Managing Conflicts in Relational Contracts

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Online Appendices

Appendix A: Main Model

Proof of Lemma 1: Part (i.): Note that $\underline{\pi}$ and $\underline{u}$ are the manager's and the worker's minmax payoffs and that they can be sustained as PPE payoffs by having each party take their outside option in each period. It is then immediate that in any PPE the bonus payments, wages, and effort are bounded. As a result, we can restrict the manager's and the worker's actions to compact sets. Standard arguments then imply that the PPE payoff set $\mathcal{E}$ is compact so that

$$u(\pi) = \max\{u, (\pi, u) \in \mathcal{E}\}.$$

Part (ii.): Since there is a public randomization device, any payoff on the line segment between $(\underline{\pi}, \underline{u})$ and $(\overline{\pi}, \underline{u})$ can be supported as a PPE payoff. It then follows that randomization between $(\pi, \underline{u})$ and $(\pi, u(\pi))$ allows us to obtain any payoff (π, u') for all $u' \in [\underline{u}, u(\pi)]$.

Part (iii.): Suppose to the contrary that $u(\overline{\pi}) > \underline{u}$. Note that since $(\overline{\pi}, u(\overline{\pi}))$ is an extremal point of $\mathcal{E}$ it must be sustained by pure actions in period 1. Consider a PPE with payoffs $(\overline{\pi}, u(\overline{\pi}))$ and associated first-period actions (e, w, b_s, b_n) and first-period continuation payoffs (π_s, π_n, u_s, u_n). Now consider an alternative strategy profile with the same first-period continuation payoffs (π_s, π_n, u_s, u_n) but in which first-period actions are given by $(\widehat{e}, b_s, b_n, w)$, where $\widehat{e} = e + \varepsilon$. It follows from the promise keeping constraints $\mathrm{PK_M}$ and $\mathrm{PK_W}$ that under this alternative strategy profile the payoffs are given by

$$\widehat{\pi} = \theta\left((1-\delta)\left(y\left(\widehat{e}\right) - (1+\alpha)(w + b_s)\right) + \delta\pi_s\right) + (1-\theta)\left((1-\delta)\left(y\left(\widehat{e}\right) - w - b_n\right) + \delta\pi_n\right)$$

and

$$\widehat{u} = \theta((1-\delta)\left(w + b_s\right) + \delta u_s) + (1-\theta)\left((1-\delta)\left(w + b_n\right) + \delta u_n\right) - (1-\delta)\, c\left(\widehat{e}\right).$$

Notice that since y is increasing, we have that $\widehat{\pi} > \overline{\pi}$. Moreover, for small enough ε, $\widehat{u} \geq \underline{u}$. It can be checked that this alternative strategy profile satisfies all the constraints in Section III.A (with the exception of the $\mathrm{IC_W}$ constraint that we ignore throughout) and therefore constitutes a PPE. Since $\widehat{\pi} > \overline{\pi}$ this contradicts the definition of $\overline{\pi}$. ■

Proof of Lemma 2: Part (i.): We proceed in two steps. First, we show that, for any $\pi_1 < \pi_2$, if both $u(\pi_1)$ and $u(\pi_2)$ can be sustained by pure actions in the stage game other than taking the outside option, then $u(\pi)$ can also be sustained by pure actions for any $\pi \in (\pi_1, \pi_2)$. Second, we then show that $u(\underline{\pi})$ is supported by a pure action in the stage game without taking the outside option. Since we know from the proof of Lemma 1 that $u(\overline{\pi})$ can sustained by pure actions, the result follows.

We prove the first step by contradiction. Consider any $\pi_1 < \pi_2$ such that $u(\pi_1)$ and $u(\pi_2)$ can be sustained by pure actions in the stage game. Take any $\pi \in (\pi_1, \pi_2)$ and suppose to the contrary that $(\pi, u(\pi))$ is not sustained by pure actions. Then there exists a $\rho \in (0,1)$, a $\widetilde{\pi}_1 \in [\pi_1, \pi)$, and a $\widetilde{\pi}_2 \in (\pi, \pi_2]$ such that (i.) $(\widetilde{\pi}_1, u(\widetilde{\pi}_1))$ and $(\widetilde{\pi}_2, u(\widetilde{\pi}_2))$ are sustained by pure actions, (ii.) $\pi = \rho\widetilde{\pi}_1 + (1-\rho)\widetilde{\pi}_2$, and (iii.) $u(\pi) = \rho u(\widetilde{\pi}_1) + (1-\rho) u(\widetilde{\pi}_2)$. Now consider a PPE with payoffs $(\widetilde{\pi}_j, u(\widetilde{\pi}_j))$, for $j = 1, 2$, and associated first-period actions $(e_j, b_{s_j}, b_{n_j}, w_j)$ and first-period continuation payoffs $(\pi_{s_j}, \pi_{n_j}, u_{s_j}, u_{n_j})$. Define $\widehat{e}$ as the effort level satisfying

$$y(\widehat{e}) = \rho y(e_1) + (1-\rho)y(e_2).$$

Since y is strictly concave, we have that

$$y(\rho e_1 + (1-\rho)e_2) > \rho y(e_1) + (1-\rho)y(e_2) = y(\widehat{e}).$$

And since y is strictly increasing this implies that

$$\widehat{e} < \rho e_1 + (1-\rho)e_2.$$

Now consider an alternative strategy profile with first-period actions $(\widehat{e}, \widehat{w}, \widehat{b}_s, \widehat{b}_n)$ and first-period continuation payoffs $(\widehat{\pi}_s, \widehat{\pi}_n, \widehat{u}_s, \widehat{u}_n)$, where $\widehat{w} = \rho w_1 + (1-\rho)w_2$ and where $\widehat{b}_s, \widehat{b}_n, \widehat{\pi}_s, \widehat{\pi}_n, \widehat{u}_s$,and $\widehat{u}_n$ are defined analogously.

Also, let $\widehat{w} = \rho w_1 + (1-\rho)w_2$ and define $\widehat{b}_s, \widehat{b}_n, \widehat{\pi}_s, \widehat{\pi}_n, \widehat{u}_s$, and $\widehat{u}_n$ analogously. It follows from the promise keeping constraints $\mathrm{PK_M}$ and $\mathrm{PK_W}$ that under this alternative strategy profile the payoffs are given by $\widehat{\pi} = \rho\widetilde{\pi}_1 + (1-\rho)\widetilde{\pi}_2$ and

$$\widehat{u} = \rho u(\widetilde{\pi}_1) + (1-\rho)u(\widetilde{\pi}_2) + (1-\delta)\left(\rho c(e_1) + (1-\rho)\, c(e_2) - c(e)\right).$$

Since $c(e)$ is strictly increasing and strictly convex, it follows that $\widehat{u} > \rho u(\widetilde{\pi}_1) + (1-\rho)u(\widetilde{\pi}_2)$. It can be checked that this alternative profile satisfies all the constraints in Section III.A and therefore constitutes a PPE. Since $\widehat{u} > \rho u(\widetilde{\pi}_1) + (1-\rho)u(\widetilde{\pi}_2)$, this proves the first step.

To prove the second step, suppose to the contrary that $u(\underline{\pi})$ cannot be sustained by pure actions other than taking the outside options. Since $(\underline{\pi}, u(\underline{\pi}))$ is an extremal point, it cannot be sustained by randomizations. The parties must therefore take their outside options in period 1, which implies that $u(\underline{\pi}) = \underline{u}$.

Now choose a payoff pair $(\pi, u(\pi))$ that is sustained by pure actions other than the outside option. Notice that such a payoff pair must exist since $(\overline{\pi}, u(\overline{\pi}))$ is an extremal point that is sustained by pure actions. Suppose $(\pi, u(\pi))$ is obtained by a PPE with first-period actions (e, w, b_s, b_n) and first-period continuation payoffs (π_s, π_n, u_s, u_n). Next, consider an alternative strategy profile with the same first-period continuation payoffs (π_s, π_n, u_s, u_n) but different first-period actions $(e, \widehat{w}, b_s, b_n)$, where $\widehat{w} = w + \varepsilon$ for some $\varepsilon > 0$. This strategy profile generates payoffs $(\widehat{\pi}, \widehat{u})$, where $\widehat{\pi} = \pi - (1-\delta)(1+\alpha\theta)\varepsilon$ and $\widehat{u} = u(\pi) + (1-\delta)\varepsilon$. It can be checked that as long as $\widehat{\pi} \geq \underline{\pi}$ or, equivalently, $\varepsilon \leq (\pi - \underline{\pi}) / ((1-\delta)(1+\alpha\theta))$, this alternative strategy profile satisfies all of the constraints in Section III.A and therefore constitute a PPE. Let $\varepsilon = (\pi - \underline{\pi}) / ((1-\delta)(1+\alpha\theta))$. The above then implies that $(\underline{\pi}, u(\pi) + (\pi - \underline{\pi}) / (1+\alpha\theta))$ is a PPE payoff, which contradicts $u(\underline{\pi}) = \underline{u}$.

Finally, notice that by sending ε to zero, the above implies that $u'_{-}(\pi) \leq -1/(1+\alpha\theta)$ for all $\pi > \underline{\pi}$. Below we will use this fact to prove part (iii.).

Part (ii.): Concavity follows directly from the availability of the public randomization device.

Part (iii.): Consider a PPE with payoffs $(\pi, u(\pi))$, where $\pi \in (\underline{\pi}, \overline{\pi})$, and associated first-period actions (e, w, b_s, b_n) and first-period continuation payoffs (π_s, π_n, u_s, u_n). Now consider an alternative profile with the same first-period continuation payoffs (π_s, π_n, u_s, u_n) but in which first-period actions are given by $(\widehat{e}, w, b_s, b_n)$, where $\widehat{e} = e + \varepsilon$ for some $\varepsilon > 0$. It follows from $\mathrm{PK_M}$ and $\mathrm{PK_W}$ that under this strategy profile the payoffs are given by $\widehat{\pi} = \pi + (1-\delta)(y(e+\varepsilon) - y(e))$ and $\widehat{u} = u(\pi) - (1-\delta)(c(e+\varepsilon) - c(e))$. For small enough ε, the alternative strategy profile satisfies all the constraints in Section III.A (with the exception of the $\mathrm{IC_W}$ constraint that we ignore throughout) and therefore constitutes a PPE.

Since $u(\pi)$ is the frontier, it must be that

$$u(\pi + (1-\delta)(y(e+\varepsilon) - y(e))) \geq \widehat{u} = u(\pi) - (1-\delta)(c(e+\varepsilon) - c(e)).$$

Sending ε to zero, we then obtain that

$$-\frac{c'(e)}{y'(e)} \leq u'_{+}(\pi).$$

Since u is concave and, as shown in the proof of part (ii.), $u'_{-}(\pi) < -1/(1+\alpha\theta)$, we have $u'_{+}(\pi) \leq u'_{-}(\pi) < -1/(1+\alpha\theta)$. This implies that

$$\frac{c'(e)}{y'(e)} \geq \frac{1}{1+\alpha\theta}.$$

Since $c'(e)/y'(e) = 0$ for $e = 0$, we therefore have $e > 0$. Next, consider another alternative strategy profile with first-period actions $(\widehat{e}, w, b_s, b_n)$, where $\widehat{e} = e - \varepsilon$ for small $\varepsilon > 0$, and first-period continuation payoffs (π_s, π_n, u_s, u_n). Proceeding in exactly the same way, it can be shown that $u'_{-}(\pi) \leq -c'(e)/y'(e)$. This implies that $u'_{+}(\pi) = u'_{-}(\pi)$ and thus u is differentiable. Since we have $u'_{-}(\pi) \leq -1/(1+\alpha\theta)$ this concludes the proof except for the claim that $-1 < u'(\pi)$. For simplicity of exposition we will prove this part after Lemma 5. The proofs of Lemmas 3-5 do not make use of this claim. ■

Proof of Lemma 3: Recall from Lemma 1 that the PPE payoff set is compact. Consider a PPE with payoffs $(\pi, u(\pi))$ and associated first-period actions (e, w, b_s, b_n) and first-period continuation payoffs (π_s, π_n, u_s, u_n). Suppose to the the contrary of the claim that $u_s < u(\pi_s)$. Now consider an alternative strategy profile with the same first-period actions but in which first-period continuation payoffs are given by $(\pi_s, \pi_n, \widehat{u}_s, u_n)$, where $\widehat{u}_s = u_s + \varepsilon$ and where $\varepsilon > 0$ is small enough such that $u_s + \varepsilon \leq u(\pi_s)$. It follows from the promise keeping constraints PK_M and PK_W that under this alternative strategy profile the payoffs are given by $\widehat{\pi} = \pi$ and $\widehat{u} = u(\pi) + \delta(1-\theta)\varepsilon > u(\pi)$. It can be checked that this alternative strategy profile satisfies all the constraints in Section III.A (with the exception of the IC_W constraint that we ignore throughout) and therefore constitutes a PPE. Since $\widehat{u} > u(\pi)$ this contradicts the definition of $u(\pi)$. Thus it must be that $u_s = u(\pi_s)$. The proof for $u_n = u(\pi_n)$ is analogous. ■

Proof of Lemma 4: Part (i.): By combining the truth-telling constraints TT_N and TT_S we have

$$(1+\alpha)(1-\delta)(b_n - b_s) \geq \delta(\pi_n - \pi_s) \geq (1-\delta)(b_n - b_s).$$

Since $(1+\alpha) > 0$, this implies that $b_n - b_s \geq 0$.

Next, consider a PPE with payoffs $(\pi, u(\pi))$ and associated first-period actions (e, w, b_s, b_n) and first-period continuation payoffs (π_s, π_n, u_s, u_n) and suppose that $b_s > 0$. Now consider an alternative strategy profile with the same continuation payoffs (π_s, π_n, u_s, u_n) but in which first-period actions are given by $(e, \widehat{w}, \widehat{b}_s, \widehat{b}_n)$, where $\widehat{w} = w + b_s$, $\widehat{b}_s = 0$, and $\widehat{b}_n = b_n - b_s$. It follows from the promise keeping constraints PK_M and PK_W that under this alternative strategy profile the payoffs are given by $(\pi, u(\pi))$. It can be checked that this alternative strategy profile satisfies all the constraints in Section III.A (with the exception of the IC_W constraint that we ignore throughout) and therefore constitutes a PPE. This proves part (i.).

Part (ii.): Consider a PPE with payoffs $(\pi, u(\pi))$ and associated first-period actions (e, w, b_s, b_n) and first-period continuation payoffs $(\pi_s, \pi_n, u(\pi_s), u(\pi_n))$. Suppose that for this PPE the TT_N is slack, that is, $\delta(\pi_n - \pi_s) > (1-\delta)(b_n - b_s)$. Together with the non-reneging constraint

NR_S this implies that NR_N is slack. Now consider an alternative strategy profile with the same first-period actions (e, w, b_s, b_n) but in which first-period continuation payoffs are given by $(\widehat{\pi}_s, \widehat{\pi}_n, u(\widehat{\pi}_s), u(\widehat{\pi}_n))$, where $\widehat{\pi}_s = \pi_s + (1-\theta)\varepsilon$ and $\widehat{\pi}_n = \pi_n - \theta\varepsilon$ for $\varepsilon > 0$. It follows from the promise keeping constraints PK_M and PK_W that under this strategy profile the payoffs are given by $\widehat{\pi} = \pi$ and

$$\widehat{u} = (1-\delta)(w + \theta b_s + (1-\theta) b_n - c(e)) + \delta(\theta u(\widehat{\pi}_s) + (1-\theta) u(\widehat{\pi}_n)).$$

From the concavity of u it then follows that

$$\widehat{u} \geq (1-\delta)(w + \theta b_s + (1-\theta) b_n - c(e)) + \delta(\theta u(\pi_s) + (1-\theta) u(\pi_n)) = u(\pi).$$

It can be checked that for sufficiently small ε this alternative strategy profile satisfies all the constraints in Section III.A (with the exception of the IC_W constraint that we ignore throughout) and therefore constitutes a PPE. Since $\widehat{u} \geq u(\pi)$ this implies that for any PPE with payoffs $(\pi, u(\pi))$ for which TT_N is not binding there exists another PPE for which TT_N is binding and which gives the parties weakly larger payoffs. ■

Proof of Lemma 5: By Lemma 2, every point on the PPE frontier can be supported by pure actions other than the outside options. Lemma 3 and the definition of u then imply that $\pi + u(\pi)$ is given by

$$\pi + u(\pi) = \max_{e,w,\pi_s,\pi_n} (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\pi_n + u(\pi_n)) - (1-\delta)\theta\alpha w$$

subject to the constraints in Section III.A (with the exception of the IC_W constraint that we ignore throughout). Notice that we have used Lemma 4 to substitute out b_s and b_n in the objective function.

To reduce the number of constraints, consider first the feasibility constraints. By Lemma 4 the non-negative constraints NN_S is no longer relevant. At the end of this proof we show that the non-negativity constraint NN_N is also no longer relevant. Moreover, in the proof of Lemma 2 we showed that $c'(e)/y'(e) \geq 1/(1+\alpha\theta)$. We therefore have $e(\pi) > 0$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$ which implies that NN_e is slack.

Next, Lemma 3 implies that SE_N and SE_S can be reduced to $\underline{\pi} \leq \pi_s \leq \overline{\pi}$ and $\underline{\pi} \leq \pi_n \leq \overline{\pi}$. The only relevant feasibility constraints are therefore given by NN_W, SE_N, and SE_S.

Next, we examine the no deviation constraints with the exception of the IC_W constraint which we ignore throughout. Part (i.) of Lemma 4 and SE_S imply that NR_N and NR_S are satisfied. Part (ii.) of Lemma 4 further implies that TT_N and TT_S are satisfied.

Next, we turn to the promise-keeping constraints. We obtain the version of the PK_M constraint in the lemma by using Lemma 4 to substitute b_s and b_n out of the original PK_M constraint. The PK_W constraint is satisfied because $\pi + u(\pi)$ is the solution to the functional equation given by the constrained maximization problem.

Finally, consider the non-negativity constraint NN_N. Notice first that NN_N is equivalent to $\pi_s \leq \pi_n$. Now consider a PPE with payoffs $(\pi, u(\pi))$ and associated first-period actions (e, w) and first-period continuation payoffs (π_s, π_n). Suppose to the contrary of the claim that $\pi_n < \pi_s$. There then exists an alternative strategy profile that satisfies all the constraints in Lemma 5 and generates a strictly larger joint surplus, leading to a contradiction. Specifically, consider an alternative strategy profile with the same first-period actions (e, w) but in which the first-period continuation payoffs are given by $(\pi_s, \widehat{\pi}_n)$, where $\widehat{\pi}_n = \pi_n + \varepsilon$ for some $\varepsilon > 0$. Since, by Lemma 2, $1 + u'(\pi) > 0$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$, the alternative strategy profile generates a strictly larger joint surplus $\pi + u(\pi)$. Moreover, it can be checked that this alternative strategy profile satisfies all the constraints in Lemma 5. As claimed above, the non-negativity constraint NN_N is therefore no longer relevant. ■

Proof of the last part of Lemma 2 ($-1 < u'(\pi)$): The Lagrangian associated with the maximization problem in Lemma 5 is given by

$$
\begin{aligned}
\pi + u(\pi) \quad = \quad & L = (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\pi_n + u(\pi_n)) - (1-\delta)\theta\alpha w \\
& + \lambda_1(\pi - (1-\delta)y(e) - \delta\pi_s + (1-\delta)(1+\theta\alpha)w) \\
& + \lambda_2(\delta\pi_s - \delta\underline{\pi}) + \lambda_3(1-\delta)w + \lambda_4(\delta\overline{\pi} - \delta\pi_n),
\end{aligned}
$$

where we use the fact that $\pi_s \leq \pi_n$ (as shown in the proof of Lemma 5) to eliminate the constraints $\pi_s \leq \overline{\pi}$ and $\pi_n \leq \underline{\pi}$. Note that this is a well-defined concave program. The first order conditions with respect to π_n, π_s, w, and e are given by

$$(1-\theta)(1 + u'(\pi_n)) - \lambda_4 = 0, \tag{FOC$_\text{N}$}$$

$$\theta(1 + u'(\pi_s)) - \lambda_1 + \lambda_2 = 0, \tag{FOC$_\text{S}$}$$

$$-\theta\alpha + \lambda_1(1+\theta\alpha) + \lambda_3 = 0, \tag{FOC$_\text{W}$}$$

and

$$y'(e) - c'(e) - \lambda_1 y'(e) = 0. \tag{FOC$_\text{e}$}$$

Furthermore, the envelope condition is given by

$$1 + u'(\pi) = \lambda_1. \tag{envelope}$$

To prove that $u'(\pi) > -1$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$, note that FOC_S and FOC_N imply that

$$\begin{aligned} \lambda_1 &= \theta(1 + u'(\pi_s)) + \lambda_2 \\ &\geq \theta(1 + u'(\pi_n)) \\ &\geq 0. \end{aligned}$$

It therefore follows from the envelope condition that $1 + u'(\pi) \geq 0$ for $\pi \in [\underline{\pi}, \overline{\pi}]$.

To finish the proof, we need to rule out that $u'(\pi) = -1$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$. Suppose the contrary and define $\Gamma \equiv \{\pi \in [\underline{\pi}, \overline{\pi}] \,|u'(\pi) = -1\}$. We now establish three facts. First, if $\pi \in \Gamma$ then $\pi_s(\pi) \in \Gamma$. This follows from FOC_S, $\lambda_1 = 0$, and $\lambda_2 \geq 0$. Second, if $\pi \in \Gamma$ then $e(\pi) = e^{FB}$, where e^{FB} is the first-best effort level that solves $y'(e) = c'(e)$. This follows from FOC_e and $\lambda_1 = 0$. Finally, if $\pi \in \Gamma$ then $w(\pi) = 0$. To see this, note that since $\lambda_1 = 0$, FOC_W implies that $\lambda_3 > 0$. It then follows from the complementarity slackness condition with respect to NN_W that $w(\pi) = 0$.

The three facts above imply that the manager can maximize her pay by always claiming to have been hit by a shock. This is a contradiction since the $(-c(e^{FB}))$ would then be smaller than his outside option. ■

Proof of Proposition 1: We first continue to ignore the IC_W constraint. For this relaxed problem we first prove parts (i.) to (v.) and then show that the optimal relational contract is unique. Finally, we show that the optimal relational contract of the relaxed problem satisfies the IC_W constraint.

Parts (i.) to (v.): The expression for b_n in part (i.) follows from the TT_N constraint. The fact that $b_n > 0$ follows from $\pi_s^* < \overline{\pi}$ which is shown in part (ii.).

For part (ii.) consider FOC_S. Suppose first that $\lambda_2 > 0$. In this case, the complementarity condition associated with SE_S implies that $\pi_s = \underline{\pi}$. Suppose next that $\lambda_2 = 0$. In this case

$$1 + u'(\pi_s) = \frac{1}{\theta}\lambda_1 = \frac{1}{\theta}(1 + u'(\pi)),$$

where the first equality follows from FOC_S and the second equality follows from the envelope condition. Since Lemma 2 implies that $1 + u'(\pi) > 0$, we have that $u'(\pi_s) > u'(\pi)$. Part (ii.) then follows from the concavity of u.

Part (iii.) follows from $1 + u'(\pi) > 0$ and FOC_N.

Part (iv.) follows from combining FOC_e and the envelope condition. Note that since $c'(e)/y'(e)$ is strictly increasing, $e^*(\pi)$ is unique.

For part (v.), recall from Lemma 2 that $u'(\pi) \leq -1/(1 + \alpha\theta)$. Suppose first that $u'(\pi) < -1/(1 + \alpha\theta)$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$. In this case, FOC_W and the envelope condition imply that $w \equiv 0$.

Suppose next that there exists a line segment with $u'(\pi) = -1/(1+\alpha\theta)$. Recall that $\widehat{e}$ is the unique effort level satisfying $c'(\widehat{e})/y'(\widehat{e}) = 1/(1+\alpha\theta)$. Part (iv.) implies that on the line segment $e = \widehat{e}$. Moreover, we must have $\pi_s = \underline{\pi}$. To see this, note that FOC_S implies that

$$\begin{aligned} \lambda_2 &= \lambda_1 - \theta(1 + u'(\pi_s)) \\ &= \frac{\alpha\theta}{1+\alpha\theta} - \theta(1 + u'(\pi_s)) \\ &\geq (1-\theta)\frac{\alpha\theta}{1+\alpha\theta}, \end{aligned}$$

where the inequality follows from $u'(\pi_s) \leq -1/(1+\alpha\theta)$. Since $\lambda_2 > 0$, the complementary slackness with respect to SE_S then implies that $\pi_s = \underline{\pi}$. Since $\pi_s(\pi) = \underline{\pi}$ and $e(\pi) = \widehat{e}$ for π on the line segment, the PK_M constraint implies that $w^*(\pi) = ((1-\delta)y(\widehat{e}) + \delta\underline{\pi} - \pi)/((1-\delta)(1+\alpha\theta))$. This proves part (v.).

Uniqueness: Note that in all of the derivations above, $e^*(\pi), w^*(\pi)$, and $\pi_n^*(\pi)$ are unique. Moreover, PK_M implies that $\pi_s^*(\pi)$ is unique. This proves that the optimal relational contract is unique as long as TT_N is binding. In Lemma 4 we showed that for any optimal relational contract there exists an equivalent one in which TT_N is binding. Next we show that TT_N has to be binding for all optimal relational contracts.

For this purpose, suppose to the contrary that there exists an optimal relational contract for which TT_N is not binding. This implies that there exists a PPE with payoffs $(\pi, u(\pi))$ and associated first-period actions (e, w, b_s, b_n) and first-period continuation payoffs $(\pi_s, \pi_n, u(\pi_s), u(\pi_n))$ such that

$$\delta(\pi_n - \pi_s) > (1-\delta)(b_n - b_s).$$

Now consider an alternative strategy profile with the same first-period actions (e, w, b_s, b_n) but in which first-period continuation payoffs are given by $(\widehat{\pi}_s, \widehat{\pi}_n, u(\widehat{\pi}_s), u(\widehat{\pi}_n))$, where $\widehat{\pi}_s = \pi_s + (1-\theta)\varepsilon$ and $\widehat{\pi}_n = \pi_n - \theta\varepsilon$. It can be checked that when $\varepsilon = [\delta(\pi_n - \pi_s) - (1-\delta)(b_n - b_s)]/\delta$, this alternative strategy profile satisfies all the constraints in Section III.A and is therefore constitutes a PPE with payoffs $(\pi, u(\pi))$. Moreover, the TT_N constraint is binding since

$$\delta(\widehat{\pi}_n - \widehat{\pi}_s) = \delta(\pi_n - \pi_s) - \delta\varepsilon = (1-\delta)(b_n - b_s).$$

Notice that under this alternative strategy profile $\widehat{\pi}_n < \overline{\pi}$. This is a contradiction since we saw above that when TT_N is binding, the optimal relational contract must have $\pi_n^*(\pi) = \overline{\pi}$.

Checking that IC_W is satisfied: Recall that the IC_W constraint is given by $\delta\underline{u} + (1-\delta)w^*(\pi) \leq u(\pi)$. The constraint is clearly satisfied when $w = 0$, that is, for all $\pi \geq \widehat{\pi}$.

For $\pi < \widehat{\pi}$, part (v.) implies that

$$\frac{\mathrm{d}(\delta \underline{u} + (1-\delta) w^*(\pi))}{\mathrm{d}\pi} = -\frac{1}{1+\alpha\theta} = u'(\pi),$$

which in turn implies that

$$\begin{aligned} & u(\pi) - (\delta \underline{u} + (1-\delta) w^*(\pi)) \\ = \;& u(\widehat{\pi}) - (\delta \underline{u} + (1-\delta) w^*(\widehat{\pi})) \\ > \;& 0. \end{aligned}$$

This shows that $\mathrm{IC_W}$ is satisfied for all $\pi \in [\underline{\pi}, \overline{\pi}]$. ■

Appendix B: The Effects of Liquidity Constraints

In this appendix we prove the results in Section V that analyzes the model with liquidity constraints. Specifically, the firm is subject to the liquidity constraint that

$$\max\{w + b_s, w + b_n\} \leq (1 + m)y$$

for some $m > 0$. This constraint significantly complicates the analysis. In particular, the PPE frontier is no longer differentiable. To make the analysis more tractable, we assume that the manager cannot pay the worker in a shock state ($\alpha = \infty$). Consequently, this implies that $w \equiv 0$ and $b_s \equiv 0$.

Since the analysis with the liquidity constraints follows similar steps as the main model, we omit the proofs for results that are obtained from identical arguments. Below, we first describe the basic properties of the PPE frontier in the background subsection and then prove the main results in the dynamics subsection. The last subsection provides several sufficient conditions that allow for further characterization of the dynamics.

1 Background

Proceeding in the same way as the main model, we can show that when the PPE frontier is sustainable by pure actions (other than the outside options), the joint surplus $\pi + u(\pi)$ is defined recursively by the following problem:

$$\pi + u(\pi) = \max_{e,\pi_s,\pi_n} (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\pi_n + u(\pi_n)) \tag{1}$$

such that

$$\delta\pi_n \leq \pi + (1-\delta) my(e), \tag{LIQ$_\mathrm{F}$}$$

$$\underline{\pi} \leq \pi_n \leq \overline{\pi}, \tag{LSE$_\mathrm{N}$}$$

$$\underline{\pi} \leq \pi_s \leq \overline{\pi}, \tag{LSE$_\mathrm{S}$}$$

and

$$\pi = (1-\delta) y(e) + \delta\pi_s. \tag{LPK$_\mathrm{M}$}$$

The problem above corresponds with that in Lemma 5 with several modifications. In particular, $\mathrm{LIQ_F}$ is the extra liquidity constraint for the firm. In addition, $w \equiv 0$ implies that $\mathrm{NN_W}$ is no longer needed. Moreover, w does not appear in the LPK_M, and relatedly, the truth-telling condition under the no-shock state is given by

$$\delta\left(\pi_n - \pi_s\right) = \left(1-\delta\right) b_n. \qquad (\text{LTT}_\text{N})$$

In addition to these modifications, there are several other differences between the main model and the model with liquidity constraint. First, unlike the main model, it is no longer true that each point on the PPE frontier can be sustained by pure actions.

LEMMA B1. *There exists a critical level of expected profits* $\pi_0 \in [\underline{\pi}, \overline{\pi})$ *such that for all* $\pi \geq \pi_0$ *the PPE frontier* $u(\pi)$ *is supported by pure actions and for all* $\pi \in (\underline{\pi}, \pi_0)$ *it is supported by randomization. Specifically, for any* $\pi < \pi_0$ *the manager and the worker randomize between terminating their relationship and playing the strategies that deliver expected payoffs* π_0 *and* $u(\pi_0)$*.*

Proof: Define π_0 as the smallest payoff of the manager at which $u(\pi_0)$ can be sustained by pure actions other than the outside options. To see that π_0 is well defined, let $\Pi = \{\pi \in [\underline{\pi}, \overline{\pi}] |$ $(\pi, u(\pi))$ can be supported by pure actions other than the outside options$\}$. We need to show that the set Π is non-empty and closed. Note that $\overline{\pi} \in \Pi$ because $(\overline{\pi}, u(\overline{\pi}))$, being an extremal point of the PPE payoff set, can be sustained by pure actions, and since $\overline{\pi} \neq \underline{\pi}$, the pure actions that support $(\overline{\pi}, u(\overline{\pi}))$ is not the outside options. To see that Π is closed, consider a convergent sequence $\{\pi_j\}_{j=1}^{\infty} \subset \Pi$ such that $\lim \pi_j = \pi$. Let $(e_j, b_{n_j}, \pi_{s_j}, \pi_{n_j}, u_{s_j}, u_{n_j})$ be the first-period actions and first-period continuation payoffs associated with $(\pi_j, u(\pi_j))$. Since the actions and continuation payoffs maximize the PPE payoff, the Maximum Theorem implies that the actions and continuation payoffs are upper hemi-continuous in expected profits. We therefore have $\lim e_j = e(\pi)$, $\lim \pi_{s_j} = \pi_s(\pi)$, $\lim \pi_{n_j} = \pi_n(\pi)$, $\lim b_{n_j} = b_n(\pi)$, $\lim u_{n_j} = u_n(\pi)$, and $\lim u_{s_j} = u_s(\pi)$. It can be checked that the profile $(e(\pi), b_n(\pi), \pi_s(\pi), \pi_n(\pi), u_s(\pi), u_n(\pi))$ satisfies all the constraints and supports $(\pi, u(\pi))$, which implies that $\pi \in \Pi$. This shows that π_0 is well defined.

Repeating the proof in Lemma 2, it is immediate that Π is convex and thus $\Pi = [\pi_0, \overline{\pi}]$. The part that for all $\pi \geq \pi_0$ the PPE frontier $u(\pi)$ is supported by pure actions follows from the definition of π_0 and the convexity of Π. It remains to show that for any $\pi \in (\underline{\pi}, \pi_0)$ the manager and the worker randomize between terminating their relationship and playing the strategies that deliver expected payoffs π_0 and $u(\pi_0)$. Note that if $\pi_0 = \underline{\pi}$, the randomization region does not exist and there is nothing to prove. Therefore, let us assume that $\pi_0 > \underline{\pi}$. Notice that $(\underline{\pi}, u(\underline{\pi}))$ is an extremal point of the PPE payoff set, and since $\pi_0 > \underline{\pi}$, $(\underline{\pi}, u(\underline{\pi}))$ must be supported by the outside options, and therefore, $(\underline{\pi}, u(\underline{\pi})) = (\underline{\pi}, \underline{u})$.

Now for any $\pi \in (\underline{\pi}, \pi_0)$, $(\pi, u(\pi))$ is sustained by randomization by the definition of π_0. Since the PPE payoff set is two-dimensional and convex and $(\pi, u(\pi))$ is at its boundary, $(\pi, u(\pi))$ can be expressed as the linear combination of two extremal points. Denote these two points by $(\pi', u(\pi'))$ and $(\pi'', u(\pi''))$, where $\pi' < \pi < \pi''$. Since these are both extremal points they are sustained by pure actions. Moreover, since $\pi' < \pi < \pi_0$ and $u(\pi')$ is supported by pure actions, we must have $\pi' = \underline{\pi}$ by the definition of π_0. Now by the concavity of u, it is clear that we can have $\pi'' = \pi_0$. We provide the argument that π'' must equal to π_0 following Lemma B2, whose proof does not depend on the uniqueness of π''. ■

Second, the PPE frontier u is no longer differentiable for all π for some parameter value m. However, since u is again concave (given the public randomization device), both the left and the right derivatives exist. This implies that the results written as equalities of derivatives can be replaced with a pair of corresponding inequalities involving left and right derivatives.

Third, while it remains true that $u'_{-}(\pi) > -1$ for all π, we no longer always have $\pi_n = \overline{\pi}$. When LIQ_{F} binds, $\pi_n < \overline{\pi}$. The lemma below gives the exact expression for π_n and proves that $u'_{-}(\pi) > -1$ for all π.

LEMMA B2. $u'_{-}(\pi) > -1$, *and the continuation payoff following a no-shock period is given by the following:*

$$\pi_n^*(\pi) = \min\{\overline{\pi}, \frac{1}{\delta}(\pi + (1-\delta)my)\}.$$

Proof: The Lagrangian of the joint surplus associated with the recursive problem is given by

$$\begin{aligned} L \;=\; & (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\pi_n + u(\pi_n)) \\ & + \lambda_1[\pi - (1-\delta)y(e) - \delta\pi_s] \\ & + \lambda_2(\delta\pi_s - \delta\underline{\pi}) + \lambda_3[\pi + (1-\delta)my(e) - \delta\pi_n] + \lambda_4(\delta\overline{\pi} - \delta\pi_n). \end{aligned}$$

The first-order conditions with respect to π_s, π_n and e are given by

$$\theta(1 + u'_{-}(\pi_s)) \geq \lambda_1 - \lambda_2 \geq \theta(1 + u'_{+}(\pi_s)) \tag{2}$$

$$(1-\theta)(1 + u'_{-}(\pi_n)) \geq \lambda_3 + \lambda_4 \geq (1-\theta)(1 + u'_{+}(\pi_n)) \tag{3}$$

$$(1 - \lambda_1 + m\lambda_3)y'(e) - c'(e) = 0. \tag{4}$$

The envelope condition is given by

$$1 + u'_{-}(\pi) \geq \lambda_1 + \lambda_3 \geq 1 + u'_{+}(\pi). \tag{5}$$

Notice that if $u'_-(\pi) > -1$ for all $\pi \in [\pi_0, \overline{\pi})$, the expression for $\pi_n^*(\pi)$ immediately follows. To see this, suppose to the contrary that for some π,

$$\pi_n^*(\pi) < \min\{\overline{\pi}, \frac{1}{\delta}[\pi + (1-\delta)my(e(\pi))]\}.$$

On the one hand, this implies that $\lambda_4(\pi) = \lambda_3(\pi) = 0$ since the associated constraints are not binding. As a result, $u'_+(\pi_n) \leq -1$ by equation (3). On the other hand, consider $\hat{\pi}_n = \pi_n + \varepsilon$ for $\varepsilon > 0$. Since $\pi_n < \overline{\pi}$, we can make ε small enough so that $\hat{\pi}_n < \overline{\pi}$. By the concavity of u, we have

$$u'_+(\pi_n) \geq u'_-(\hat{\pi}_n) > -1,$$

where the second inequality comes from the fact that $u'_-(\pi) > -1$ for all $\pi \in [\pi_0, \overline{\pi})$. This is a contradiction because we have shown that $u'_+(\pi_n) \leq -1$.

Now we prove that $u'_-(\pi) > -1$ for all $\pi \in [\pi_0, \overline{\pi})$ in two steps. In step 1, we show that $u'_-(\pi) \geq -1$ for all $\pi \in [\pi_0, \overline{\pi})$. Suppose to the contrary that $u'_-(\pi) < -1$ for some $\pi \in [\pi_0, \overline{\pi})$. Then there is a PPE with payoff $(\pi, u(\pi))$ such that its first-period actions are denoted by (e, b_n) and its first-period continuation payoffs are denoted by (π_s, π_n, u_s, u_n). Notice that $\lambda_3(\pi)$ has to be zero because otherwise we have

$$\pi_n = \frac{1}{\delta}[\pi + (1-\delta)\, my\,(e)] > \pi.$$

Since $u'_-(\pi_n) \geq -1$ by equation (3), this implies that $u'_-(\pi) \geq u'_-(\pi_n) \geq -1$, contradicting the claim that $u'_-(\pi) < -1$.

Next, since $\lambda_2(\pi) \geq 0$, (2) and (5) imply that

$$1 + u'_-(\pi) \geq \lambda_1 \geq \theta(1 + u'_+(\pi_s)).$$

Since $u'_-(\pi) < -1$, the above inequality means that $u'_+(\pi_s) < -1$. Notice that if $\pi_s < \pi_n$, the concavity of u implies that $u'_+(\pi_s) \geq u'_-(\pi_n) \geq -1$, which contradicts $u'_+(\pi_s) < -1$.

As a result, we must have $\pi_s = \pi_n$. $\mathrm{LTT_N}$ then implies that $b_n = 0$, and the worker's promise-keeping condition then gives that

$$u(\pi) = \delta u(\pi_s) - (1-\delta)c(e).$$

This implies $u(\pi_s) > u(\pi)$, and consequently, $\pi_s < \pi$ given u is concave and $u'_-(\pi) < -1$.

Now consider an alternative strategy profile with first-period actions (e, b_n) and first-period continuation payoffs $(\hat{\pi}_s, \hat{\pi}_n, \hat{u}_s, \hat{u}_n)$, where $\hat{\pi}_k = \pi_k + \varepsilon$ and $\hat{u}_k = u(\hat{\pi}_k)$, for $k = s, n$ and $\varepsilon > 0$.

It can be checked that for small enough ε, the alternative strategy profile is a PPE with payoffs $(\hat{\pi}, \hat{u})$, where

$$\begin{aligned}\hat{\pi} &= \pi + \delta\varepsilon \\ \hat{u} &= u(\pi) + \delta[u(\hat{\pi}_s) - u(\pi_s)].\end{aligned}$$

By the definition of u, we have $\hat{u} \leq u(\hat{\pi})$. Sending ε to zero, we get

$$u'_+(\pi) \geq u'_+(\pi_s).$$

This means $u'_+(\pi) = u'_+(\pi_s)$ since we have shown that $\pi_s < \pi$. Hence, the PPE frontier contains a line segment in $[\pi_s, \pi]$ with a slope strictly below -1. Define the left end point of the line segment as π_l. For all $\pi > \pi_l$, it must be that $\pi_s^*(\pi) = \pi_n^*(\pi) = \pi_l$ because otherwise it contradicts $u'_-(\pi_n) \geq -1$. Moreover, equation (4) and the envelope condition (5) implies that $e(\pi)$ is constant since $\lambda_3(\pi) = 0$. But this contradicts $\mathrm{LPK_M}$

$$\pi = (1-\delta)\, y(e) + \delta\pi_s,$$

since the left hand side is strictly increasing in π and the right hand side is constant. This finishes proving that $u'_-(\pi) \geq -1$ for all $\pi \in [\pi_0, \overline{\pi})$.

Now in step 2, we show that it is impossible that $u'_-(\pi) = -1$ for some $\pi \in [\pi_0, \overline{\pi})$. Suppose the contrary. Then the PPE frontier contains a line segment with slope -1 on the right side. Again define π_l as the left end point of this segment. In order to derive a contradiction, we first show the following result:

$$\text{for any } \pi > \pi_l, \text{ it must be that } e^*(\pi) \equiv e^{FB} \text{ and } \pi_s(\pi) > \pi_l. \tag{6}$$

Note that if $\pi > \pi_l$, it is easy to see that $\lambda_3 = \lambda_4 = 0$ (because otherwise $u'_-(\pi) \geq u'_-(\pi_n) > -1$). In addition, since u is differentiable at π, and $u'(\pi) = -1$, it must be that $\lambda_1 = 0$. Then from (4) we have $e^*(\pi) = e^{FB}$, which solves $y'\left(e^{FB}\right) = c'\left(e^{FB}\right)$. To see that $\pi_s(\pi) > \pi_l$, suppose $\pi_s(\pi) < \pi_l$, then $u'_+(\pi_s) \geq u'_-(\pi_s + \varepsilon) > -1$, for $0 < \varepsilon < \pi_l - \pi_s$. This contradicts condition (2),

$$0 \geq \theta(1 + u'_+(\pi_s)) - \lambda_1 + \lambda_2,$$

due to the facts that $\lambda_1 = 0$ and $\lambda_2 \geq 0$. Suppose $\pi_s(\pi) = \pi_l$, since $e(\pi) = e^{FB}$ for all $\pi > \pi_l$, we have $\pi_s(\pi - \varepsilon) < \pi_s(\pi) = \pi_l$ for some small positive ε. This becomes the case when $\pi_s(\pi) < \pi_l$ for some $\pi > \pi_l$. Similar to the previous analysis, we can derive a contradiction, which completes the proof for result (6).

Finally, result (6) implies that the manager can maximize her pay by always announcing that the state is a shock state, because the worker always makes an effort of e^{FB} and the manager never makes a payment. But this is a contradiction because in this case, the worker's payoff is $-c(e^{FB})$, which is smaller than his outside option. ■

Now we can prove the uniqueness of π'' in Lemma B1.

Proof: Suppose the contrary. Let $u = L(x)$ be the line that passes through $(\underline{\pi}, \underline{u})$ and $(\pi, u(\pi))$, for $x \in [\underline{\pi}, \overline{\pi}]$. Then we have that $u(\pi) = L(\pi)$ and $u(\pi'') = L(\pi'')$. We will derive a contradiction in three steps.

In step 1, we show that $u(x) = L(x)$ for all $x \in [\underline{\pi}, \pi'']$. To see this, note that the payoffs on the line $u = L(x)$ are PPE payoffs, and therefore, $u(x) \geq L(x)$ by the definition of u. Now if $u(x') > L(x')$ for some x', then let $(\pi, \hat{u})$ be the weighted average of $(\underline{\pi}, \underline{u})$ and $(x', u(x'))$, and we obtain that $\hat{u} > L(\pi) = u(\pi)$, which is a contradiction. Thus, the PPE frontier contains a line segment in the left. Let $\pi_r \geq \pi'' > \pi_0$ be the right end point of this segment. Since $u(\overline{\pi}) = \underline{u}$, it cannot be that $\pi_r = \overline{\pi}$.

In step 2, we show that $e(x) \equiv e$ for all $x \in [\pi_0, \pi_r]$. Suppose to the contrary that $e(x_1) \neq e(x_2)$, then following the proof of part (ii.) of Lemma 2, we can find a PPE with payoffs $(\rho x_1 + (1-\rho)x_2, \hat{u})$ for $\rho \in (0, 1)$, such that $\hat{u} > u(\rho x_1 + (1-\rho)x_2)$, which contradicts the definition of u.

In step 3, let $(\pi_r, u(\pi_r))$ be supported by a PPE with first-period actions (e_r, b_{nr}) and first-period continuation payoffs $(\pi_{sr}, \pi_{nr}, u_{sr}, u_{nr})$. We show that $\pi_{nr} \leq \pi_r$. Suppose the contrary. The definition of π_r then implies that $u'_{-}(\pi_{nr}) < u'_{-}(\pi_r)$. Since $e(x) \equiv e$ for all $x \in [\pi_0, \pi_r]$ by step 2, the worker's promise-keeping condition ($\pi = (1-\delta)\, y\,(e) + \delta \pi_s$) then implies that π_s strictly increases with π. Since $\pi_s(\pi_0) \geq \underline{\pi}$, it must be that $\pi_{sr} > \underline{\pi}$. Now consider an alternative strategy profile with first-period actions (e_r, b_{nr}) and first-period continuation payoffs $(\hat{\pi}_{sr}, \hat{\pi}_{nr}, \hat{u}_{sr}, \hat{u}_{nr})$ where $\hat{\pi}_{sr} = \pi_{sr} - \varepsilon$, and $\hat{\pi}_{nr} = \pi_{nr} - \varepsilon$, for $\varepsilon > 0$. When ε is small enough, it can be checked that the alternative strategy profile is a PPE with associated payoff $(\hat{\pi}, \hat{u})$, where

$$\begin{aligned}
\hat{\pi} &= \pi_r - \delta\varepsilon \\
\hat{u} &= u(\pi_r) + \theta\delta[u(\hat{\pi}_{sr}) - u(\pi_{sr})] + (1-\theta)\delta[u(\hat{\pi}_{nr}) - u(\pi_{nr})].
\end{aligned}$$

The definition of u gives $\hat{u} \leq u(\hat{\pi})$. Sending ε to zero, we get

$$u'_{-}(\pi_r) \leq \theta u'_{-}(\pi_{sr}) + (1-\theta)u'_{-}(\pi_{nr}).$$

By the definition of π_r and the concavity of u, the above inequality holds only when $u'_{-}(\pi_{sr}) = u'_{-}(\pi_{nr}) = u'_{-}(\pi_r)$, a contradiction.

Finally, the result of step 3 contradicts Lemma B1, which states that

$$\pi_n^*(\pi) = \min\{\overline{\pi}, \frac{1}{\delta}[\pi + (1-\delta)my(e^*(\pi))]\} > \pi, \text{ for all } \pi < \overline{\pi}. \quad \blacksquare$$

Next, the proof that $e > 0$ in the model with liquidity constraints is different from that in the main model. We state this result as a separate lemma below.

LEMMA B3. *For each* $\pi \geq \pi_0$, $e^*(\pi) > 0$.

Proof: We first prove that $e^*(\pi) > 0$ for $\pi > \pi_0$. Suppose to the contrary that there exists a $\pi > \pi_0$ with $e^*(\pi) = 0$. The liquidity constraint then implies that $b_n = 0$, and LTT_N implies that $\pi_s^*(\pi) = \pi_n^*(\pi)$. As a result, LPK_M implies that $\pi_s^*(\pi) = \pi_n^*(\pi) = \pi/\delta > \pi$. We now derive a contradiction in three steps.

In step 1, we show that $u'_-(\pi) = u'_-(\pi/\delta)$, so u is a line segment in $[\pi, \pi/\delta]$. To see this, suppose to the contrary that $u'_-(\pi) > u'_-(\pi/\delta)$. Now consider an alternative strategy profile with first-period actions (e, b_n) and first-period continuation payoffs $(\hat{\pi}_s, \hat{\pi}_n, \hat{u}_s, \hat{u}_n)$, where $\hat{\pi}_k = \pi_k - \varepsilon$ and $\hat{u}_k = u(\hat{\pi}_k)$, for $k = s, n$. It can be checked that for small enough ε, the alternative strategy profile is PPE with $(\hat{\pi}, \hat{u})$, where

$$\begin{aligned} \hat{\pi} &= \pi - \delta\varepsilon \\ \hat{u} &= u(\pi) + \delta[u(\pi/\delta - \varepsilon) - u(\pi/\delta)]. \end{aligned}$$

By the definition of u, we have

$$u(\pi - \delta\varepsilon) \geq u(\pi) + \delta[u(\pi/\delta - \varepsilon) - u(\pi/\delta)].$$

Sending ε to 0, we get that

$$u'_-(\pi) \leq u'_-(\pi/\delta),$$

which contradicts the assumption that $u'_-(\pi) > u'_-(\pi/\delta)$.

In step 2, we show that u is a line segment in $[\pi_0, \pi/\delta]$. To see this, for each $\pi' < \pi$, consider a strategy profile with first-period actions $(\hat{e}, \hat{b}_n)$ and first-period continuation payoffs $(\hat{\pi}_s, \hat{\pi}_n, \hat{u}_s, \hat{u}_n)$ such that $\hat{e}(\pi') = \hat{b}_n(\pi') = 0$, $\hat{\pi}_k(\pi') = \pi'/\delta$, and $\hat{u}_k = u(\hat{\pi}_k)$, for $k = s, n$. It can be checked that these strategy profiles are PPEs and that their payoffs lie on the left extension of the line segment between $(\pi, u(\pi))$ and $(\pi/\delta, u(\pi/\delta))$. The concavity of u then implies that these payoffs are on the PPE frontier. It follows that u is a line segment in $[\pi_0, \pi/\delta]$. This implies that the PPE frontier contains a line segment in the left. Let π_r be the right end point of this segment.

In step 3, we derive a contradiction on the left derivative of u for payoffs near $\delta\pi_r$. To do this, first note that the same construction as in step 2 implies that $e^*(\delta\pi_r)=0$, $\pi_s^*(\delta\pi_r)=\pi_n^*(\delta\pi_r)=\pi_r$. It then follows that for any $\pi'\in(\delta\pi_r,\delta\pi_r+\varepsilon)$ for small enough $\varepsilon>0$, we must have $\pi_n^*(\pi')>\pi_r$ by $\mathrm{LIQ_F}$ and $\pi_s^*(\pi')>\pi_0$ by the continuity of π_s^*. This implies that $u'_-(\pi_n^*(\pi'))<u'_-(\pi')$ by the definition of π_r. In addition, $\pi_s^*(\pi')>\pi_0$, so $u'_-(\pi_s^*(\pi'))$ exists. Moreover, $u'_-(\pi_s^*(\pi'))\leq u'_-(\pi')$ since u is a line segment in $[\pi_0,\pi_r]$. The inequalities on π_s^* and π_n^* then imply that

$$(1-\theta)u'_-(\pi_n^*(\pi'))+\theta u'_-(\pi_s^*(\pi'))<u'_-(\pi').$$

Now, starting at $(\pi',u(\pi'))$, consider an alternative strategy profile with the same first-period actions, but whose first-period continuation payoffs are given by $(\pi_s^*(\pi')-\varepsilon,u(\pi_s^*(\pi')-\varepsilon))$ and $(\pi_n^*(\pi')-\varepsilon,u(\pi_n^*(\pi')-\varepsilon))$. For small enough ε, it can be checked that this alternative strategy profile is a PPE. Sending ε to 0, we get

$$(1-\theta)u'_-(\pi_n^*(\pi'))+\theta u'_-(\pi_s^*(\pi'))\geq u'_-(\pi'),$$

which contradicts the earlier inequality. This proves that we cannot have $e(\pi)=0$ for $\pi>\pi_0$.

Finally, suppose to the contrary that $e^*(\pi_0)=0$. As a result $\pi_s^*(\pi_0)=\pi_n^*(\pi_0)>\pi_0$. The same argument as above then implies that u is a line segment in $[\pi_0,\pi_n^*(\pi)]$, and we can derive the same contradiction as above. ■

The next lemma shows that the PPE frontier for $\pi\geq\pi_0$ can be divided into (at most) three regions. In the right region, the liquidity constraints are slack. In the left region, the liquidity constraints are binding and $\pi_n<\overline{\pi}$. In the middle region, the liquidity constraints are binding and $\pi_n=\overline{\pi}$.

LEMMA B4. *There exists π_1 and π_2 with $\pi_0\leq\pi_1\leq\pi_2<\overline{\pi}$ such that the following holds: (i.) if $\pi>\pi_2$, $\pi_n^*(\pi)=\overline{\pi}$ and $\pi+(1-\delta)my(e^*)>\delta\overline{\pi}$, (ii.) if $\pi\in[\pi_1,\pi_2]$, $\pi_n^*(\pi)=\overline{\pi}$ and $\pi+(1-\delta)my(e^*)=\delta\overline{\pi}$, and (iii.) if $\pi<\pi_1$, $\pi_n^*(\pi)<\overline{\pi}$ and $\pi+(1-\delta)my(e^*)<\delta\overline{\pi}$.*

Proof: To prove part (i.), it suffices to show that for any $\pi'\geq\pi$, $\pi+(1-\delta)my(e^*(\pi))>\delta\overline{\pi}$ implies $\pi'+(1-\delta)my(e^*(\pi'))>\delta\overline{\pi}$. Now take a manager's payoff π with $\pi+(1-\delta)my(e^*(\pi))>\delta\overline{\pi}$, the same argument as in Lemma 2 shows that u is differentiable at π with

$$u'(\pi)=-\frac{c'(e^*(\pi))}{y'(e^*(\pi))}.$$

To see this, consider a PPE with payoff $(\pi,u(\pi))$ and first-period actions (e,b_n) and first-period continuation payoffs
Consider an alternative strategy profile with the same first-period continuation payoffs, but with a

different first-period actions $(\hat{e}, b_n)$ where $\hat{e} = e + \varepsilon$ for some $\varepsilon > 0$. For small enough ε, it can be checked that this alternative strategy profile is a PPE with payoffs $(\hat{\pi}, \hat{u})$, where

$$\begin{aligned} \hat{\pi} &= \pi + (1-\delta)\left(y(e+\varepsilon) - y(e)\right) \\ \hat{u} &= u(\pi) - (1-\delta)\left(c(e+\varepsilon) - c(e)\right). \end{aligned}$$

The definition of u implies that $\hat{u} \leq u(\hat{\pi})$. As a result,

$$u\left(\pi + (1-\delta)\left(y(e+\varepsilon) - y(e)\right)\right) \geq u(\pi) - (1-\delta)\left(c(e+\varepsilon) - c(e)\right).$$

Sending ε to zero, we obtain

$$u'_+(\pi) \geq -\frac{c'(e)}{y'(e)}.$$

Next, consider an alternative strategy profile with again the same first-period continuation payoffs, but with a different first-period actions $(\hat{e}, b_n)$ where $\hat{e} = e - \varepsilon$ for some $\varepsilon > 0$. By Lemma B3, $e > 0$. Therefore, if ε is sufficiently small, we have that (i.) $\hat{e} > 0$; (ii.) $\pi + (1-\delta)my(\hat{e}) > \delta\overline{\pi}$. One can then check that this alternative strategy is a PPE with payoff $(\hat{\pi}, \hat{u})$, where

$$\begin{aligned} \hat{\pi} &= \pi + (1-\delta)\left(y(e-\varepsilon) - y(e)\right) \\ \hat{u} &= u(\pi) - (1-\delta)\left(c(e-\varepsilon) - c(e)\right). \end{aligned}$$

Similar to the analysis above, we have $\hat{u} \leq u(\hat{\pi})$, which implies

$$u\left(\pi + (1-\delta)\left(y(e-\varepsilon) - y(e)\right)\right) \geq u(\pi) - (1-\delta)\left(c(e-\varepsilon) - c(e)\right).$$

Sending ε to zero, we obtain

$$u'_-(\pi) \leq -c'(e)/y'(e).$$

Finally, the concavity of u implies that $u'_-(\pi) \geq u'_+(\pi)$, which, combined with the above results, gives that

$$u'_-(\pi) = u'_+(\pi) = -c'(e)/y'(e).$$

Given that $u'(\pi) = -c'(e)/y'(e)$, the concavity of u then implies that

$$\frac{c'(e(\pi'))}{y'(e(\pi'))} > \frac{c'(e(\pi))}{y'(e(\pi))}.$$

As a result, $e(\pi') \geq e(\pi)$, and, thus, $\pi' + (1-\delta)my(e(\pi')) > \delta\overline{\pi}$. This proves part (i.).

Given part (i.), we prove parts (ii.) and (iii.) simultaneously by showing that if $\pi_n(\pi) = \overline{\pi}$, then for all $\pi' > \pi$, $\pi_n(\pi') = \overline{\pi}$. Suppose the contrary. This implies that there exists a pair of $\pi' > \pi$ such that

$$\pi'_n = \pi' + (1-\delta)my(e(\pi')) < \pi + (1-\delta)my(e(\pi)) = \overline{\pi}.$$

Now consider an alternative strategy profile with first-period actions $(\widehat{e}, \widehat{b}_n)$ and first-period continuation payoffs $(\widehat{\pi}_s, \widehat{\pi}_n)$ such that $\widehat{e} = e(\pi') + \varepsilon$, $\widehat{\pi}_s = \pi_s(\pi')$, and $\widehat{\pi}_n$ and $\widehat{b}_n$ are chosen appropriately so that the liquidity constraint remains to bind. It can be checked that this alternative strategy profile constitutes a PPE, and its payoffs fall weakly below the PPE frontier. Sending ε to zero, we get

$$\frac{c'(e(\pi'))}{y'(e(\pi'))} \geq (m+1)(1-\theta)(1+u'_+(\pi_n(\pi')) - u'_+(\pi').$$

Similarly, at π, decrease $e(\pi)$ to $e(\pi) - \varepsilon$, keep $\pi_s(\pi)$ the same, and decrease $\pi_n(\pi)$ correspondingly, we get

$$\frac{c'(e(\pi))}{y'(e(\pi))} \leq (m+1)(1-\theta)(1+u'_-(\overline{\pi})) - u'_-(\pi).$$

Since u is concave, we have $u'_+(\pi_n(\pi')) > u'_-(\overline{\pi})$ and $u'_+(\pi') \leq u'_-(\pi)$. The two inequalities above then imply that

$$\frac{c'(e(\pi'))}{y'(e(\pi'))} \geq \frac{c'(e(\pi))}{y'(e(\pi))},$$

and, thus, $e(\pi') \geq e(\pi)$. But this contradicts

$$\pi' + (1-\delta)my(e(\pi')) < \pi + (1-\delta)my(e(\pi)). \quad \blacksquare$$

Notice that Lemma 6 follows immediately from Lemma B1 and B4. Also notice that the right region always exist (so that $\pi_2 < \overline{\pi}$) because $\pi + (1-\delta)my(e(\pi)) > \delta\overline{\pi}$ for all $\pi > \delta\overline{\pi}$. In contrast, the middle region or the left region does not always exist. This can occur, for example, when m is large and when $\underline{\pi}$ is large. In this case, the firm's liquidity constraint is always slack, and we return to the main model. At the end of this section, we provide sufficient conditions for the existence of the left and the middle region.

2 Dynamics of the Optimal Relational Contract

In this subsection, we study the dynamics of the optimal relational contract. The following lemma characterizes the effort and continuation payoff functions associated with the optimal relational contract.

LEMMA B5. *There exists a unique set of effort and continuation payoffs* $(e^*(\pi), \pi_s^*(\pi), \pi_n^*(\pi))$ *that satisfies the following:*

(i.) *For* $\pi > \pi_2$, *the PPE frontier is differentiable with*

$$\frac{c'(e^*)}{y'(e^*)} = -u'(\pi)$$

and

$$\theta u'_+(\pi_s^*) - (1-\theta) \le u'(\pi) \le \theta u'_-(\pi_s^*) - (1-\theta).$$

In this region, both e^* and π_s^* weakly increase with π.

(ii.) For $\pi \in [\pi_1, \pi_2]$, if $m \neq 0$, then

$$y(e^*) = \frac{\delta\overline{\pi} - \pi}{(1-\delta)m} \text{ and } \pi_s^* = \frac{(m+1)\pi - \delta\overline{\pi}}{\delta m}.$$

In this region, e^* strictly decreases with π and π_s^* strictly increases with π.

If $m = 0$, then $\pi_1 = \pi_2 = \delta\overline{\pi}$. u is not differentiable at $\delta\overline{\pi}$, and e^* and π_s^* satisfy

$$-u'_+(\delta\overline{\pi}) \le \frac{c'(e^*)}{y'(e^*)} \le -u'_-(\delta\overline{\pi})$$

and

$$\theta u'_+(\pi_s^*) - (1-\theta) \le u'(\pi) \le \theta u'_-(\pi_s^*) - (1-\theta).$$

(iii.) For $\pi \in [\pi_0, \pi_1)$, e^*, π_s^*, and π_n^* satisfy

$$\begin{aligned}(1+m)(1-\theta)(1+u'_+(\pi_n^*)) - \frac{c'(e^*)}{y'(e^*)} &\le u'_+(\pi) \le u'_-(\pi) \\ &\le (1+m)(1-\theta)(1+u'_-(\pi_n^*)) - \frac{c'(e^*)}{y'(e^*)}.\end{aligned} \tag{L-e-n}$$

When $\pi_s^* > \underline{\pi}$,

$$\begin{aligned}(1+m)(\theta u'_+(\pi_s^*) - (1-\theta)) + \frac{c'(e^*)}{y'(e^*)} &\le m u'_+(\pi) \le m u'_-(\pi) \\ &\le (1+m)(\theta u'_-(\pi_s^*) - (1-\theta)) + \frac{c'(e^*)}{y'(e^*)},\end{aligned} \tag{L-e-s}$$

and

$$\theta u'_+(\pi_s^*) + (1-\theta) u'_+(\pi_n^*) \le u'_+(\pi) \le u'_-(\pi) \le \theta u'_-(\pi_s^*) + (1-\theta) u'_-(\pi_n^*). \tag{L-s-n}$$

In this region, π_s^* weakly increases in π.

Proof: Notice that for $\pi > \pi_2$, the differentiability of the payoff frontier and that $u'(\pi) = -c'(e^*)/y'(e^*)$ are both established in the proof of Lemma B4. In addition, the inequalities in this lemma are all equalities if u were differentiable. In this case, the equalities can be obtained directly from the Kuhn-Tucker conditions of Lagrangian associated with the constrained maximization problem (1). The formal proof of the inequalities is standard and is omitted here. Below, we show that u is not differentiable at $\delta\overline{\pi}$ and that π_s^* is weakly increasing in $[\pi_0, \pi_1]$.

To see that u is not differentiable at $\delta\overline{\pi}$ when $m=0$, note that $\pi=\delta\overline{\pi}$ is the only point in the middle region. On the one hand, part (i.) implies that

$$u'_{+}(\delta\overline{\pi}) = -\frac{c'(e^{*})}{y'(e^{*})}.$$

On the other hand, L-e-n implies that

$$u'_{-}(\delta\overline{\pi}) \geq -\frac{c'(e^{*})}{y'(e^{*})} + (1+m)(1-\theta)(1+u'_{+}(\pi_n^{*})).$$

Notice that from Lemma B2, we have $u'_{+}(\pi_n^{*}) > -1$. Therefore, $u'_{-}(\delta\overline{\pi}) > u'_{+}(\delta\overline{\pi})$, so u is not differentiable at $\delta\overline{\pi}$.

Next, to see that π_s^{*} is weakly increasing for $\pi \in [\pi_0, \pi_1]$, we assume that u is differentiable at π and $\pi_s^{*}(\pi)$ to ease exposition, and the argument can be adapted to the non-differentiable case. When u is differentiable at π and $\pi_s^{*}(\pi)$, L-e-s can be written as

$$-(1+m)(1-\theta) + (1+m)\theta u'(\pi_s^{*}) + \frac{c'(e^{*})}{y'(e^{*})} = mu'(\pi).$$

As π increases, the right hand side of the equation above weakly decreases. Now suppose to the contrary that π_s^{*} decreases. It follows that $u'(\pi_s^{*})$ weakly increases. Moreover, when π increases and π_s^{*} decreases, e^{*} increases by the LPK_{M}. Consequently, $c'(e^{*})/y'(e^{*})$ strictly increases. In summary, if π_s^{*} decreases, the left hand side strictly increases. This is a contradiction because the right hand side weakly decreases.

Finally, we show that $(e^{*}(\pi), \pi_s^{*}(\pi), \pi_n^{*}(\pi))$ is unique. Suppose first that $\pi > \pi_2$. Then $\pi_n^{*}(\pi) = \overline{\pi}$, $e^{*}(\pi)$ is uniquely determined by $u'(\pi)$, and $\pi_s^{*}(\pi)$ is uniquely determined by LPK_{M}. Next, suppose that $\pi \in [\pi_1, \pi_2]$. Then $\pi_n^{*}(\pi) = \overline{\pi}$, $e^{*}(\pi)$ is uniquely given by LIQ_{F}, and $\pi_s^{*}(\pi)$ is uniquely given by LPK_{M}. Finally, for $\pi \in [\pi_0, \pi_1)$ we prove uniqueness by contradiction. Suppose to the contrary that there are two different PPEs that both generate $(\pi, u(\pi))$. Let the first-period action and the first-period continuation payoffs associated with these two PPEs as $(e_1, b_1, \pi_{s_1}, \pi_{n_1}, u_{s_1}, u_{n_1})$ and $(e_2, b_2, \pi_{s_2}, \pi_{n_2}, u_{s_2}, u_{n_2})$ respectively. It suffices to show that these two vectors are identical.

Define $\widetilde{e}$ as the effort level satisfying

$$y(\widetilde{e}) = \frac{1}{2}y(e_1) + \frac{1}{2}y(e_2).$$

Notice that since y is strictly increasing and strictly concave, $\widetilde{e} \leq \frac{1}{2}e_1 + \frac{1}{2}e_2$. Next, define $\widetilde{b}_n = \frac{1}{2}b_{n_1} + \frac{1}{2}b_{n_2}$, $\widetilde{\pi}_s = \frac{1}{2}\pi_{s_1} + \frac{1}{2}\pi_{s_2}$, and $\widetilde{\pi}_n, \widetilde{\pi}_s$ and $\widetilde{\pi}_n$ analogously. Now consider an alternative strategy profile with first-period actions $(\widetilde{e}, \widetilde{b}_n)$ and the first-period continuation payoffs $(\widetilde{\pi}_s, \widetilde{\pi}_n, \widetilde{u}_s, \widetilde{u}_n)$. It can be checked that the alternative strategy profile satisfies all the constraints in Section 1 and therefore constitute a PPE. Moreover, its payoff is given by $(\widetilde{\pi},\widetilde{u})$, where $\widetilde{\pi}=\pi$, and

$$\widetilde{u}=u(\pi)+(1-\delta)\left(\frac{1}{2}c(e_1)+\frac{1}{2}c(e_2)-c(\widehat{e})\right).$$

Since $c(e)$ is strictly increasing and strictly convex, it follows that $\widetilde{u}>u(\pi)$ unless $e_1=e_2$. The effort level $e^*(\pi)$ is therefore unique. The constraints $\mathrm{LPK_M}$ and $\mathrm{LIQ_F}$ then imply the uniqueness of $b_n^*(\pi),\pi_s^*(\pi)$ and $\pi_n^*(\pi)$. This implies that the two PPEs are identical and proves uniqueness.■

Since $\pi_s^*(\pi)$ is weakly increasing in π in all three regions, the continuity of $\pi_s^*(\pi)$ then implies that $\pi_s^*(\pi)$ is weakly increasing for all $\pi\in[\pi_0,\overline{\pi}]$. In contrast, $e^*(\pi)$ is decreasing in the middle region. In other words, the worker's effort level increases as the manager's payoff decreases.

The following two lemmas state the properties of the manager's continuation payoff functions for $\pi\geq\pi_0$ in the optimal relational contract.

LEMMA B6. *For all $\pi\in[\pi_0,\overline{\pi})$, $\pi_n^*(\pi)>\pi$.*

Proof: It is clear that $\pi_n^*(\pi)>\pi$ for $\pi\in[\pi_1,\overline{\pi})$ since $\pi_n^*(\pi)=\overline{\pi}$. For $\pi\in[\pi_0,\pi_1)$, suppose the contrary is true. The continuity of π_n^* then implies that there exists a largest $\pi<\overline{\pi}$ satisfying $\pi_n^*(\pi)=\pi$. Notice that $u'_+(\pi_s^*(\pi))\geq u'_+(\pi_n^*(\pi))=u'_+(\pi)$. Therefore, the first inequality in L-s-n implies that $u'_+(\pi_s^*)=u'_+(\pi_n^*)$. In other words, u is a line segment between $\pi_s^*(\pi)$ and $\pi_n^*(\pi)$. Define π_r as the right end point of this segment, and we must have $\pi_r<\overline{\pi}$. Because otherwise, the monotonicity of π_s^* implies that $u'_+(\pi_s^*(\pi'))=u'_+(\pi_n^*(\pi'))=u'_+(\pi')$ for all $\pi'>\pi$. This implies that $u'_-(\pi_s^*(\pi'))=-1$ for $\pi'>\pi_2$ by part (i.) of Lemma B5. But this contradicts $u'_-(\pi_s^*(\pi'))>-1$ by Lemma B2.

Now given $\pi_r<\overline{\pi}$, notice that the continuity of π_n^* then implies that there exists a $\pi''\in(\pi,\pi_r)$ such that $\pi_n^*(\pi'')>\pi_r$. In addition, $\pi_s^*(\pi'')\in(\pi_s^*(\pi),\pi_r)$ since π_s^* increases with π. At π'', however, $u'_-(\pi_s^*(\pi''))=u'_-(\pi'')>u'_-(\pi_s^*(\pi''))$, violating the last inequality in L-s-n. ■

LEMMA B7. *For all $\pi\in(\pi_0,\overline{\pi}]$, $\pi_s^*(\pi)<\pi$.*

Proof: It is clear that $\pi_s^*(\pi)<\pi$ for $\pi\geq\pi_1$. For $\pi<\pi_1$, suppose to the contrary there exists a manager's payoff π with $\pi_s^*(\pi)\geq\pi$. By L-s-n, u must be a line segment between π and $\pi_n^*(\pi)$. Let π_r be the right end point of this line segment. Using the same argument as in Lemma B6, we must have $\pi_r<\overline{\pi}$. Now by Lemma B6 and the monotonicity of π_s, there exists a $\pi'\in(\pi,\pi_r)$ such that $\pi_n^*(\pi')>\pi_r$ and $\pi_s^*(\pi')\in(\pi,\pi_+)$. This implies that $u'_-(\pi_s^*(\pi'))=u'_-(\pi')>u'_-(\pi_n^*(\pi'))$, again violating the last inequality in L-s-n. ■

The next lemma provides further information on the dynamics by characterizing π_0.

LEMMA B8. $\pi_s^*(\pi_0) = \underline{\pi}$ *if* $\pi_0 > \underline{\pi}$.

Proof: Suppose $\pi_0 > \underline{\pi}$ and to the contrary $\pi_s^*(\pi_0) > \underline{\pi}$. L-s-n implies that $u'(\pi_0) = u'(\pi_s^*) = u'(\pi_n^*)$. We same argument in Lemma B6 then implies that there exists a $\pi_r > \pi_0$ such that u is a line segment in $[\underline{\pi}, \pi_r]$. Notice that this is a contradiction by the proof of the uniqueness of π'' in the last part of Lemma B1. ■

Proof of Proposition 2: Notice that part (i.) follows from part (iii.) and LTT_N. Part (ii.) is a restatement of Lemma B7 and B8. Part (iii.) follows directly from Lemma B2 and B4. Finally, part (iv.) follows from Lemma B5. ■

3 Sufficient Conditions

In this subsection, we first provide a condition for the left region to exist, implying that the liquidity constraint is relevant. A sufficient condition for $\pi_0 > \underline{\pi}$ is given next. Finally, we provide a sufficient condition for the existence of the middle region.

Define $\overline{\pi}^u$ as the maximal equilibrium payoff of the manager in the main model.

LEMMA B9. *The PPE frontier contains more than the right region, i.e.,* $(\pi_2 > \underline{\pi})$ *if and only if the following Condition L holds:*

$$\delta\overline{\pi}^u > (1+m)\underline{\pi}. \tag{L}$$

Proof: The PPE frontier contains more than the right region when the liquidity constraint is violated in the main model for some π. By Proposition 1, this is equivalent to that the liquidity constraint is violated at $\underline{\pi}$. In our main model, $\pi_n^*(\underline{\pi}) = \overline{\pi}^u$ and $\pi_s^*(\underline{\pi}) = \underline{\pi}$. In addition, NR_S states that

$$\underline{\pi} = \delta\pi_s^*(\underline{\pi}) + (1-\delta)y(e^*(\underline{\pi})).$$

This implies that $y(e^*(\underline{\pi})) = (1-\delta)\underline{\pi}$. Therefore, the liquidity constraint that $\delta\pi_n^* \leq \pi + (1-\delta)my(e^*)$ is equivalent to

$$\delta\overline{\pi}^u \leq (1+m)\underline{\pi}. \quad ■$$

Next, we describe a sufficient condition for $\pi_0 > \underline{\pi}$.

LEMMA B10. *Suppose Condition L holds.* $\pi_0 > \underline{\pi}$ *if* $m < \frac{\theta}{1-\theta}$ *and*

$$\underline{u} > \frac{(1-\theta)(1+m)}{\theta - m(1-\theta)} \frac{y'(\underline{e}) - c'(\underline{e})}{y'(\underline{e})}\underline{\pi} - c(\underline{e}),$$

where $\underline{e}$ is the unique effort level satisfying $y(\underline{e}) = \underline{\pi}$.

Proof: To prove that $\pi_0 > \underline{\pi}$ if the conditions above hold, we proceed as if u were differentiable to simplify the exposition. The argument can be adapted for the non-differentiable case by replacing the equalities involving derivatives with inequalities involving left and right derivatives. Now suppose to the contrary $\pi_0 = \underline{\pi}$, define the Lagrangian as

$$\begin{aligned}
\pi + u(\pi) &= L = (1-\delta)\left(y\left(e\right) - c\left(e\right)\right) + \theta\delta\left(\pi_s + u\left(\pi_s\right)\right) + (1-\theta)\,\delta\left(\pi_n + u\left(\pi_n\right)\right) \\
&\quad + \lambda_1(\pi - (1-\delta)\,y\left(e\right) - \delta\pi_s) \\
&\quad + \lambda_2(\pi + (1-\delta)my(e) - \delta\pi_n) \\
&\quad + \lambda_3(\delta\pi_s - \delta\underline{\pi}) + \lambda_4\delta(\overline{\pi} - \pi_n).
\end{aligned}$$

The first-order conditions are given by

$$\theta(1 + u'\left(\pi_s\right)) - \lambda_1 + \lambda_3 = 0. \qquad (\text{FOC}_\text{S})$$

$$(1-\theta)\left(1 + u'\left(\pi_n\right)\right) - \lambda_2 - \lambda_4 = 0. \qquad (\text{FOC}_\text{N})$$

$$(1 - \lambda_1 + m\lambda_2)y'\left(e\right) - c'\left(e\right) = 0. \qquad (\text{FOC}_\text{e})$$

The envelope condition is given by

$$1 + u'(\pi) = \lambda_1 + \lambda_2. \qquad \text{(envelope condition)}$$

We now proceed in two steps. In step 1, we provide an upper bound to $u'(\underline{\pi})$. To do this, notice that Condition L implies that $\pi_n(\underline{\pi}) < \overline{\pi}$. FOC_N then implies that at $\pi = \underline{\pi}$,

$$(1-\theta)(1 + u'\left(\pi_n(\underline{\pi})\right)) = \lambda_2.$$

By the envelope condition,

$$1 + u'(\underline{\pi}) = \lambda_1 + \lambda_2.$$

Since u is concave, $u'(\underline{\pi}) \geq u'\left(\pi_n\right)$. This then implies that $\lambda_2 \leq (1-\theta)\lambda_1/\theta$, and it follows that

$$1 + u'(\underline{\pi}) \leq \lambda_1 + \frac{1-\theta}{\theta}\lambda_1 = \frac{\lambda_1}{\theta}.$$

Next, we provide an upper bound of λ_1. Note that at $\underline{\pi}$, Proposition 2 implies that $\pi_s(\underline{\pi}) = \underline{\pi}$, and consequently, by LTT_N, $y(e(\underline{\pi})) = \underline{\pi}$. In other words, $e(\underline{\pi}) = \underline{e}$. FOC_e then implies that

$$\frac{y'\left(\underline{e}\right) - c'\left(\underline{e}\right)}{y'(\underline{e})} = \lambda_1 - m\lambda_2 \geq \frac{\theta - m(1-\theta)}{\theta}\lambda_1,$$

where the inequality follows because $\lambda_2 \leq (1-\theta)\lambda_1/\theta$ as shown above.

Combining the two inequalities above and noting $\theta - m(1-\theta) > 0$, we then obtain

$$1 + u'(\underline{\pi}) \leq \frac{\lambda_1}{\theta} \leq \frac{1}{\theta - m(1-\theta)} \frac{y'(\underline{e}) - c'(\underline{e})}{y'(\underline{e})},$$

which concludes step 1.

In step 2, we derive a contradiction on the joint payoff at $\underline{\pi}$ using the upper bound in step 1. Since the liquidity constraint binds at $\underline{\pi}$,

$$\delta\pi_n(\underline{\pi}) = \underline{\pi} + (1-\delta)my(\underline{e}) = (1 + (1-\delta)m)\underline{\pi}.$$

It follows that

$$\delta(\pi_n(\underline{\pi}) - \pi_s(\underline{\pi})) = (1-\delta)(1+m)\underline{\pi}.$$

The concavity of u then implies that

$$\begin{aligned}
& u(\pi_n(\underline{\pi})) - u(\pi_s(\underline{\pi})) \\
\leq\ & u'(\pi_s(\underline{\pi})))(\pi_n(\underline{\pi}) - \pi_s(\underline{\pi})) \\
=\ & u'(\underline{\pi})\left(\frac{(1-\delta)(1+m)}{\delta}\right)\underline{\pi}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \underline{\pi} + u(\underline{\pi}) \\
=\ & (1-\delta)(y(\underline{e}) - c(\underline{e})) + \delta(\pi_s(\underline{\pi}) + u(\pi_s(\underline{\pi}))) \\
& + (1-\theta)\delta((\pi_n(\underline{\pi}) + u(\pi_n(\underline{\pi})) - ((\pi_s(\underline{\pi}) + u(\pi_s(\underline{\pi}))) \\
\leq\ & (1-\delta)(\underline{\pi} - \underline{c}) + \delta(\underline{\pi} + u(\underline{\pi})) + (1-\theta)(1-\delta)(1+m)(1+u'(\underline{\pi}))\underline{\pi}.
\end{aligned}$$

Rearranging the above and substituting the inequality at the end of step 1, we get

$$\underline{\pi} + u(\underline{\pi}) \leq \underline{\pi} - \underline{c} + \frac{(1-\theta)(1+m)}{\theta - m(1-\theta)} \frac{y'(\underline{e}) - c'(\underline{e})}{y'(\underline{e})}\underline{\pi},$$

which contradicts the condition in the lemma. ■

Finally, we provide a sufficient condition for the middle region to exist.

LEMMA B11. *Suppose Condition L holds. The middle region exists, i.e.,* ($\pi_2 > \pi_1$) *if* $m < \theta/(1-\theta)$ *and* $(1+m)^2(1-\theta)\theta/m < 1$.

Proof: Condition L implies that the PPE frontier contains more than the right region. Suppose to the contrary that the middle region does not exist. Let π_d be the payoff that divides the left and the right region. The same argument in Proposition 2 shows that u is not differentiable at π_d with

$$\begin{aligned} u'_+(\pi_d) &= -\frac{c'(e)}{y'(e)}, \text{ and} \\ u'_-(\pi_d) &= (1+m)(1-\theta)(1+u'(\overline{\pi})) - \frac{c'(e)}{y'(e)}. \end{aligned}$$

Let $\Delta u'(\pi_d) = u'_+(\pi_d) - u'_-(\pi_d) > 0$. L-e-s then implies that

$$\Delta u'(\pi_d) \leq \frac{(1+m)\theta}{m} \Delta u'(\pi_s(\pi_d)).$$

In other words, u is not differentiable at $\pi_s(\pi_d)$.

Note that by L-e-n, we have

$$\Delta u'(\pi_s(\pi_d)) \leq (1+m)(1-\theta)\Delta u'(\pi_n(\pi_s(\pi_d))).$$

This implies that u is not differentiable at $\pi_n(\pi_s(\pi_d))$.

Since u is differentiable for all $\pi \in (\pi_d, \overline{\pi}]$, the above implies that either $\pi_n(\pi_s(\pi_d)) = \pi_d$ or $\pi_n(\pi_s(\pi_d)) \in (\pi_s(\pi_d), \pi_d)$. In the later case, we can show, using the same argument as above, that either $\pi_n^2(\pi_s(\pi_d)) = \pi_d$ or $\pi_n^2(\pi_s(\pi_d)) \in (\pi_n(\pi_s(\pi_d)), \pi_d)$, where the superscript denotes that applying π_n twice. Since $\pi_n > \pi$, the sequence of π_n^k is monotone in k. It follows that there exists some K such that

$$\pi_d = \pi_n^K(\pi_s(\pi_d)).$$

Note that for all $k \leq K$, we have by above

$$\Delta u'(\pi_n^k(\pi_s(\pi_d))) \leq (1+m)(1-\theta)\Delta u'(\pi_n^{k+1}(\pi_s(\pi_d))).$$

Linking this chain of inequalities, we obtain

$$\Delta u'(\pi_d) \leq \frac{(1+m)\theta}{m}(1+m)^K(1-\theta)^K \Delta u'(\pi_d).$$

This is a contradiction because by assumption $(1+m)(1-\theta) < 1$, and $(1+m)^2(1-\theta)\theta/m < 1$, so for all $K \geq 1$

$$\frac{(1+m)\theta}{m}(1+m)^K(1-\theta)^K < 1. \quad \blacksquare$$

Appendix C: Discussion

In this appendix, we prove the results in Section VI.

4 Folk Theorem and Conditions for First-Best

While the first-best is not achievable, the Folk Theorem holds: any interior payoff of the feasible payoff set belongs to the PPE payoff set as $\delta \to 1$.

PROPOSITION C1. *Define* $\pi^{FB} \equiv y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u}$. *For all* $\pi \in [\underline{\pi}, \pi^{FB})$ *and* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) \in (0,1)$ *such that for all* $\delta \geq \delta(\varepsilon)$,

$$u(\pi) + \pi > y\left(e^{FB}\right) - c\left(e^{FB}\right) - \varepsilon.$$

Proof: We prove the equivalent statement that for all π, the expected surplus destruction $D(\pi|\delta) \equiv y(e^{FB}) - c(e^{FB}) - \pi - u(\pi)$ goes to 0 as $\delta \to 1$. Since $u'(\pi) > -1$ by Lemma 2, $D(\pi|\delta)$ is decreasing in π and is thus maximized at $\pi = \underline{\pi}$. As a result, it suffices to show that $D(\underline{\pi}|\delta)$ goes to 0 as $\delta \to 1$.

Define d as the surplus destruction in the first-period play of the optimal relational contract at $\underline{\pi}$. It follows from Proposition 1 that

$$D(\underline{\pi}|\delta) = (1-\delta)\, d + \delta\left((1-\theta)\, D(\overline{\pi}|\delta) + \theta D(\underline{\pi}|\delta)\right),$$

or equivalently,

$$D(\underline{\pi}|\delta) = \frac{(1-\delta)\, d + \delta(1-\theta)\, D(\overline{\pi}|\delta)}{1-\delta\theta}.$$

This implies that

$$\lim_{\delta\to 1} D(\underline{\pi}|\delta) = \lim_{\delta\to 1} D(\overline{\pi}|\delta).$$

Since $D(\overline{\pi}|\delta) = y\left(e^{FB}\right) - c\left(e^{FB}\right) - \overline{\pi} - \underline{u} = \pi^{FB} - \overline{\pi}$, the proof is complete if we can show that for any $\varepsilon > 0$, $(\pi^{FB} - \varepsilon, \underline{u})$ can be sustained as a PPE payoff as $\delta \to 1$.

To do this, consider the following sequence of conjectured relational contracts in which the players take their outside options forever if any party deviates, and on the equilibrium path they choose $e = e^{FB}$, $w = 0$, and

$$b_t = \begin{cases} 0 & \text{if } \Theta = s \text{ and } t < t_n + T \\ \left(c\left(e^{FB}\right) + \underline{u}\right)\left(1 + \delta^{-1} + \delta^{-2} + ... + \delta^{-(t-1-t_n)}\right) & \text{otherwise,} \end{cases}$$

where t_n is the largest previous period (t) in which the bonus (b_t) is positive and T is an exogenously given deadline.

This sequence of conjectured relational contracts is the same as the long-term contracts described in Proposition C4 with the extra requirement that the manager must not renege on the bonuses. When the manager can commit to the bonus, we show in Proposition C4 that the conjecture relational contracts above are incentive contractible for both players. We avoid repeating the argument here but mention that the proof of Proposition C4 does not depend on previous results.

Notice that for all T, each conjectured relational contract above gives the worker a payoff of $\underline{u}$. Define $\pi(T)$ as the associated manager's payoff. The proof of Proposition C4 implies that the normalized expected destruction of surplus is given by

$$\delta\theta\frac{1-\theta\delta}{(1-\delta)}\frac{\theta^T\left(1-\delta^T\right)}{\left(1-(\theta\delta)^{T+1}\right)}\alpha c\left(e^{FB}\right) < T\theta^T\alpha c\left(e^{FB}\right).$$

Notice that $T\theta^T\alpha c\left(e^{FB}\right)$ goes to 0 as $T\to\infty$. As a result, for any $\varepsilon>0$, there exists a $T(\varepsilon)$, independent of δ, such that $\pi(T(\varepsilon))>\pi^{FB}-\varepsilon$. Moreover, for the fixed $T(\varepsilon)$, the maximal bonus is bounded above by $2^T\left(c(e^{FB})+\underline{u}\right)$ for all $\delta\in(1/2,1)$. It follows that as $\delta\to 1$, it is incentive compatible for the manager to commit to the bonus, so the conjectured relational contract is a PPE. This shows that $\left(\pi^{FB}-\varepsilon,\underline{u}\right)$ can be sustained as a PPE payoff and finishes the proof. ■

Our next proposition shows that to obtain the first-best, it is necessary to have (partial) public observability of the *no-shock* states rather than the *shock* states. Specifically, suppose that when the firm is not hit by a shock, with probability $p\in[0,1)$ it becomes publicly known that the firm's opportunity costs are low. When the firm is hit by a shock, with probability $q\in[0,1)$ it becomes publicly known that the firm's opportunity costs are high. First-best can then be achieved (for sufficiently high discount factors) if and only if $p>0$.

PROPOSITION C2. *If $p=0$, there does not exist a PPE in which the joint payoff of the manager and worker is equal to $y\left(e^{FB}\right)-c\left(e^{FB}\right)$. Otherwise, when*

$$\delta\geq\frac{c\left(e^{FB}\right)+\underline{u}}{c\left(e^{FB}\right)+\underline{u}+(1-\theta)\,p\left(y\left(e^{FB}\right)-c\left(e^{FB}\right)-\underline{u}-\underline{\pi}\right)},$$

there exists a PPE such that the joint payoff of the manager and worker is equal to $y\left(e^{FB}\right)-c\left(e^{FB}\right)$.

Proof: Since $q\geq 0$, define b_{sk} as the bonus payment when it is publicly known that the firm is hit by a shock and b_{su} stands for the bonus payment when the shock is unknown to the worker. Similarly, define π_{sk} and π_{su} as the associated continuation payoffs.

First consider $p = 0$. Suppose to the contrary that the first-best can be obtained. Let π_f be the smallest PPE payoff of the manager in which first-best is achieved. We derive a contradiction below by showing $\pi_f > y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u}$, which is the maximal feasible payoff of the manager that gives the worker a payoff of least $\underline{u}$.

Notice that if the first-best is obtained at π_f, we must have $w = b_{sk} = b_{su} = 0$ and $e = e^{FB}$. Moreover, the continuation payoffs must weakly exceed π_f since π_f is the smallest PPE payoff to obtain the first-best.

The promise-keeping constraint of the manager then implies that

$$\begin{aligned} \pi_f &= (1-\delta)\left(y\left(e^{FB}\right) - (1-\theta)\, b_n\right) + \delta\left(\theta q \pi_{sk} + \theta(1-q)\,\pi_{su} + (1-\theta)\,\pi_n\right) \\ &\geq (1-\delta)\, y\left(e^{FB}\right) + \delta\left(\theta q \pi_{sk} + (1-\theta q)\,\pi_{su}\right) \\ &\geq (1-\delta)\, y\left(e^{FB}\right) + \delta \pi_f, \end{aligned}$$

where the first inequality follows from the manager's truth-telling constraint in the no-shock state $(\delta(\pi_n - \pi_{su}) \geq (1-\delta)(b_n - b_{su}) = (1-\delta)\, b_n)$ and the second inequality follows because the continuation payoffs weakly exceed π_f. But this implies $\pi_f \geq y\left(e^{FB}\right) > y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u}$, which is a contradiction as noted above.

Next, consider $p > 0$. We construct a PPE that reaches the first-best. In particular, along the equilibrium path, let $w = b_{sk} = b_{su} = 0$ and $e = e^{FB}$. Moreover, when it is publicly known that the firm's opportunity costs are low, the manager pays out a bonus $b_{nk} = \left(c\left(e^{FB}\right) + \underline{u}\right) / \left((1-\theta)\, p\right)$. Notice that the equilibrium play does not depend on the manager's private information. When any party deviates, the play reverts to the unique subgame perfect Nash equilibrium within each period and the parties take their outside options in all future periods.

To check that this an equilibrium, we need to show, first, that the worker will put in effort and is willing to participate:

$$c\left(e^{FB}\right) \leq (1-\theta)\, p b_{nk} \text{ and } -c\left(e^{FB}\right) + (1-\theta) p b_{nk} \geq \underline{u}.$$

Given $b_{nk} = \left(c\left(e^{FB}\right) + \underline{u}\right) / \left((1-\theta)\, p\right)$, both inequalities are clearly satisfied.

Moreover, we need to show that the manager will not renege on the bonus and is willing to participate. The participation constraint is clearly implied by the non-reneging constraint, which is given by

$$(1-\delta)\, b_{nk} \leq \delta\left(y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u} - \overline{\pi}\right).$$

Given $b_{nk} = \left(c\left(e^{FB}\right) + \underline{u}\right) / \left((1-\theta)\, p\right)$, the inequality above is equivalent to

$$\delta \geq \frac{c\left(e^{FB}\right) + \underline{u}}{c\left(e^{FB}\right) + \underline{u} + (1-\theta)\, p\left(y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u} - \overline{\pi}\right)},$$

which is the condition in the proposition. Therefore, the first-best can be obtained if the condition above is satisfied. ■

5 Benchmarks: Public Information and Long-term Contracts

In this subsection, we analyze the dynamics of the relationship when the state of the world is public information. We characterize the PPE frontier, and for each payoff pair on the frontier, we state the associated effort, base wage, bonuses, and the continuation payoffs. This essentially specifies the dynamics of the relationship. Since the analysis is similar to and simpler than that in the private information case, we only state and prove the main results.

LEMMA C1. *With public information, the PPE frontier satisfies the following. For each PPE payoff of the manager π,*

$$\pi + u(\pi) = \max_{e,w,\pi_s,b_n} (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\overline{\pi} + u(\overline{\pi})) - (1-\delta)\theta\alpha w \tag{Public Program}$$

subject to

$$\pi = (1-\delta)(y(e) - (1+\theta\alpha)w - (1-\theta)b_n) + \delta(\theta\pi_s + (1-\theta)\overline{\pi}), \tag{PK$_\text{M}$}$$

$$w \geq 0, \text{ and} \tag{NN$_\text{W}$}$$

$$(1-\delta)b_n \leq \delta(\overline{\pi} - \underline{\pi}). \tag{NR$_\text{S}$}$$

Lemma C1 directly corresponds to Lemma 5 in the main model. As in the main model, $b_s \equiv 0$ and $\pi_n \equiv \overline{\pi}$, so they do not appear as choice variables. Notice that unlike the main model, there can be multiple choices for π_n. The multiplicity arises only when the first-best is achievable and does not affect the dynamics. We choose $\pi_n \equiv \overline{\pi}$ to better connect our results here to those in the main model. Another difference is that the maximization problem does not contain the truth-telling constraint in the no-shock state ($(1-\delta)(b_n - b_s) = \delta(\pi_n - \pi_s)$) since information is public. This implies that b_n is now a choice variable included in the program. As a result, we need to include the non-reneging constraint in the no-shock state. Since the proof of Lemma C1 is essentially the same as that of Lemma 5, we omit it here. The next proposition is the main result of this subsection.

PROPOSITION C3. *For any level of expected profit π, the PPE payoff frontier $u(\pi)$ and associated actions and continuation payoffs satisfy the followings:*

(i.) *For all* $\pi \geq \underline{\pi}$,

$$u'(\pi) \leq -\frac{1}{1+\theta\alpha}.$$

(ii.) *For all* $\pi \geq \underline{\pi}$, *the associated effort level is given by*

$$\frac{c'(e)}{y'(e)} = -u'(\pi).$$

(iii.) *(The middle region) When* $u'(\pi) \in (-1, -1/(1+\theta\alpha))$, *the associated wage, shock-state continuation payoff and the bonus are given by*

$$w = 0,\ \pi_s = \pi,\ \text{and}\ b_n = \frac{\delta}{1-\delta}\left(\overline{\pi} - \underline{\pi}\right).$$

In this region, $u'(\pi)$ *is strictly decreasing.*

(iv.) *(The right region) When* $u'(\pi) = -1$, *the PPE payoff frontier reaches the first-best at* π, *i.e.,* $u(\pi) = y\left(e^{FB}\right) - c\left(e^{FB}\right) - \pi$. *The associated wage and effort are given by*

$$w = 0\ \text{and}\ e = e^{FB}.$$

There can be multiple choices of π_s *and* b_n. *One such choice is*

$$\pi_s = \pi\ \text{and}\ b_n = \frac{(1-\delta)\,y(e) - (1+\delta\theta)\,\pi - \delta(1-\theta)\,\overline{\pi}}{(1-\delta)(1-\theta)},$$

where $\overline{\pi} = y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u}$.
The right region exists if and only if

$$\frac{1-\delta}{1-\theta}\left(c\left(e^{FB}\right) + \underline{u}\right) \leq \delta\left(y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u} - \underline{\pi}\right).$$

In this case, its left boundary is given by $\left((1-\delta)y\left(e^{FB}\right) + (1-\theta)\,\delta\underline{\pi}\right)/(1-\delta\theta)$.

(v.) *(The left region) When* $u'(\pi) = -1/(1+\theta\alpha)$, *the effort level and the bonus are given by*

$$e = \widehat{e},\ \text{and}\ b_n = \frac{\delta}{1-\delta}\left(\overline{\pi} - \underline{\pi}\right),$$

where recall $\widehat{e}$ *is the unique effort level satisfying* $c'(e)/y'(e) = 1/(1+\alpha\theta)$. *There can be multiple choices of* π_s *and* w. *One such choice is*

$$\pi_s = \pi\ \text{and}\ w = \frac{1}{(1-\delta)(1+\alpha\theta)}\left((1-\delta)\,y(\widehat{e}) + \delta(1-\theta)\,\underline{\pi} - (1-\delta\theta)\,\pi\right).$$

The left region exists if $y(\widehat{e}) > \underline{\pi}$. *In this case, the right boundary of this region is given by* $\left((1-\delta)\,y(\widehat{e}) + \delta(1-\theta)\,\underline{\pi}\right)/(1-\delta\theta)$.

Proof: Using Lemma C1, we define the Lagrangian associated with $\pi + u(\pi)$ as

$$\begin{aligned} L &= (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\overline{\pi} + u(\overline{\pi})) - (1-\delta)\theta\alpha w \\ &\quad + \lambda_1(\pi - (1-\delta)(y(e) - (1+\theta\alpha)w - (1-\theta)b_n) - \delta(\theta\pi_s + (1-\theta)\overline{\pi})) \\ &\quad + \lambda_2(1-\delta)w + \lambda_3(\delta(\overline{\pi} - \underline{\pi}) - (1-\delta)b_n). \end{aligned}$$

The first-order conditions and the envelope condition are given by

$$1 + u'(\pi_s) = \lambda_1. \qquad (\text{FOC}_\text{S})$$

$$-\theta\alpha + \lambda_1(1+\theta\alpha) + \lambda_2 = 0. \qquad (\text{FOC}_\text{W})$$

$$y'(e) - c'(e) = \lambda_1 y'(e). \qquad (\text{FOC}_\text{e})$$

$$\lambda_1(1-\theta) = \lambda_3. \qquad (\text{FOC}_\text{N})$$

$$1 + u'(\pi) = \lambda_1. \qquad (\text{envelope})$$

For part (i.), notice that FOC_W implies $\lambda_1 \le \theta\alpha/(1+\theta\alpha)$. The envelope condition then implies that $u'(\pi) \le -1/(1+\alpha\theta)$.

For part (ii.), notice that FOC_e implies $\lambda_1 = (y'(e) - c'(e))/y'(e)$. Using this expression to substitute for λ_1 in the envelope condition, we obtain the formula for effort.

For part (iii.), we first derive the expression for b_n. Notice that if $u'(\pi) \in (-1, -1/(1+\theta\alpha))$, the envelope condition implies that $\lambda_1 > 0$. It then follows from FOC_N that $\lambda_3 > 0$. As a result, the complementarity slackness condition associated with λ_3 implies that $b_n = \delta(\overline{\pi} - \underline{\pi})/(1-\delta)$.

Next, we show that $w = 0$. To see this, notice that when $u'(\pi) < -1/(1+\theta\alpha)$, the envelope condition implies that $-\theta\alpha + \lambda_1(1+\theta\alpha) < 0$. Consequently, FOC_W implies that $\lambda_2 > 0$. As a result, the complementarity slackness condition associated with λ_2 implies that $w = 0$.

Finally, we show that $\pi_s = \pi$. Comparing the FOC_S with the envelope condition, we obtain that $u'(\pi_s) = u'(\pi)$. This implies that $\pi_s = \pi$ unless there is an interval of manager's payoffs in which $u'(\pi)$ is constant. To see that such an interval cannot exist, suppose the contrary. Notice that by using the expressions for b_n and w, we can rewrite PK_M as

$$\pi = (1-\delta)y(e) + \delta(\theta\pi_s + (1-\theta)\underline{\pi}).$$

Moreover, part (ii.) then implies that $y(e)$ is a constant in the interval. PK_M then implies that for each π in the interval, $d\pi_s/d\pi = 1/\delta\theta > 1$. But this is a contradiction unless the length of the interval is zero. This proves $\pi_s = \pi$.

For part (iv.), notice that when $u'(\pi) = -1$, the envelope condition implies that $\lambda_1 = 0$. It then follows from FOC_W that $\lambda_2 > 0$, and therefore, the associated complementarity condition implies that $w = 0$. Notice that $\lambda_1 = 0$ also implies $e = e^{FB}$ by FOC_e, so efficient actions are taken in this region. Moreover, $\lambda_1 = 0$ implies that $u'(\pi_s) = -1$ by FOC_S. Now by choosing $\pi_s = \pi$ and $\pi_n = \overline{\pi}$, the PPE payoff frontier in the region $[\pi, \overline{\pi}]$ becomes self-generating and reaches the first-best. Notice that the expression for b_n follows from PK_M.

Finally, for this region to exist, a necessary and sufficient condition is that, at $\pi = \overline{\pi}$, the non-reneging constraint is satisfied. PK_M at $\overline{\pi}$ implies that

$$b_n(\overline{\pi}) = \frac{y\left(e^{FB}\right) - \overline{\pi}}{1-\theta} = \frac{c\left(e^{FB}\right) + \underline{u}}{1-\theta},$$

where the second inequality uses that the first-best is obtained at $\overline{\pi}$. Substituting this into the non-reneging constraint, we obtain the necessary and sufficient condition in part (iv.). Moreover, the left boundary of the region must satisfy that (a.) its shock-state continuation payoff must remains at the boundary and (b.) its non-reneging constraint must bind. Substituting these into the PK_M, we obtain the expression for the left boundary in part (iv.).

For part (v.), notice that $e = \widehat{e}$ by part (ii.). In addition, since $\lambda_1 > 0$, FOC_N implies that $\lambda_3 > 0$, and therefore, the associated complementarity slackness condition implies that $(1-\delta) b_n = \delta(\overline{\pi} - \underline{\pi})$. The PK_M becomes $\pi = (1-\delta) y(\widehat{e}) - (1+\alpha\theta) w + \delta(\theta\pi_s + (1-\theta)\underline{\pi})$ by substituting for e and b_n. Since $u'(\pi)$ is constant in this region, there is some flexibility is choosing π_s. To be consistent with our choice in the middle and the right region, we choose $\pi_s = \pi$. The PK_M above then gives the expression for w in the proposition.

Finally, for this region to exist, a necessary and sufficient condition is that at $\pi = \underline{\pi}$, we have $w > 0$. Note that $\pi_s(\underline{\pi}) = \underline{\pi}$, the PK_M above then implies that $w > 0$ is equivalent to $\underline{\pi} < y(\widehat{e})$. Moreover, the right boundary of this region must satisfy that (a.) its shock-state continuation payoff must remains on the boundary and (b.) $w = 0$. Substituting these into the PK_M, we obtain the expression for the right boundary in part (v.). ■

We finish by showing that if long-term contracts are feasible, the first-best can be arbitrarily approximated. Define $h^t \equiv \{y_1, ...y_t\}$ as the history of outputs, $m^t \equiv \{m_1, ..m_t\}$ as the history of reports, and denote $b_t\left(h^t, m^t\right)$ as the manager's payment to the worker in period t. Let $t_n(t)$ be the last period before t in which the manager reports a no-shock state, and $t_n(t) = 0$ if the manager has never reported a no-shock state.

PROPOSITION C4. *As T approaches ∞, the following sequence of contracts approaches first-best. The worker chooses $e_t = e^{FB}$. The manager reports the state of the world truthfully and pays out*

$b_t\left(h^t, m^t\right) = 0$ *if* $h^t \neq \{y^{FB}, .., y^{FB}\}$, *or* $m_t = s$ *and* $t < t_n(t) + T$ *and otherwise*

$$b_t\left(h^t, m^t\right) = \left(c\left(e^{FB}\right) + \underline{u}\right)\left(1 + \delta^{-1} + ... + \delta^{-(t-1-t_n(t))}\right)$$

Proof: To simplify the exposition, we normalize $\underline{u}$ to zero. We first show that these contracts are incentive compatible for sufficiently large T and then show that the surplus destruction goes to zero under this sequence of contracts. To check the worker's incentive constraints for effort provisions are satisfied, notice that the worker's payoff is 0 if $e_t = e^{FB}$. Moreover, any other effort choice of the worker leads to a non-positive payoff. Therefore, it is optimal for the worker to accept the contract and choose $e_t = e^{FB}$ for all t.

To check the manager's incentive to report the state of the world truthfully, notice that the manager's payoff is equal to the value of the relationship since the worker's payoff is always zero. As a result, the manager's payoff is maximized when the surplus destruction, i.e., the expected payment in shock states, is minimized. This immediately implies that in a no-shock state the manager will report $m_t = n$ and pay the bonus.

To check that the manager will report $m_t = s$ in a shock state, first notice that the contract repeats itself whenever a no-shock state is reported. This implies that the optimal reporting strategy repeats itself following each restart. Since the manager will always report truthfully in a no-shock state (and, thus, triggers to contract to restart), her reporting strategy is then completely determined by her reports following τ consecutive shock periods for each $\tau \leq T$. Moreover, since the contract restarts itself following each no-shock state is reported, the manager's strategy is determined by the smallest number of consecutive shock-states following which a no-shock state is reported. Denote this number by N, and notice that if $N = T$, this means that the manager is truth-telling. Define D_i^N as the normalized expected surplus destruction (associated with N) if the manager has reported shock states in the past i^{th} periods. Notice that D_i^N satisfies the following equations:

$$\begin{aligned} D_i^N &= (1-\theta)\,\delta D_0^N + \theta\delta D_{i+1}^N, \quad \text{for } i < N, \\ D_N^N &= (1-\delta)\,\theta\left(1 + \delta^{-1} + ... + \delta^{-(N-1)}\right)\alpha c\left(e^{FB}\right) + \delta D_0^N. \end{aligned}$$

Solving these equations, we obtain

$$D_0^N = \delta\theta \frac{1-\theta\delta}{(1-\delta)} \frac{\theta^N\left(1-\delta^N\right)}{\left(1-(\theta\delta)^{N+1}\right)} \alpha c\left(e^{FB}\right).$$

Notice that $\theta^N \left(1-\delta^N\right) / \left(1-(\theta\delta)^{N+1}\right)$ is decreasing in N when $1-\theta > \delta^{N-1}$. Moreover, as N goes to infinity, $\theta^N \left(1-\delta^N\right) / \left(1-(\theta\delta)^{N+1}\right)$ goes to zero. Therefore, when T is sufficiently large, D_0^N is minimized at $N = T$. It follows that the manager's truth-telling constraints in the shock states are satisfied for sufficiently large T. This finishes showing that the contracts are incentive compatible.

Finally, it can be checked that D_0^T goes to zero as T goes to infinity, so this sequence of contracts approximates first-best. ■