The Inflation-Output Trade-off with Downward Wage Rigidities

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Web Appendix
B Supplemental Appendix

In this appendix we derive equations (26) and (27). The Hamilton-Jacobi-Bellman equation in state 1 is given by

\begin{align*}
\rho V_1(\cdot)dt &= \pi(w_t(j, i), W_t(i), \tilde{Y}_t(i))dt + V_{1,w}(w_t(j, i), W_t(i), \tilde{Y}_t(i))dw_t(j, i) \\
&+ V_{1,W}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t dW_t(i) + V_{1,y}(w_t(j, i), W_t(i), \tilde{Y}_t(i))\theta'(i)dt \\
&\frac{1}{2} V_{1,yy}(w_t(j, i), W_t(i), \tilde{Y}_t(i))\tilde{Y}_t^2(i)\sigma^2(i)dt + \frac{1}{2} V_{1,WW}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t(dW_t(i))^2 \\
&+ V_{1,yy}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t dW_t(i)d\tilde{Y}_t(i) + \lambda V_2(w_t(j, i), W_t(i), \tilde{Y}_t(i))dt + \\
&- \lambda V_1(w_t(j, i), W_t(i), \tilde{Y}_t(i))dt
\end{align*}

where \( V_1(\cdot) = V_1(w_t(j, i), W_t(i), \tilde{Y}_t(i)) \) which results from the standard application of Itô’s Lemma, with the addition of the last two terms that account for the possibility of switching to state 2 where \( \tilde{Y}_t(i) \) denotes the level of the state variable \( \tilde{Y}_t(i) \) in state 2.

In deriving the above equation, we have used the fact that in state 1, \( dw_t(j, i) \) has finite variation implying \( (dw_t(j, i))^2 = dw_t(j, i)dw_t(i) = dw_t(j, i)d\tilde{Y}_t(i) = 0 \). At the optimum, \( V_{1,w}(w_t(j, i), W_t(i), \tilde{Y}_t(i))dw_t(j, i) = 0 \). Differentiating the above equation with respect to \( w_t(j, i) \), we obtain

\begin{align*}
\rho V_{1,w}(\cdot)dt &= \pi(w_t(j, i), W_t(i), \tilde{Y}_t(i))dt + V_{1,W}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t dW_t(i) \\
&+ V_{1,y}(w_t(j, i), W_t(i), \tilde{Y}_t(i))\theta'(i)dt + \frac{1}{2} V_{1,yy}(w_t(j, i), W_t(i), \tilde{Y}_t(i))\tilde{Y}_t^2(i)\sigma^2(i)dt \\
&+ \frac{1}{2} V_{1,WW}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t(dW_t(i))^2 \\
&+ V_{1,yy}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t dW_t(i)d\tilde{Y}_t(i) + \\
&\lambda V_2(w_t(j, i), W_t(i), \tilde{Y}_t(i))dt - \lambda V_{1,w}(w_t(j, i), W_t(i), \tilde{Y}_t(i))dt.
\end{align*}

We can now use the fact that the equilibrium will be symmetric across all \( j \), so that \( dw_t(j, i) = dW_t(i) \) which implies that also \( dW_t(i) \) has finite variation. We can then simplify the above expression to

\begin{align*}
\rho v_{1,w}(W_t(i), \tilde{Y}_t(i)) &= \pi(w_t(i), \tilde{Y}_t(i)) + v_{1,y}(W_t(i), \tilde{Y}_t(i))\tilde{Y}_t(i)\theta'(i) \\
&+ \frac{1}{2} v_{1,yy}(W_t(i), \tilde{Y}_t(i))\tilde{Y}_t^2(i)\sigma^2(i) + \lambda v_2(w_t(i), \tilde{Y}_t(i)) + \\
&- \lambda v_{1,w}(W_t(i), \tilde{Y}_t(i))
\end{align*}

where we have defined \( v_{1,w}(W_t(i), \tilde{Y}_t(i)) \equiv V_{1,w}(w_t(j, i), W_t(i), \tilde{Y}_t(i)) \) and used the smooth-pasting condition \( v_{1,w}(W_t(i), \tilde{Y}_t(i))dW_t(i) = 0 \). Finally, noting that \( v_2(w_t(i), \tilde{Y}_t(i)) = 0 \)
in state 2 because wage setters can adjust their wages, we can then obtain equation (26) in the text.

The Hamilton-Jacobi-Bellman equation in state 2 is given by

\[
\rho V_2(\cdot) dt = \pi(w_t(j, i), W_t(i), Y_t'(i))dt + V_{2,w}(w_t(j, i), W_t(i), Y_t'(i))dW_t(j, i) + V_{2,W}(w_t(j, i), W_t(i), Y_t'(i))E_t dW_t(i) + V_{2,y}(w_t(j, i), W_t(i), Y_t'(i))\theta'(i)dt + \frac{1}{2} V_{2,yy}(w_t(j, i), W_t(i), Y_t'(i))(Y_t'(i))^2\sigma^2(i)dt + \frac{1}{2} V_{2,WW}(w_t(j, i), W_t(i), Y_t'(i))E_t (dW_t(i))^2 + \frac{1}{2} V_{2,ww}(w_t(j, i), W_t(i), Y_t'(i))E_t(dw_t(j, i))^2 + V_{2,ww}(w_t(j, i), W_t(i), Y_t'(i))E_t(dw_t(j, i)dW_t(i)) + V_{2,yW}(w_t(j, i), W_t(i), Y_t'(i))E_t dw_t(j, i)dY_t'(i) + \phi V_1(w_t(j, i), W_t(i), Y_t'(i))dt - \phi V_2(w_t(j, i), W_t(i), Y_t'(i))dt
\]

where we have defined \( V_2(\cdot) = V_2(w_t(j, i), W_t(i), Y_t'(i)) \), \( \theta'(i) = \theta + 1/2 \cdot \sigma^2(i) \) and \( \sigma^2(i) = \sigma_2^2(i) + \sigma_6^2 + \sigma_7^2(i) \) and noted that the state variable \( \tilde{Y}_t'(i) \) from state 2 to 1 is continuous.

Optimality condition requires \( V_{2,w}(w_t(j, i), W_t(i), Y_t'(i)) = 0 \) (and therefore its differential is also zero) and simplifies the above condition to

\[
\rho V_2(\cdot) dt = \pi(w_t(j, i), W_t(i), Y_t'(i))dt + V_{2,W}(w_t(j, i), W_t(i), Y_t'(i))E_t dW_t(i) + V_{2,y}(w_t(j, i), W_t(i), Y_t'(i))\theta'(i)dt + \frac{1}{2} V_{2,yy}(w_t(j, i), W_t(i), Y_t'(i))(\tilde{Y}_t'(i))^2\sigma^2(i)dt + \frac{1}{2} V_{2,WW}(w_t(j, i), W_t(i), Y_t'(i))E_t (dW_t(i))^2 + \frac{1}{2} V_{2,ww}(w_t(j, i), W_t(i), Y_t'(i))E_t(dw_t(j, i))^2 + \phi V_1(w_t(j, i), W_t(i), Y_t'(i))dt - \phi V_2(w_t(j, i), W_t(i), Y_t'(i))dt
\]

By taking the derivative with respect to \( w_t(j, i) \) and noting that the resulting equilibrium is symmetric we can obtain

\[
\rho v_2(W_t(i), \tilde{Y}_t'(i))dt = \pi_w(W_t(i), \tilde{Y}_t'(i))dt + v_{2,w}(W_t(i), \tilde{Y}_t'(i))E_t dW_t(i) + v_{2,y}(W_t(i), \tilde{Y}_t'(i))\theta'(i)dt + \frac{1}{2} v_{2,yy}(W_t(i), \tilde{Y}_t'(i))(\tilde{Y}_t'(i))^2\sigma^2(i)dt + \frac{1}{2} v_{2,WW}(W_t(i), \tilde{Y}_t'(i))E_t (dW_t(i))^2 + v_{2,yW}(W_t(i), \tilde{Y}_t'(i))E_t dw_t(i)d\tilde{Y}_t'(i) + \phi v_1(W_t(i), \tilde{Y}_t'(i))dt - \phi v_2(W_t(i), \tilde{Y}_t'(i))dt
\]
where we have defined \( v_2(W_t(i), \tilde{Y}_t'(i)) \equiv V_{z,w}(w_t(j,i), W_t(i), \tilde{Y}_t'(i)) \). Since \( v_2(W_t(i), \tilde{Y}_t'(i)) = 0 \) together with its differential we can get

\[
\pi_w(W_t(i), \tilde{Y}_t'(i)) + \phi v_1(W_t(i), \tilde{Y}_t'(i)) = 0
\]

which is condition (27) in the text.

We can further elaborate on equation (B.13) noting that we can write it as

\[
\begin{bmatrix}
1 - \mu \left( \frac{\tilde{Y}_t'(i)}{W_t(i)} \right)^{1+\eta} \\
- \frac{\phi \mu}{(\rho + \lambda) - \theta'(1 + \eta) - \frac{1}{2}(1 + \eta)\eta\sigma_y^2(i)} \left( \frac{\tilde{Y}_t'(i)}{W_t(i)} \right)^{1+\eta} + \frac{k(i)}{k_w} \left( \frac{\tilde{Y}_t'(i)}{W_t(i)} \right)^{\gamma(i)}
\end{bmatrix} = 0
\]

where we have used the results of the previous subsection of the appendix. Indeed they still apply to derive \( v_1(W_t(i), \tilde{Y}_t'(i)) \) as discussed in the text with the caveat that now the total discount factor to is \( (\rho + \lambda) \) instead of \( \rho \). Clearly, a solution of the above equation is of the form

\[
W_t(i) = \tilde{c}(i)^{\mu \frac{1}{1+\eta}} \tilde{Y}_t(i) \epsilon_t(i)
\]

which determines the wages in sector \( i \) in state 2 where \( \tilde{c}_i \) solves the equation

\[
[1 - \tilde{c}(i)^{-(1+\eta)}] + \frac{\phi}{(\rho + \lambda)} - \frac{\phi}{(\rho + \lambda) - \theta'(1 + \eta) - \frac{1}{2}(1 + \eta)\eta\sigma_y^2(i)} \tilde{c}(i)^{-(1+\eta)} + \frac{k(i)}{k_w} \tilde{c}(i)^{-\gamma(i)} \mu^{\frac{\gamma(i)}{1+\eta}} = 0,
\]

where

\[
\frac{k(i)}{k_w} = \frac{(1 + \eta)}{\gamma(i) \left[ (\rho + \lambda) - \theta'(1 + \eta) - \frac{1}{2}(1 + \eta)\eta\sigma_y^2(i) \right]} \mu^{\frac{\gamma(i)}{1+\eta}} c(i)^{\gamma(i)-(1+\eta)}.
\]

Note again that in state 1, in the case of adjustment, wages are adjusted to

\[
W_t(i) = c(i)^{\mu \frac{1}{1+\eta}} \tilde{Y}_t(i) \epsilon_{t1}(i)
\]

where \( \epsilon_{t1}(i) \) represent the realization of \( \epsilon_t \) at time \( t_1 < t \), which is the last time before \( t \) at which state 2 occurred.