Appendix to: Baylis, Kathy, Don Fullerton, and Daniel H. Karney, "Leakage, Welfare, and Cost-Effectiveness of Carbon Policy" (*AER Papers and Proceedings*, May 2013).

Proof of Theorem 1: See Fullerton *et al* (2012).

Proof of Theorem 2: Totally differentiate $C = C_X + C_Y$ to get $dC = dC_X + dC_Y$, and then multiply and divide through by appropriate terms, using the hat notation:

$$\hat{C} = \beta_X \hat{C}_X + \beta_Y \hat{C}_Y$$

where $\beta_Y = C_Y / C$ and $1 = \beta_X + \beta_Y$. Recall that $\hat{\tau}_Y > 0$ implies $\hat{C}_Y < 0$ always, and thus negative leakage $\left(\hat{C}_X < 0\right)$ clearly leads to a fall in total carbon emissions. ■

Proof of Theorem 3: Using $\hat{C} = \beta_X \hat{C}_X + \beta_Y \hat{C}_Y$, insert solutions of Fullerton *et al* (2012):

$$\hat{C} = \beta_X \left[\alpha_Y (\sigma_U - \sigma_Y)\theta_{YC}\right]\hat{\tau}_Y + \beta_Y \left[-(\alpha_X \sigma_U + \alpha_Y \sigma_Y)\theta_{YC} - \theta_{YK}\sigma_Y\right]\hat{\tau}_Y$$

and simplify to yield:

$$\hat{C} = \left\{\sigma_U (\alpha_Y - \beta_Y)\theta_{YC} - \sigma_Y (\alpha_Y \theta_{YC} + \beta_Y \theta_{YK})\right\}\hat{\tau}_Y .$$

By inspection, $\alpha_Y < \beta_Y$ is a sufficient condition for $\hat{C} < 0$. However, the condition $\alpha_Y < \beta_Y$ implies both $(C_Y / K_Y) > (C/K)$ and $(C/K) > (C_X / K_X)$. ■

Proof of Theorem 4: We know $\hat{C} = \left\{\sigma_U (\alpha_Y - \beta_Y)\theta_{YC} - \sigma_Y (\alpha_Y \theta_{YC} + \beta_Y \theta_{YK})\right\}\hat{\tau}_Y$. For total carbon emissions to increase, it follows that:

$$\hat{C} = \left\{\sigma_U (\alpha_Y - \beta_Y)\theta_{YC} - \sigma_Y (\alpha_Y \theta_{YC} + \beta_Y \theta_{YK})\right\}\hat{\tau}_Y > 0$$

$$\Leftrightarrow \frac{\sigma_U}{\sigma_Y} > \frac{(\alpha_Y \theta_{YC} + \beta_Y \theta_{YK})}{(\alpha_Y - \beta_Y)\theta_{YC}}.$$

From the proof of Theorem 3, we know that $\hat{C} > 0$ requires $\alpha_Y > \beta_Y$, and so the denominator of that last expression is positive. But the positive numerator necessarily exceeds the positive denominator, so we have $\frac{\sigma_U}{\sigma_Y} > \frac{(\alpha_Y \theta_{YC} + \beta_Y \theta_{YK})}{(\alpha_Y - \beta_Y)\theta_{YC}} > 1$. ■

Derivation of Equation 2: Totally differentiate the utility function to yield:

$$dU = U_X dX + U_Y dY + U_C dC$$

Next, substitute in the first-order conditions from the utility maximization problem:

$$dU = \lambda p_X dX + \lambda p_Y dY + U_C dC$$

where λ is the multiplier on the budget constraint. Divide through by λ, so the left-hand side is the dollar value of the change in utility:

$$\frac{dU}{\lambda} = p_X dX + p_Y dY + \frac{U_C}{\lambda} dC$$

Continue by substituting in the totally differentiated production functions for *X* and *Y*, and the totally differentiated definition of total carbon:

$$\frac{dU}{\lambda} = p_X \left(X_K dK_X + X_C dC_X \right) + p_Y \left(Y_K dK_Y + Y_C dC_Y \right) + \frac{U_C}{\lambda} \left(dC_X + dC_Y \right)$$

where X_K is the marginal product of *X* (K_X,C_X) with respect to K_X. Other terms are similarly defined. Next, use first-order conditions from profit maximization (e.g. $p_K = p_X X_K$):

$$\frac{dU}{\lambda} = p_K dK_X + \tau_X dC_X + p_K dK_Y + \tau_Y dC_Y + \frac{U_C}{\lambda} \left(dC_X + dC_Y \right)$$

Recall that $dK_X = -dK_Y$, so:

$$\frac{dU}{\lambda} = \tau_X dC_X + \tau_Y dC_Y + \frac{U_C}{\lambda} \left(dC_X + dC_Y \right)$$

Define $\mu_C \equiv -U_C / \lambda$ as the marginal environmental damage from carbon, and rearrange terms:

$$\frac{dU}{\lambda} = \left(\tau_X - \mu_C \right) dC_X + \left(\tau_Y - \mu_C \right) dC_Y$$

Multiply and divide though by C_X and C_Y, respectively:

$$\frac{dU}{\lambda} = \left(\tau_X - \mu_C \right) C_X \hat{C}_X + \left(\tau_Y - \mu_C \right) C_Y \hat{C}_Y$$

In the first term, any increase in C_X reduces welfare if the existing τ_X is not high enough to correct for carbon damages μ_C (and similarly for C_Y). However, our measure of the policy's cost ignores benefits from reduced carbon damages, and so we delete the μ_C parameter:

$$\frac{dU}{\lambda} = \tau_X C_X \hat{C}_X + \tau_Y C_Y \hat{C}_Y .$$

Finally, multiply by –1, which converts a negative gain to a positive change in deadweight loss.

Derivation of Equation 3: From $\Delta DWL = -\left(\tau_X C_X \hat{C}_X + \tau_Y C_Y \hat{C}_Y \right)$, insert solutions for $\hat{C}_X$ and $\hat{C}_Y$:

$$\Delta DWL = -\left(\tau_X C_X \left[\alpha_Y \left(\sigma_U - \sigma_Y \right) \theta_{YC} \right] \hat{\tau}_Y + \tau_Y C_Y \left[-\left(\alpha_X \sigma_U + \alpha_Y \sigma_Y \right) \theta_{YC} - \theta_{YK} \sigma_Y \right] \hat{\tau}_Y \right)$$

Note that $\delta_i \equiv \tau_i C_i / R$. We can replace $\tau_i C_i = \delta_i R$, where R is total tax revenue:

$$\Delta DWL = -R\{\delta_X[\alpha_Y(\sigma_U - \sigma_Y)\theta_{YC}] - \delta_Y[(\alpha_X\sigma_U + \alpha_Y\sigma_Y)\theta_{YC} + \theta_{YK}\sigma_Y]\}\hat{\tau}_Y$$

Rearranging to collect the terms σ_U and σ_Y we obtain

$$\Delta DWL = R\{\sigma_U[\alpha_X - \delta_X]\theta_{YC} + \sigma_Y[\alpha_Y\theta_{YC} + \delta_Y\theta_{YK}]\}\hat{\tau}_Y .$$

Proof of Theorem 5: From $\Delta DWL = R\{\sigma_U[\alpha_X - \delta_X]\theta_{YC} + \sigma_Y[\alpha_Y\theta_{YC} + \delta_Y\theta_{YK}]\}\hat{\tau}_Y$, by inspection, $\alpha_X > \delta_X$ is a sufficient condition for $\Delta DWL>0$. However, the condition $\alpha_X > \delta_X$ necessarily means both $(\tau_Y C_Y/K_Y) > (R/K)$ and $(R/K) > (\tau_X C_X/K_X)$). ■

Proof of Theorem 6: Use $\Delta DWL = -(\tau_X C_X\hat{C}_X + \tau_Y C_Y\hat{C}_Y)$ and $\hat{C}_Y < 0$. Thus, negative leakage $(\hat{C}_X < 0)$ clearly leads to an increase in deadweight loss. ■

Proof of Theorem 7: From equation (3), a reduction in deadweight loss requires:

$$\Delta DWL = R\{\sigma_U[\alpha_X - \delta_X]\theta_{YC} + \sigma_Y[\alpha_Y\theta_{YC} + \delta_Y\theta_{YK}]\}\hat{\tau}_Y < 0$$

$$\Leftrightarrow \frac{\sigma_U}{\sigma_Y} < -\frac{[\alpha_Y\theta_{YC} + \delta_Y\theta_{YK}]}{[\alpha_X - \delta_X]\theta_{YC}}, \text{ where the latter term equals } \frac{[\alpha_Y\theta_{YC} + \delta_Y\theta_{YK}]}{[\delta_X - \alpha_X]\theta_{YC}}.$$

Also, Theorem 6 says that negative leakage always means a positive change in deadweight loss. Thus, $\Delta DWL<0$ requires that leakage be positive ($\sigma_U > \sigma_Y$). Accordingly, σ_U/σ_Y exceeds one, and:

$$\Leftrightarrow 1 < \frac{\sigma_U}{\sigma_Y} < \frac{[\alpha_Y\theta_{YC} + \delta_Y\theta_{YK}]}{[\delta_X - \alpha_X]\theta_{YC}}. \blacksquare$$

Derivation of Equation 4: We construct the MCA by dividing $-\Delta DWL$ by the totally differentiated change in total carbon emissions to yield:

$$\text{MCA} = \frac{(dU/\lambda)}{dC} = \frac{\tau_X C_X\hat{C}_X + \tau_Y C_Y\hat{C}_Y}{C_X\hat{C}_X + C_Y\hat{C}_Y}.$$

Next, substitute to closed-form expressions for the baseline $\hat{C}_X$ and $\hat{C}_Y$:

$$\frac{(dU/\lambda)}{dC} = \frac{\tau_X C_X[\alpha_Y(\sigma_U - \sigma_Y)\theta_{YC}\hat{\tau}_Y] - \tau_Y C_Y[(\alpha_X(\sigma_U - \sigma_Y)\theta_{YC} + \sigma_Y)\hat{\tau}_Y]}{C_X[\alpha_Y(\sigma_U - \sigma_Y)\theta_{YC}\hat{\tau}_Y] - C_Y[(\alpha_X(\sigma_U - \sigma_Y)\theta_{YC} + \sigma_Y)\hat{\tau}_Y]}$$

and simplify:

$$\frac{(dU/\lambda)}{dC} = \frac{\theta_{YC}(\sigma_U - \sigma_Y)[\alpha_Y\tau_X C_X - \alpha_X\tau_Y C_Y] - \sigma_Y\tau_Y C_Y}{\theta_{YC}(\sigma_U - \sigma_Y)[\alpha_Y C_X - \alpha_X C_Y] - \sigma_Y C_Y}.$$

noting that $\hat{\tau}_Y$ cancels from numerator and denominator. Continuing, define $\delta_X \equiv \tau_X C_X/R$ and $\delta_Y \equiv \tau_Y C_Y/R$, where $\delta_X + \delta_Y = 1$ and R is the total tax revenue; that is, δ_X is the share of

tax revenue collected from sector *X*. Similarly, define the carbon share in sector X as $\beta_X \equiv C_X / C$ and define β_Y analogously. Rewrite the MCA in terms delta- and beta-shares:

$$\frac{(dU/\lambda)}{dC} = \frac{\theta_{YC}(\sigma_U - \sigma_Y)[\alpha_Y \delta_X - \alpha_X \delta_Y]R - \sigma_Y \delta_Y R}{\theta_{YC}(\sigma_U - \sigma_Y)[\alpha_Y \beta_X - \alpha_X \beta_Y]C - \sigma_Y \beta_Y C}.$$

Finally, factor (R/C) and simplify the term in square brackets to yield equation (4) in the text.

Proof of Theorem 8: To start, observe $\tau_X > \tau_Y$ if and only if $\beta_Y > \delta_Y$ and $\delta_X > \beta_X$. In that case, the theorem to be proven says that:

$$\text{MCA} = \frac{\tau_X C_X \hat{C}_X + \tau_Y C_Y \hat{C}_Y}{C_X \hat{C}_X + C_Y \hat{C}_Y} < \frac{R}{C}.$$

The denominator is negative by assumption, so we need to show:

$$\tau_X C_X \hat{C}_X + \tau_Y C_Y \hat{C}_Y > \left(C_X \hat{C}_X + C_Y \hat{C}_Y\right)\left(\frac{R}{C}\right)$$

$$\Leftrightarrow \left(\tau_X C_X \hat{C}_X + \tau_Y C_Y \hat{C}_Y\right)\left(\frac{1}{R}\right) > \left(C_X \hat{C}_X + C_Y \hat{C}_Y\right)\left(\frac{1}{C}\right)$$

$$\Leftrightarrow \delta_X \hat{C}_X + \delta_Y \hat{C}_Y > \beta_X \hat{C}_X + \beta_Y \hat{C}_Y$$

$$\Leftrightarrow \hat{C}_X > \frac{(\beta_Y - \delta_Y)}{(\delta_X - \beta_X)}\hat{C}_Y \qquad \Leftrightarrow \hat{C}_X > \frac{((1-\beta_X)-(1-\delta_X))}{(\delta_X - \beta_X)}\hat{C}_Y$$

$$\Leftrightarrow \hat{C}_X > \frac{(-\beta_X + \delta_X)}{(\delta_X - \beta_X)}\hat{C}_Y \qquad \Leftrightarrow \hat{C}_X > \hat{C}_Y.$$

Insert solutions for $\hat{C}_X$ and $\hat{C}_Y$, and we still need to show that $\hat{C}_X > \hat{C}_Y$:

$$[\sigma_U \alpha_Y \theta_{YC} - \sigma_Y \alpha_Y \theta_{YC}]\hat{\tau}_Y > [-(\alpha_X \sigma_U + \alpha_Y \sigma_Y)\theta_{YC} - \theta_{YK}\sigma_Y]\hat{\tau}_Y$$

$$\Leftrightarrow \alpha_Y \sigma_U \theta_{YC} + \alpha_X \sigma_U \theta_{YC} > -\theta_{YK}\sigma_Y$$

$$\Leftrightarrow \theta_{YC}\sigma_U > -\theta_{YK}\sigma_Y.$$

Thus, it is always true that $\hat{C}_X > \hat{C}_Y$. ■

Derivation of Equation 5: We use $\Delta DWL = R\{\sigma_U[\alpha_X - \delta_X]\theta_{YC} + \sigma_Y[\alpha_Y \theta_{YC} + \delta_Y \theta_{YK}]\}\hat{\tau}_Y$ and set it equal to zero, which means:

$$\sigma_U = \sigma_Y \frac{[\alpha_Y \theta_{YC} + \delta_Y \theta_{YK}]}{[\alpha_Y - \delta_Y]\theta_{YC}} = \sigma_Y \frac{[\alpha_Y \theta_{YC} + \delta_Y (1 - \theta_{YC})]}{[\alpha_Y - \delta_Y]\theta_{YC}} = \sigma_Y \left[1 + \frac{\delta_Y}{(\alpha_Y - \delta_Y)\theta_{YC}}\right].$$