

# The Inflation-Output Trade-off with Downward Wage Rigidities

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Web Appendix

## B Supplemental Appendix

In this appendix we derive equations (26) and (27). The Hamilton-Jacobi-Bellman equation in state 1 is given by

$$\begin{aligned} \rho V_1(\cdot)dt &= \pi(w_t(j, i), W_t(i), \tilde{Y}_t(i))dt + V_{1,w}(w_t(j, i), W_t(i), \tilde{Y}_t(i))dw_t(j, i) \\ &\quad + V_{1,W}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t dW_t(i) + V_{1,y}(w_t(j, i), W_t(i), \tilde{Y}_t(i))\tilde{Y}_t(i)\theta'(i)dt \\ &\quad + \frac{1}{2}V_{1,yy}(w_t(j, i), W_t(i), \tilde{Y}_t(i))\tilde{Y}_t^2(i)\sigma^2(i)dt + \frac{1}{2}V_{1,WW}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t(dW_t(i))^2 \\ &\quad + V_{1,yW}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t dW_t(i)d\tilde{Y}_t(i) + \lambda V_2(w_t(j, i), W_t(i), \tilde{Y}_t'(i))dt + \\ &\quad - \lambda V_1(w_t(j, i), W_t(i), \tilde{Y}_t(i))dt \end{aligned}$$

where  $V_1(\cdot) = V_1(w_t(j, i), W_t(i), \tilde{Y}_t(i))$  which results from the standard application of Ito's Lemma, with the addition of the last two terms that account for the possibility of switching to state 2 where  $\tilde{Y}_t'(i)$  denotes the level of the state variable  $\tilde{Y}_t(i)$  in state 2. In deriving the above equation, we have used the fact that in state 1,  $dw_t(j, i)$  has finite variation implying  $(dw_t(j, i))^2 = dw_t(j, i)dW_t(i) = dw_t(j, i)d\tilde{Y}_t(i) = 0$ . At the optimum,  $V_{1,w}(w_t(j, i), W_t(i), \tilde{Y}_t(i))dw_t(j, i) = 0$ . Differentiating the above equation with respect to  $w_t(j, i)$ , we obtain

$$\begin{aligned} \rho V_{1,w}(\cdot)dt &= \pi_w(w_t(j, i), W_t(i), \tilde{Y}_t(i))dt + V_{1,Ww}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t dW_t(i) \\ &\quad + V_{1,yw}(w_t(j, i), W_t(i), \tilde{Y}_t(i))\tilde{Y}_t(i)\theta'(i)dt + \frac{1}{2}V_{1,yyw}(w_t(j, i), W_t(i), \tilde{Y}_t(i))\tilde{Y}_t^2(i)\sigma^2(i)dt \\ &\quad + \frac{1}{2}V_{1,WWw}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t(dW_t(i))^2 + \\ &\quad + V_{1,yWw}(w_t(j, i), W_t(i), \tilde{Y}_t(i))E_t dW_t(i)d\tilde{Y}_t(i) + \\ &\quad \lambda V_{2,w}(w_t(j, i), W_t(i), \tilde{Y}_t'(i))dt - \lambda V_{1,w}(w_t(j, i), W_t(i), \tilde{Y}_t(i))dt. \end{aligned}$$

We can now use the fact that the equilibrium will be symmetric across all  $j$ , so that  $dw_t(j, i) = dW_t(i)$  which implies that also  $dW_t(i)$  has finite variation. We can then simplify the above expression to

$$\begin{aligned} \rho v_{1,w}(W_t(i), \tilde{Y}_t(i)) &= \pi_w(W_t(i), \tilde{Y}_t(i)) + v_{1,y}(W_t(i), \tilde{Y}_t(i))\tilde{Y}_t(i)\theta'(i) \\ &\quad + \frac{1}{2}v_{1,yy}(W_t(i), \tilde{Y}_t(i))\tilde{Y}_t^2(i)\sigma^2(i) + \lambda v_{2,w}(W_t(i), \tilde{Y}_t'(i)) + \\ &\quad - \lambda v_{1,w}(W_t(i), \tilde{Y}_t(i)), \end{aligned}$$

where we have defined  $v_{1,w}(W_t(i), \tilde{Y}_t(i)) \equiv V_{1,w}(w_t(j, i), W_t(i), \tilde{Y}_t(i))$  and used the smooth-pasting condition  $v_{1,w}(W_t(i), \tilde{Y}_t(i))dW_t(i) = 0$ . Finally, noting that  $v_{2,w}(W_t(i), \tilde{Y}_t'(i)) = 0$

in state 2 because wage setters can adjust their wages, we can then obtain equation (26) in the text.

The Hamilton-Jacobi-Bellman equation in state 2 is given by

$$\begin{aligned}
\rho V_2(\cdot)dt &= \pi(w_t(j, i), W_t(i), \tilde{Y}'_t(i))dt + V_{2,w}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))dw_t(j, i) \\
&+ V_{2,W}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))E_t dW_t(i) + V_{2,y}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))\tilde{Y}'_t(i)\theta''(i)dt \\
&+ \frac{1}{2}V_{2,yy}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))(\tilde{Y}'_t(i))^2\sigma_a^2(i)dt + \\
&+ \frac{1}{2}V_{2,WW}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))E_t(dW_t(i))^2 + \\
&+ \frac{1}{2}V_{2,ww}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))E_t(dw_t(j, i))^2 + \\
&+ V_{2,wW}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))E_t(dw_t(j, i)dW_t(i)) \\
&+ V_{2,yW}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))E_t dW_t(i)d\tilde{Y}'_t(i) + \\
&+ V_{2,wy}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))E_t(dw_t(j, i)d\tilde{Y}'_t(i)) \\
&+ \phi V_1(w_t(j, i), W_t(i), \tilde{Y}'_t(i))dt - \phi V_2(w_t(j, i), W_t(i), \tilde{Y}'_t(i))dt.
\end{aligned}$$

where we have defined  $V_2(\cdot) = V_2(w_t(j, i), W_t(i), \tilde{Y}'_t(i))$ ,  $\theta''(i) = \theta + 1/2 \cdot \sigma_a^2(i)$  and  $\sigma_a^2(i) = \sigma_\xi^2(i) + \sigma_y^2 + \sigma_v^2(i)$  and noted that the state variable  $\tilde{Y}'_t(i)$  from state 2 to 1 is continuous.

Optimality condition requires  $V_{2,w}(w_t(j, i), W_t(i), \tilde{Y}'_t(i)) = 0$  (and therefore its differential is also zero) and simplifies the above condition to

$$\begin{aligned}
\rho V_2(\cdot)dt &= \pi(w_t(j, i), W_t(i), \tilde{Y}'_t(i))dt + V_{2,W}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))E_t dW_t(i) + \\
&+ V_{2,y}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))\tilde{Y}'_t(i)\theta''(i)dt + \frac{1}{2}V_{2,yy}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))(\tilde{Y}'_t(i))^2\sigma_a^2(i)dt \\
&+ \frac{1}{2}V_{2,WW}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))E_t(dW_t(i))^2 + \\
&+ V_{2,yW}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))E_t dW_t(i)d\tilde{Y}'_t(i) + \\
&+ \phi V_1(w_t(j, i), W_t(i), \tilde{Y}'_t(i))dt - \phi V_2(w_t(j, i), W_t(i), \tilde{Y}'_t(i))dt.
\end{aligned}$$

By taking the derivative with respect to  $w_t(j, i)$  and noting that the resulting equilibrium is symmetric we can obtain

$$\begin{aligned}
\rho v_2(W_t(i), \tilde{Y}'_t(i))dt &= \pi_w(W_t(i), \tilde{Y}'_t(i))dt + v_{2,W}(W_t(i), \tilde{Y}'_t(i))E_t dW_t(i) + \\
&+ v_{2,y}(W_t(i), \tilde{Y}'_t(i))\tilde{Y}'_t(i)\theta''(i)dt + \frac{1}{2}v_{2,yy}(W_t(i), \tilde{Y}'_t(i))(\tilde{Y}'_t(i))^2\sigma_a^2(i)dt \\
&+ \frac{1}{2}v_{2,WW}(W_t(i), \tilde{Y}'_t(i))E_t(dW_t(i))^2 \\
&+ v_{2,yW}(W_t(i), \tilde{Y}'_t(i))E_t dW_t(i)d\tilde{Y}'_t(i) + \\
&+ \phi v_1(W_t(i), \tilde{Y}'_t(i))dt - \phi v_2(W_t(i), \tilde{Y}'_t(i))dt,
\end{aligned}$$

where we have defined  $v_2(W_t(i), \tilde{Y}'_t(i)) \equiv V_{2,w}(w_t(j, i), W_t(i), \tilde{Y}'_t(i))$ . Since  $v_2(W_t(i), \tilde{Y}'_t(i)) = 0$  together with its differential we can get

$$\pi_w(W_t(i), \tilde{Y}'_t(i)) + \phi v_1(W_t(i), \tilde{Y}'_t(i)) = 0 \quad (\text{B.13})$$

which is condition (27) in the text.

We can further elaborate on equation (B.13) noting that we can write it as

$$\left[ 1 - \mu \left( \frac{\tilde{Y}'_t(i)}{W_t(i)} \right)^{1+\eta} \right] + \frac{\phi}{(\rho + \lambda)} +$$

$$- \frac{\phi \mu}{(\rho + \lambda) - \theta'(1 + \eta) - \frac{1}{2}(1 + \eta)\eta\sigma_y^2(i)} \left( \frac{\tilde{Y}'_t(i)}{W_t(i)} \right)^{1+\eta} + \frac{k(i)}{k_w} \left( \frac{\tilde{Y}'_t(i)}{W_t(i)} \right)^{\gamma(i)} = 0$$

where we have used the results of the previous subsection of the appendix. Indeed they still apply to derive  $v_1(W_t(i), \tilde{Y}'_t(i))$  as discussed in the text with the caveat that now the total discount factor to is  $(\rho + \lambda)$  instead of  $\rho$ . Clearly, a solution of the above equation is of the form

$$W_t(i) = \tilde{c}(i)\mu^{\frac{1}{1+\eta}}\tilde{Y}_t\xi_t(i)\varepsilon_t(i)$$

which determines the wages in sector  $i$  in state 2 where  $\tilde{c}_i$  solves the equation

$$\left[ 1 - \tilde{c}(i)^{-(1+\eta)} \right] + \frac{\phi}{(\rho + \lambda)} - \frac{\phi}{(\rho + \lambda) - \theta'(i)(1 + \eta) - \frac{1}{2}(1 + \eta)\eta\sigma^2(i)} \tilde{c}(i)^{-(1+\eta)} + \phi \frac{k(i)}{k_w} \tilde{c}(i)^{-\gamma(i)} \mu^{-\frac{\gamma(i)}{1+\eta}} = 0,$$

where

$$\frac{k(i)}{k_w} = \frac{(1 + \eta)}{\gamma(i) \left[ (\rho + \lambda) - \theta'(i)(1 + \eta) - \frac{1}{2}(1 + \eta)\eta\sigma^2(i) \right]} \mu^{\frac{\gamma(i)}{1+\eta}} c(i)^{\gamma(i)-(1+\eta)}.$$

Note again that in state 1, in the case of adjustment, wages are adjusted to

$$W_t(i) = c(i)\mu^{\frac{1}{1+\eta}}\tilde{Y}_t\xi_t(i)\varepsilon_{t_1}(i)$$

where  $\varepsilon_{t_1}(i)$  represent the realization of  $\varepsilon_t$  at time  $t_1 < t$ , which is the last time before  $t$  at which state 2 occurred.