

SUPPLEMENTARY ONLINE MATERIAL FOR ROBSON AND KAPLAN
NOTES ON THE ANALYSIS

It follows readily that any solution to the problem described in Section III must also solve the following problem

$$(5) \quad \max_{K \geq K_0, w(\cdot) \geq 0} \int_0^{\infty} e^{-(\mu+r)t} (F - \beta K w) dt - \alpha(K - K_0) = \gamma \quad \text{subject to} \quad \frac{dQ}{dt} = w - \rho Q.$$

Simplification is possible since the optimal fertility profile is indeterminate in this example, subject only to satisfying the Euler-Lotka equation, (4).

Given that $Q_0 = c/2d$ and that the cost of quality maintenance is linear in w , the optimal trajectory in (5) for w involves allowing quality to run down until it reaches its optimal steady state level, and then maintaining it at that level. That is, for some $Q^* \geq 0$, $w = 0$ until $t = t^*$ but $w = \rho Q^*$ thereafter, where $Q_0 e^{-\rho t^*} = Q^*$. Let $Q(t)$ be the overall time path of quality this implies. Thus, the problem (5) reduces to

$$\max_{K^*, Q^*} V(K^*, Q^*) = \gamma \quad \text{where}$$

$$V(K^*, Q^*) = \int_0^{\infty} F(K^*, Q(t)) e^{-(\mu+r)t} dt - \frac{\beta \rho K^* Q^*}{\mu + r} \left(\frac{Q^*}{Q_0} \right)^{\frac{\mu+r}{\rho}} - \alpha(K^* - K_0)$$

Dropping the asterisks on K and Q , and since the upper bound is never binding, the first-order Kuhn-Tucker condition for choice of K for this problem is then

$$V_K \leq 0 \text{ and } V_K(K - K_0) = 0 \text{ so } a - 2bK - \left(\frac{Q}{Q_0}\right)^{\frac{\mu+r}{\rho}} \beta \rho Q - \alpha(\mu+r) \leq 0 \text{ and}$$

$$\left(a - 2bK - \left(\frac{Q}{Q_0}\right)^{\frac{\mu+r}{\rho}} \beta \rho Q - \alpha(\mu+r) \right) (K - K_0) = 0$$

The first-order condition for choice of Q is similarly

$$V_Q \leq 0 \text{ and } V_Q Q = 0 \text{ so that } c - 2dQ - (\mu + r + \rho)\beta K \leq 0 \text{ and}$$

$$(c - 2dQ - (\mu + r + \rho)\beta K)Q = 0.$$

There cannot be an interior solution for Q if $\beta > \frac{c}{(\mu + r + \rho)K_0}$; rather

$Q = 0$ in this case. There must, on the other hand, be an interior solution for K as long as α and K_0 are small enough to satisfy $a > \alpha(\mu + r) + 2bK_0$.

Finally, for completeness, it is shown that any solution of (5) is also a solution to the problem posed in Section III. Note first that

$$\frac{d}{dr} \left(\max_{K \geq K_0, w(\cdot) \geq 0} \left[\int_0^\infty e^{-(\mu+r)t} (F - \beta K w) dt - \alpha(K - K_0) \right] \right) = - \int_0^\infty t e^{-(\mu+r)t} (F - \beta K w) dt$$

Since, furthermore

$$\frac{d}{dt} \left(t \int_t^\infty e^{-(\mu+r)\tau} (F - \beta K w) d\tau \right) = \int_t^\infty e^{-(\mu+r)\tau} (F - \beta K w) d\tau - t e^{-(\mu+r)t} (F - \beta K w)$$

then

$$\int_0^\infty t e^{-(\mu+r)t} (F - \beta K w) dt = \int_0^\infty dt \int_t^\infty e^{-(\mu+r)\tau} (F - \beta K w) d\tau > 0$$

because choosing $w = 0$ ensures that $\int_t^\infty e^{-(\mu+r)\tau} F d\tau > 0$. Hence

$$(6) \quad \frac{d}{dr} \left(\max_{K \geq K_0, w(\cdot) \geq 0} \left[\int_0^\infty e^{-(\mu+r)t} (F - \beta K w) dt - \alpha(K - K_0) \right] \right) < 0.$$

Consider now any solution of the problem in (5). Since increasing r must be infeasible given (6), the desired result follows.