Searching and Learning by Trial-and-Error

Web Appendix

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Proposition 6  Consider two industries with mappings $\psi_1$ and $\psi_2$, and drift-variance pairs of $(\mu, \sigma^2)$ and $(\tau \mu, \tau \sigma^2)$, respectively, and that have identical status quo outcomes. In expectation, search in these industries is identical, with the exception that the expected size of experimental steps in Industry 2 is equal to $\frac{1}{\tau}$ that in Industry 1.

Proof of Proposition 6: Suppose at time $t$ the industries face the respective histories (denoting industry by a subscript):

$$
\begin{align*}
    h^t_1 &= \{(0, \psi_1(0)), (p_1, \psi_1(p_1)), (p_2, \psi_1(p_2)), \ldots, (p_{t-1}, \psi_1(p_{t-1}))\}, \\
    h^t_2 &= \{(0, \psi_2(0)), \left(\frac{1}{\tau}p_1, \psi_2\left(\frac{1}{\tau}p_1\right)\right), \left(\frac{1}{\tau}p_2, \psi_2\left(\frac{1}{\tau}p_2\right)\right), \ldots, \left(\frac{1}{\tau}p_{t-1}, \psi_2\left(\frac{1}{\tau}p_{t-1}\right)\right)\},
\end{align*}
$$

with $\psi_1(p_t) = \psi_2\left(\frac{1}{\tau}p_t\right)$ for $t = 1, 2, \ldots, t$ and $p_0 = 0$. Straightforward algebra establishes that for each product, $p \in \mathbb{R}$, the distribution of possible outcomes in Industry 1 is equal to the distribution of outcomes in Industry 2 for product $\frac{1}{\tau}p$; that is $\psi_1(p) \equiv \psi_2\left(\frac{1}{\tau}p\right)$. Facing the identical set of alternatives, the equilibrium choice across industries satisfies $p^*_t = \frac{1}{\tau}p^*_{t,1}$. As the industries share a common history at $t = 1$, the result follows by induction.

Corollary 5  With drift uncertainty, the equilibrium strategy of entrepreneur 1 is:

(i) Stable at $p^*_1 = p^*_0$ if $\psi(p^*_0) \in [0, \alpha]$.

(ii) Experimental if $\psi(p^*_0) > \alpha$, where $E(\psi(p^*_1|h^1)) > \alpha$ and strictly increasing in $\psi(p^*_0)$.

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Proof of Corollary 5: Setting $\mu_1 = \mu - \nu$, $\mu_2 = \mu + \nu$ and denoting utility here with a tilde, the derivatives of expected utility become:

$$
\frac{dE\tilde{u}(z)}{dz} = -\mu_1 [\psi(p^*_0) + \mu_1 (z - p^*_0)] - \mu_2 [\psi(p^*_0) + \mu_2 (z - p^*_0)] - \sigma^2,
$$

$$
= -2\mu [\psi(p^*_0) + \mu_1 (z - p^*_0)] - 2\nu^2 (z - p^*_0) - \sigma^2,
$$

$$
\frac{d^2E\tilde{u}(z)}{dz^2} = -\mu_1 - \mu_2 < -2\mu^2 < 0.
$$

Part i follows from $\frac{dE\tilde{u}(z)}{dz}|_{z=p^*_0} = \frac{dEu(z)}{dz}|_{z=p^*_0}$. As $\frac{dEu(z)}{dz}|_{z=p^*_0} > 0$ implies $\frac{dE\tilde{u}(z)}{dz}|_{z=p^*_0} > 0$, part ii follows because $\frac{dE\tilde{u}(z)}{dz} < \frac{dEu(z)}{dz}$ for all $z > p^*_0$.

**Corollary 6** With drift uncertainty, the equilibrium strategy for the second entrepreneur is:

(i) Stuck at $p^*_2 = p^*_0$ if $\psi(p^*_1) > \hat{\gamma}_2$ where the cut-point $\hat{\gamma}_2$ is less extreme than in the baseline model; i.e., $\hat{\gamma}_2 < \gamma_2$, for $\gamma_2$ defined in Proposition 2.

(ii) Experimental if $\psi(p^*_1)$ is in a neighborhood of $\alpha$.

**Proof of Corollary 6:** By stochastic dominance and $E\psi(p^*_1) < \psi(p^*_0)$, a realization $\psi(p^*_1) > \psi(p^*_0)$ induces more weight on $\mu_2$ in posterior beliefs. Similarly, a realization $\psi(p^*_1) \approx \alpha < E\psi(p^*_1)$ puts more weight on $\mu_1$. Both results then follow from straightforward algebra.

**Proposition 7** For a given known starting point, $(p^*_0, \psi(p^*_0))$, there is a $\delta'$ such that the monotonic-triangulating-stable dynamic holds if $\delta \in [0, \delta']$.

**Proof of Proposition 7:** For the monotonic-triangulating-stable dynamic to not hold, the entrepreneur must in some period choose an experimental product over a known product that has a (weakly) better expected value. Let the deviation in period $t$ be to $z$ with $p$ the nearest known product, and suppose $\psi(p) > 0$.

The current period cost of the deviation is variance and possibly expected value. The variance cost alone increases linearly in $z$ if uncertainty is open-ended and at least quadratically if on a bridge. The benefit of the deviation accrues in subsequent periods. By the single period deviation principle, a realization of $\psi(z) > 0$ leaves the optimal strategy in effect: the deviation is thereafter ignored if $\psi(z) > \psi(p)$ and potentially stabilized at for $\psi(z) \in [0, \psi(p)]$. For realizations $\psi(z) < 0$ behavior is unclear, but is clearly inferior to
obtaining the ideal outcome 0 in every subsequent period. The benefit from the deviation is, therefore, bounded by:

$$\zeta(z|\psi(p), \delta) = \frac{\delta}{1 - \delta} \left[ \int_0^{\psi(p)} (\psi(p)^2 - x^2) \phi(x|0, z\sigma^2) \, dx + (1 - \Phi(\psi(p)|0, z\sigma^2)) \psi(p)^2 \right],$$

where $\Phi$ and $\phi$ are the cdf and pdf of the normal distribution with mean 0 and variance $z\sigma^2$ (supposing uncertainty is open-ended). The benefit $\zeta$ is continuous in $z$, and bounded by $\psi(p)^2$, which itself is bounded by $\psi(p_0^*)^2$. Thus, $\frac{d}{dz}\zeta(z|\psi(p), \delta)$ is bounded for given starting point $(p_0^*, \psi(p_0^*))$, and as $\zeta(0|\psi(p), \delta) = 0$, the deviation is not profitable for sufficiently small $\delta$.

**Proposition 8** The monotonic-triangulating-stable dynamic holds for any weakly concave utility function.

**Proof of Proposition 8:** An entrepreneur with weakly concave utility strictly prefers the expected value of a lottery with certainty to the lottery itself. The result then follows by comparing expected values of products in each phase.