I'_i)$ such that: For all z such that $a \in \psi_i(z)$: $i \notin g_y(z)$.
- (b) Else play some $a \in A^q$.

The above strategy is well-defined for any agent in any personal-clock auction, by inspection of Definition 15.

Consider any deviating strategy S'_i . At any earliest point of departure, the agent will have encountered an information set consistent with either Clause 1 or Clause 2 of Definition 15. Suppose that the agent has encountered an information set covered by Clause 1.

Take some earliest point of departure $I_i \in \alpha(S_i, S'_i)$. Notice that, by (1.a) of Definition 15, *no matter what strategy i plays*, conditional on reaching I_i , either agent i is not in the allocation and receives \bar{t}_i , or agent i is in the allocation and receives a transfer $\hat{t}_i \leq \tilde{t}_i(I_i)$.

Suppose $\theta_i + \tilde{t}_i(I_i) > \bar{t}_i$. Note that under S_i , conditional on reaching I_i , the agent either is not in the allocation and receives \bar{t}_i , or is in the allocation and receives a transfer strictly above $\bar{t}_i - \theta_i$. If $S'_i(I_i) \in A_q$ (*i.e.* if agent i quits), then the best outcome under S'_i is no better than the worst outcome under S_i . If $S'_i(I_i) \notin A_q$, then, since $S'_i(I_i) \neq S_i(I_i)$, $|A(I_i) \setminus A^q| > 1$. Then, by (1.d.iii) of Definition 15, \tilde{t}_i will fall no further. So $S_i(I_i)$ guarantees that i will be in the allocation and receive transfer $\tilde{t}_i(I_i)$. But, by (1.a) of Definition 15, the best possible outcome under S'_i conditional on reaching I_i is no better, so the obvious dominance inequality holds.

Suppose $\theta_i + \tilde{t}_i(I_i) \leq \bar{t}_i$. Then, under S_i , conditional on reaching I_i , agent i is not in the allocation and has transfer \bar{t}_i .⁴ However, under S'_i , either the outcome is the same, or agent i is in the allocation for some transfer $\hat{t}_i \leq \tilde{t}_i(I_i) \leq \bar{t}_i - \theta_i$. Thus, the best possible outcome under S'_i is no better than the worst possible outcome under S_i , and the obvious dominance inequality holds.

The argument proceeds symmetrically for Clause 2.

Notice that the above strategies result in *some* allocation and *some* payments, as a function of the type profile. We define these to be (f_y, f_t) , such that G OSP-implements (f_y, f_t) . □

2 Alternative Empirical Specifications

Here we report alternative empirical specifications for the experiment.

⁴By (1.d.ii) of Definition 15, either i will have quit in the past, or will have an opportunity to quit now, which he exercises.

A natural measure of errors would be to take the sum, for $k = 1, 2, 3, 4$, of the absolute difference between the k th highest bid and the k th highest value. However we do not observe the highest bid under AC, and we often do not observe the highest bid under AC+X. We could instead take the sum for $k = 2, 3, 4$ of the absolute difference between the k th highest bid and the k th highest value, averaged as before in five-round blocks. Table 2 reports the results.

Table 2: mean(sum(abs(k th bid - k th value))), for $k = 2, 3, 4$

Format	Rounds	SP	OSP	p -value
Auction	1-5	32.63 (4.64)	9.89 (1.89)	< .001
	6-10	16.28 (2.73)	5.53 (0.91)	.001
+X Auction	1-5	17.04 (3.70)	6.18 (1.06)	.011
	6-10	14.21 (3.70)	4.74 (0.75)	.022

For each auction, we sum the absolute differences between the k th bid and the k th value, for $k = 2, 3, 4$. We then take the mean of this over each 5-round block. We then compute standard errors counting each group's 5-round mean as a single observation. (18 observations per cell.) p -values are computed using a two-sample t -test, allowing for unequal variances.

Another measure of errors would be to take the sum of the absolute difference between each bidder's bid and *that bidder's* value, dropping all highest bidders for symmetry. Table 3 reports the results.

Table 4 reports the results of Table 4 in the main text, except that the p -values are calculated using the Wilcoxon rank-sum test.

29.0% of rank-order lists are incorrect under SP-RSD. 2.6% of choices are incorrect under OSP-RSD. However, this is not a fair comparison; rank-order lists mechanically allow us to spot more errors than single choices. To compare like with like, we compute the proportion of incorrect choices we would have observed, if subjects played OSP-RSD as though they were implementing the submitted rank-order lists for SP-RSD. This is a cautious measure; it counts errors under SP-RSD only if they alter the outcome. Table 5 reports the results.

In Table 6, we compute the mean difference between the second-highest bid and the second-highest value, by auction format and by five-round blocks. This summarizes the average *direction* of deviations (positive or negative) from equilibrium play. I had no prior hypothesis about these outcome variables, but report this analysis for completeness.

Table 7 breaks down the rate that subjects submit incorrect rank-order lists in one-

Table 3: mean(sum(abs(i 's bid - i 's value))), dropping highest bidders

Format	Rounds	SP	OSP	p -value
Auction	1-5	35.13 (5.20)	10.18 (1.88)	< .001
	6-10	15.46 (2.85)	4.89 (0.72)	.002
+X Auction	1-5	17.88 (4.10)	5.58 (1.01)	.009
	6-10	14.20 (3.72)	4.64 (0.83)	.022

For each auction, we sum the absolute differences between each bidder's bid and their value, dropping the highest bidder. We then take the mean of this over each 5-round block. We then compute standard errors counting each group's 5-round mean as a single observation. (18 observations per cell.) p -values are computed using a two-sample t -test, allowing for unequal variances.

Table 4: Proportion of serial dictatorships not ending in dominant strategy outcome, p -values calculated using Wilcoxon rank-sum test

	SP	OSP	p -value
Rounds 1-5	43.3%	7.8%	.0001
Rounds 6-10	28.9%	6.7%	.0010

This is the same as Table 4 in the main text, except that the p -values are calculated using the Wilcoxon rank-sum test.

Table 5: Proportion of incorrect choices under serial dictatorship: SP (imputed) vs OSP (actual)

	SP	OSP	p -value
Rounds 1-5	17.8% (3.5%)	2.6% (1.1%)	< .001
Rounds 6-10	10.7% (2.0%)	2.6% (1.3%)	.002
p -value	.078	1.000	

For each group in SP-RSD, for each period, we simulate the three choices that we would have observed under OSP-RSD. For each group, for each 5-round block, we record the proportion of choices that are incorrect. We then compute standard errors counting each group-block pair as a single observation. (18 observations per cell.) When comparing SP to OSP, we compute p -values using a two-sample t -test. When comparing early to late rounds of the same game, we compute p -values using a paired t -test. In the sample for OSP-RSD, there are 7 incorrect choices in the first five rounds and 7 incorrect choices in the last five rounds.

shot SP-RSD by demographic category.

Table 6: mean(2nd bid - 2nd value)

Format	Rounds	SP	OSP
Auction	1-5	-1.59	0.58
		(2.07)	(0.77)
	6-10	4.22	0.78
		(1.28)	(0.40)
+X Auction	1-5	1.62	0.15
		(0.70)	(0.47)
	6-10	2.97	0.02
		(0.97)	(0.31)

For each group, we take the mean difference between the second-highest bid and the second-highest value over each 5-round block. We then compute standard errors counting each group’s 5-round mean as a single observation. (18 observations per cell, standard errors in parentheses.)

3 Quantal Response Equilibrium

Quantal response equilibrium (QRE) is defined for normal form games (McKelvey and Palfrey, 1995). Agent quantal response equilibrium (AQRE) adapts QRE to extensive form games (McKelvey and Palfrey, 1998); agents play a perturbed best response at each information set, given correct beliefs about the (perturbed) play of other agents and their future selves.

Standard specifications of AQRE predict extreme under-bidding in ascending auctions with fine bid increments. I here provide a simple proof that, for any private-value ascending auction, there exist bid increments fine enough that logit-AQRE predicts that all bids will be close to 0 with probability close to 1. The proof also works for a class of related models, that allow agents to have arbitrary beliefs about the strategies of other

Table 7: Incorrect lists in one-shot SP-RSD, by demographic categories

	Yes	No	<i>p</i> -value
Economics major	46%	31%	.131
STEM major	32%	34%	.876
Took a course in game theory	34%	33%	.876
Male	37%	28%	.214

This table displays the proportion of subjects who submitted incorrect rank-order lists in one-shot SP-RSD, by self-reported demographic categories. There are 192 subjects in this treatment. *p*-values are computed with Fisher’s exact test.

agents and their future selves.

Definition 1. An *equilibrium with τ -logit errors* specifies, for every information set, some (arbitrary) beliefs about opponent strategies and the agent's own continuation strategy. Given those beliefs, let $\{v_1, \dots, v_J\}$ denote the implied continuation values for actions $\{a_1, \dots, a_J\}$ at information set I_i . The probability that agent i chooses a_j at I_i is equal to $\frac{e^{\tau v_j}}{\sum_{l=1}^J e^{\tau v_l}}$ for some $\tau \geq 0$, where this randomization is independent across information sets and agents.

This contains logit-AQRE (McKelvey and Palfrey, 1998) as a special case. It also includes naïve logit models, such that agents believe that their opponents and their future selves play dominant strategies.

We now consider the following environment: There is a single object and some finite set of bidders, with types (private values) in the interval $[0, 1]$ and quasi-linear utility. Types are drawn according to some joint cdf $F : [0, 1]^{|N|} \rightarrow [0, 1]$.

Definition 2. In a *k -increment ascending auction*, the price rises step by step through the set $\{\frac{0}{k}, \frac{1}{k}, \frac{2}{k}, \dots\}$. Agents observe only the clock price, and whether the auction has ended. At each price, active agents (in some arbitrary order) choose In or Out. Agents who choose Out become inactive; they make no payments and do not receive the object. At any point when only one agent is active, the auction ends, and the active agent wins the object at the current clock price.

Proposition 3. For any $\epsilon > 0$ and any τ , there exists \bar{k} such that, for all $k \geq \bar{k}$, in any equilibrium with τ -logit errors of the k -increment ascending auction, the probability that the closing price is above ϵ is less than ϵ .

Proof. Let $M_k^\epsilon := \max\{j \in \mathbb{Z} \text{ such that } \frac{j}{k} \leq \epsilon\}$; this is the index of the highest bid increment below ϵ . Take any agent i , any type θ_i , any k -increment auction, and any equilibrium with τ -logit errors. Let $v_{i,j,k,\theta_i}^{\text{In}}$ denote the continuation value (given i 's arbitrary beliefs) of playing In at clock price $\frac{j}{k}$.

$$\begin{aligned} & P([i \text{ does not quit before } \epsilon] \cap [\text{some } j \in N \setminus i \text{ does not quit before } \epsilon]) \\ & \leq P(i \text{ does not quit before } \epsilon \mid \text{some } j \in N \setminus i \text{ does not quit before } \epsilon) \\ & = \prod_{j=0}^{M_k^\epsilon} \frac{e^{\tau v_{i,j,k,\theta_i}^{\text{In}}}}{e^{\tau v_{i,j,k,\theta_i}^{\text{In}}} + e^0} \leq \prod_{j=0}^{M_k^\epsilon} \frac{e^\tau}{e^\tau + e^0} \quad (63) \end{aligned}$$

The last inequality holds because, for *any* beliefs, i 's payoff from the k -increment ascending auction is bounded above by 1. Moreover,

$$\begin{aligned}
& P(\text{closing price above } \epsilon) \\
&= P\left(\bigcup_{i \in N} [i \text{ does not quit before } \epsilon] \cap [\text{some } j \in N \setminus i \text{ does not quit before } \epsilon]\right) \\
&\leq \sum_{i \in N} P([i \text{ does not quit before } \epsilon] \cap [\text{some } j \in N \setminus i \text{ does not quit before } \epsilon]) \\
&\leq |N| \prod_{j=0}^{M_k^\epsilon} \frac{e^\tau}{e^\tau + e^0} \quad (64)
\end{aligned}$$

$\lim_{k \rightarrow \infty} M_k^\epsilon = \infty$, so $\lim_{k \rightarrow \infty} \prod_{j=0}^{M_k^\epsilon} \frac{e^\tau}{e^\tau + e^0} = 0$. Consequently, we can pick \bar{k} large enough that, for all $k \geq \bar{k}$, the right-hand side of the above equation is less than ϵ , which completes the proof. \square

As Proposition 3 indicates, in any ascending auction with fine enough bid increments, models such as logit-AQRE or naïve logit predict extreme underbidding.

In the auctions in my experiment, the range of types is \$120 and bid increments are 25 cents. It is not feasible to compute AQRE for these auctions. Players have 481 possible types and there are 601 possible bids. Consequently, the pure strategy space in 2P and 2P+X is of the order of $601^{481} \approx 10^{1336}$, and larger yet in AC and AC+X, since there are multiple payoff-equivalent strategies of each kind (specifying different actions at information sets that are never reached).

However, I compute logit-AQRE for the following simpler games: There are 4 agents, with types drawn independently and uniformly at random from 0.25 to 20 (in 0.25 increments). Agents either play a second-price auction with 80 possible bids (equal to the types), or an ascending clock auction with 80 clock prices.

I compute symmetric logit-AQRE for $\tau \in \{0, 2^{-4}, 2^{-2}, 1, 2^2, 2^4\}$, and sample 10^6 plays for each game-parameter pair. Table 8 displays the average *absolute* difference between the second-highest bid and the second-highest value, by τ and auction format. For every computed parameter value, AQRE predicts larger errors in the ascending auction than in the second-price auction, which is the opposite of the effect in the experimental data.

For the simplified auctions that I computed, Table 9 displays the average difference between the second-highest bid and the second-highest value predicted by AQRE. Comparing Table 8 and Table 9 indicates that the large errors in ascending auctions under

Table 8: AQRE computed for simplified auctions

τ	mean(abs(2nd bid - 2nd value))	
	2nd-Price	Ascending
0	4.57	11.61
2^{-4}	4.34	11.52
2^{-2}	3.79	11.06
1	2.69	9.32
2^2	1.63	5.91
2^4	0.91	2.67

Table 9: Directional Errors under AQRE

τ	mean(2nd bid - 2nd value)	
	2nd-Price	Ascending
0	0.01	-11.61
2^{-4}	-0.33	-11.52
2^{-2}	-0.83	-11.06
1	-0.75	-9.32
2^2	-0.32	-5.91
2^4	-0.06	-2.65

AQRE are driven almost entirely by under-bidding.

As Figure 2 and Figure 3 (in the main text) illustrate, there is no evidence of systematic under-bidding in AC and AC+X. Table 6 shows this rigorously.

For further robustness, I compute a ‘naïve’ alternative to AQRE: Agents make logistic errors at each information set, while believing that their opponents (and their future selves) play their dominant strategies⁵. Table 10 displays the average *absolute* difference

⁵In ascending auctions, there are multiple payoff-equivalent dominant strategies, which prescribe different actions at prices above the agents’ values. For the purpose of these computations, I assume that agents believe that they will quit at all such information sets.

Table 10: Naïve logistic errors computed for simplified auctions

τ	mean(abs(2nd bid - 2nd value))	
	2nd-Price	Ascending
0	4.57	11.61
2^{-4}	4.35	11.59
2^{-2}	3.89	11.52
1	2.86	10.71
2^2	1.68	6.02
2^4	0.91	2.43

between the second-highest bid and the second-highest value, by τ and auction format. This model also predicts larger mistakes in ascending auctions than in second-price auctions.

The preceding results suggest that models in which agents make payoff-sensitive mistakes *independently* at every information set are substantially at odds with the data. They predict that ascending auctions with fine increments will induce extreme underbidding, and that errors should be larger in ascending auctions than in second-price auctions. One could invent new theories that permit non-independent errors across information sets or different accuracy parameters for different games, but this introduces many new degrees of freedom, and may have little testable content.

References

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Obviously Strategy-Proof Mechanisms

Online Appendix: Experiment Instructions

Shengwu Li

WELCOME

This is a study about decision-making. Money earned will be paid to you in cash at the end of the experiment. This study is about 90 minutes long.

We will pay you \$5 for showing up, and \$15 for completing the experiment. Additionally, you will be paid in cash your earnings from the experiment. If you make choices in this experiment that lose money, we will deduct this from your total payment. However, your total payment (including your show-up payment and completion payment) will always be at least \$20.

You have been randomly assigned into groups of 4. This experiment involves 3 games played for real money. You will play each game 10 times with the other people in your group.

We will give you instructions about each game just before you begin to play it. Your choices in one game will not affect what happens in other games.

There is *no deception* in this experiment. Every game will be exactly as specified in the instructions. Anything else would violate the IRB protocol under which we run this study. (IRB Protocol 34876)

Please do not use electronic devices or talk with other volunteers during this study. If we do find you using electronic devices or talking with other volunteers, the rules of the study require us to deduct \$20 from your earnings.

If you have questions at any point, please raise your hand and we will answer your questions privately.

GAME 1

In this game, you will bid in an auction for a money prize. The prize may have a different dollar value for each person in your group. You will play this game for 10 rounds. All dollar amounts in this game are in 25 cent increments.

At the start of each round, we display your value for this round's prize. If you win the prize, you will earn the value of the prize, minus any payments from the auction.

Your **value for the prize** will be calculated as follows:

1. For each group we will draw a **common value**, which will be between \$10.00 and \$100.00. Every number between \$10.00 and \$100.00 is equally likely to be drawn.
2. For each person, we will also draw a **private adjustment**, which will be between \$0.00 and \$20.00. Every number between \$0.00 and \$20.00 is equally likely to be drawn.

In each round, your **value for the prize** is equal to the **common value** plus your **private adjustment**. At the start of each round, you will learn your total value for the prize, but not the common value or the private adjustment.

This means that each person in your group may have a different value for the prize. However, when you have a high value, it is more likely that other people in your group have a high value.

The auction proceeds as follows: First, you will learn your value for the prize. Then, the auction will start. We will display a price to everyone in your group, that starts low and counts upwards in 25 cent increments, up to a maximum of \$150.00. At any point, you can choose to leave the auction, by clicking the button that says "Stop Bidding".

ACTIVE BIDDERS	4
PRICE	35.25
Your value for the prize	48.00
Stopped bidding at	

When there is only one bidder left in the auction, that bidder will win the prize at **the current price**. This means that we will **add** to her earnings her **value for the prize**, and **subtract** from her earnings **the current price**. All other bidders' earnings will not change.

At the end of each auction, we will show you the prices where bidders stopped, and the winning bidder's profits. If there is a tie for the highest bidder, no bidder will win the object.

GAME 2

In this game, you will bid in an auction for a money prize. You will play this game for 10 rounds.

Your **value for the prize** will be generated as before.

However, each round, we will also draw a new number, **X**, for each group.

The rules of the auction are different, as follows:

The price will count up from a low value, and you can choose to leave the auction at any point, by clicking the button that says "Stop Bidding". When there is only one bidder left in the auction, the price will **continue to rise for another X dollars**, and then freeze.

If the last bidder **stays in the auction until the price freezes**, then she will win the prize at **the final price**. This means that we will **add** to her earnings her **value for the prize**, and **subtract** from her earnings **the final price**. All other bidders' earnings will not change.

If the last bidder stops bidding before the price freezes, then no bidder will win the prize. In that case, no bidder's earnings will change.

X will be **between \$0.00 and \$3.00**, with every 25 cent increment equally likely to be drawn. You will be told your value for the prize at the start of each round, but will not be told **X**. At the end of each round, we will tell you the value of **X**.

GAME 3

You will play this game for 10 rounds. In each round of this game, there are four prizes, labeled A, B, C, and D. Prizes will be worth between \$0.00 and \$1.25. For each prize, its value will be **the same for all the players** in your group.

At the start of each round, you will learn the value of each prize. You will also learn your priority score, which is a random number between 1 and 10. Every whole number between 1 and 10 is equally likely to be chosen.

The game proceeds as follows:

1. The player with the highest priority score will pick one prize.
2. The player with the second-highest priority score will pick one of the prizes that remains.
3. The player with the third-highest priority score will pick one of the prizes that remains.
4. The player with the lowest priority score will be assigned whatever prize remains.

If two players have the same priority score, we will break the tie randomly.

When it is your turn to pick, you will have 30 seconds to make your choice. You do this by selecting a prize and then clicking the button that says “Confirm Choice”. Your choices will not count unless you click the button that says “Confirm Choice”.

Prize	Value (\$)
A	0.75
B	0.50
C	1.25
D	0.00

Pick one of the prizes

A

B

C

D

Confirm Choice

If you do not make a choice by the end of 30 seconds, we will assign prizes as though you picked whichever prize is earliest in the alphabet.

At the end of each round, we will **add** to your earnings the value of the prize you picked.

WELCOME

This is a study about decision-making. Money earned will be paid to you in cash at the end of the experiment. This study is about 90 minutes long.

We will pay you \$5 for showing up, and \$15 for completing the experiment. Additionally, you will be paid in cash your earnings from the experiment. If you make choices in this experiment that lose money, we will deduct this from your total payment. However, your total payment (including your show-up payment and completion payment) will always be at least \$20.

You have been randomly assigned into groups of 4. This experiment involves 3 games played for real money. You will play each game 10 times with the other people in your group.

We will give you instructions about each game just before you begin to play it. Your choices in one game will not affect what happens in other games.

There is *no deception* in this experiment. Every game will be exactly as specified in the instructions. Anything else would violate the IRB protocol under which we run this study. (IRB Protocol 34876)

Please do not use electronic devices or talk with other volunteers during this study. If we do find you using electronic devices or talking with other volunteers, the rules of the study require us to deduct \$20 from your earnings.

If you have questions at any point, please raise your hand and we will answer your questions privately.

GAME 1

In this game, you will bid in an auction for a money prize. The prize may have a different dollar value for each person in your group. You will play this game for 10 rounds. All dollar amounts in this game are in 25 cent increments.

At the start of each round, we display your value for this round's prize. If you win the prize, you will earn the value of the prize, minus any payments from the auction.

Your **value for the prize** will be calculated as follows:

1. For each group we will draw a **common value**, which will be between \$10.00 and \$100.00. Every number between \$10.00 and \$100.00 is equally likely to be drawn.
2. For each person, we will also draw a **private adjustment**, which will be between \$0.00 and \$20.00. Every number between \$0.00 and \$20.00 is equally likely to be drawn.

In each round, your **value for the prize** is equal to the **common value** plus your **private adjustment**. At the start of each round, you will learn your total value for the prize, but not the common value or the private adjustment.

This means that each person in your group may have a different value for the prize. However, when you have a high value, it is more likely that other people in your group have a high value.

The auction proceeds as follows: First, you will learn your value for the prize. Then you can choose a bid in the auction. Each person in your group will submit their bids privately and at the same time. You do this by typing your bid into a text box and clicking 'confirm bid'. You will have 90 seconds to make your decision, and can revise your bid as many times as you like. At the end of 90 seconds, your final bid will be the one that counts.

Your value for the prize	48.00
Current bid	0.00
Enter bid here:	<input type="text"/>
<input type="button" value="Confirm Bid"/>	

All bids must be between \$0.00 and \$150.00, and in 25 cent increments.

The highest bidder will win the prize, and make a payment equal to **the second-highest bid**. This means that we will **add** to her earnings her **value for the prize**, and **subtract** from her earnings **the second-highest bid**. All other bidders' earnings will not change.

At the end of each auction, we will show you the bids, ranked from highest to lowest, and the winning bidder's profits. If there is a tie for the highest bidder, no bidder will win the object.

GAME 2

In this game, you will bid in an auction for a money prize. You will play this game for 10 rounds.

Your **value for the prize** will be generated as before.

However, each round, we will also draw a new number, **X**, for each group.

The rules of the auction are different, as follows:

All bidders will submit their bids privately and at once. However, the highest bidder will win the prize if and only if their bid **exceeds the second-highest bid by more than X**.

If the highest bidder wins the prize, she will make a payment equal to **the second-highest bid plus X**. This means that we will **add** to her earnings her **value for the prize**, and **subtract** from her earnings **the second-highest bid plus X**. All other bidders' earnings will not change.

If the highest bid does not exceed the second-highest bid by **more than X**, then no bidder will win the prize. In that case, no bidder's earnings will change.

X will be **between \$0.00 and \$3.00**, with every 25 cent increment equally likely to be drawn. You will be told your value for the prize at the start of each round, but will not be told **X**. At the end of each round, we will tell you the value of **X**.

GAME 3

You will play this game for 10 rounds. In each round of this game, there are four prizes, labeled A, B, C, and D. Prizes will be worth between \$0.00 and \$1.25. For each prize, its value will be **the same for all the players** in your group.

At the start of each round, you will learn the value of each prize. You will also learn your priority score, which is a random number between 1 and 10. Every whole number between 1 and 10 is equally likely to be chosen.

The game proceeds as follows: We will ask you to list the prizes, **in any order of your choice**. All players will submit their lists privately and at the same time.

After all the lists have been submitted, we will assign prizes using the following rule:

1. The player with the highest priority score will be assigned the top prize on his list.
2. The player with the second-highest priority score will be assigned the top prize on his list, among the prizes that remain.
3. The player with the third-highest priority score will be assigned the top prize on his list, among the prizes that remain.
4. The player with the lowest priority score will be assigned whatever prize remains.

If two players have the same priority score, we will break the tie randomly.

You will have 90 seconds to form your list. You do this by typing a number, from 1 to 4, next to each prize, and then clicking the button that says “Confirm Choices”. Each prize must be assigned a different number, from 1 (top) to 4 (bottom). Your choices will not count unless you click the button that says “Confirm Choices”.

Prize	Value (\$)	Choose (1-4)	Rank
A	0.75	<input type="text" value="1"/>	
B	0.50	<input type="text"/>	
C	1.25	<input type="text"/>	
D	0.00	<input type="text"/>	

Your priority score is 5.
Rank the prizes in any order from 1 to 4.

[Confirm Choices](#)

If you do not produce a list by the end of 90 seconds, we will assign prizes as though you reported the list in order A-B-C-D.

At the end of each round, we will **add** to your earnings the value of the prize you were assigned.