The proof of Proposition 4 builds on that of Proposition 5, so these two results are proved in reverse order.

**A. PROOF OF PROPOSITION 5:**

Consider a given path of play in the equilibrium described in Proposition 3. Until there are more than $k^*$ signals available for decoding, a new conversation is initiated each time a conversant fails to generate an additional signal. Since player A is assumed to start with an initial signal, $c$ is the random number of failures in a series of independent Bernoulli ($p$) trials before there are $I(k^*)$ successes. A random variable such as $c$ with

$$\Pr(c = i) = \binom{i + I(k^*) - 1}{I(k^*) - 1} p^{I(k^*)} (1 - p)^i \text{ for } i = 0, 1, 2, \ldots$$  

(A.1)

is said to follow the negative binomial distribution. It is then straightforward to show that $E(c) = I(k^*)(1 - p)/p$ (see George Casella and Roger L. Berger (2002), p. 95-96).

Next, note that the random variable $k$ follows a geometric distribution with
(A.2) \[ \Pr(k = i) = (1 - p)p^{i-1} \text{ for } i = 1, 2, \ldots. \]

A simple calculation confirms that \( E(k) = 1/(1 - p) \) (see Casella and Berger (2002), p. 97).

B. PROOF OF PROPOSITION 4:

First, recall that \( \text{cov}(c, k) = E(ck) - E(c)E(k) \). By Proposition 5, we have 

\[ E(c)E(k) = I(k^*)/p. \]

To calculate \( E(ck) \), we can use the fact that 

\[ E(ck) = E[E(c|k)k] \]

where \( E(c|k) \) denotes the conditional expectation of \( c \) given \( k \). Next note that:

\[ E(c|k) = \begin{cases} 
1 + \left(I(k^*) + 1 - k\right) \frac{(1 - p)}{p} & \text{if } k \leq I(k^*) \\
0 & \text{if } k > I(k^*) + 1.
\end{cases} \]

Thus, for \( k \leq I(k^*) \), we carry out Bernoulli \( (p) \) trials until there are \( I(k^*) + 1 - k \) successes, so the expression for \( E(c|k) \) follows from Proposition 5 (we must add one since we are effectively conditioning on \( c \geq 1 \)); while for \( k > I(k^*) + 1 \), B will not initiate a conversation with C so that \( E(c|k) = 0 \). Note that \( E(c|k) \) is decreasing in \( k \). Calculating, we then obtain:

\[ E[E(c|k)k] = \sum_{i=1}^{I(k^*)} \left[ 1 + \left(I(k^*) + 1 - i\right) \frac{(1 - p)}{p} \right] \left(1 - p\right) p^{i-1} = \frac{I(k^*)}{p} - \frac{1 - p^{I(k^*)}}{1 - p}, \]
so that \( \text{cov}(c, k) = -(1 - p^{r(k^*)})/(1 - p) < 0 \). After computing \( \text{var}(c) \) and \( \text{var}(k) \) (see Casella and Berger (2002), p. 95-97), and rearranging the resulting expression for 
\[
\rho_{ck} = \frac{\text{cov}(c, k)}{\sqrt{\text{var}(c) \text{var}(k)}},
\]
we have:

(A.5) \[
\rho_{ck} = -\left(1 - p^{r(k^*)}\right)\sqrt{\frac{p}{I(k^*)(1 - p)}} < 0.
\]

C. COMPARATIVE STATICS PROPERTIES OF \( E(c) \) AND \( \Phi\{E(c)\} \):

Let us begin with the comparative statics of \( k^* \) with respect to \( p \) and \( \theta \). Let

(A.6) \[
F(k, p, \beta, \theta) = \left[\frac{\theta(1 - p) + p(1 - \beta)}{1 - p\beta}\right] \beta^k - \theta
\]

and note that the function \( k^*(p, \beta, \theta) \) is defined implicitly by \( F(k^*(p, \beta, \theta), p, \beta, \theta) = 0 \).

Since \( \partial F(k, p, \beta, \theta) / \partial k < 0 \), applying the Implicit Function Theorem and noting that

(A.7) \[
\frac{\partial F(k^*, p, \beta, \theta)}{\partial p} = \frac{(1 - \theta)(1 - \beta)}{(1 - p\beta)^2} \beta^{k^*} > 0, \quad \text{and}
\]

(A.8) \[
\frac{\partial F(k^*, p, \beta, \theta)}{\partial \theta} = \frac{(1 - p)}{(1 - p\beta)} \beta^{k^*} - 1 < 0,
\]
we see that \( \frac{\partial k^*}{\partial p} > 0 \) and \( \frac{\partial k^*}{\partial \theta} < 0 \). Since \( I(k^*) \) is weakly increasing in \( k^* \), it follows immediately from the above results that \( E(c) = I(k^*)(1 - p)/p \) is weakly decreasing in \( \theta \).

Finally, recall that the upper bound on \( E[c] \) is given by \( \Phi\{E[c]\} = k^*(1 - p)/p \).

Differentiating with respect to \( p \) yields

\[
\frac{\partial \Phi\{E[c]\}}{\partial p} = \frac{\partial k^*}{\partial p} \frac{(1 - p)}{p} - \frac{k^*}{p^2}.
\]

Computing \( \frac{\partial k^*}{\partial p} \) and noting that \( (1 - p)/(1 - p\beta) < 1 \), we have

\[
\frac{\partial k^*}{\partial p} \frac{(1 - p)}{p} = \left[ -\frac{1}{\ln[\beta]} \frac{(1 - \theta)(1 - \beta)}{(1 - p)(1 - \beta + p(1 - \beta))} \right] \frac{(1 - p)}{p} \leq -\frac{1}{p\ln[\beta]} \left[ \frac{(1 - \theta)(1 - \beta)}{\theta(1 - p) + p(1 - \beta)} \right].
\]

Finally since \( -\ln[x] > 1 - x \) for \( 0 < x < 1 \), it follows that

\[
\frac{k^*}{p^2} = \frac{1}{p^2} \log \left[ \frac{(1 - p\beta)\theta}{\theta(1 - p) + p(1 - \beta)} \right] \leq -\frac{1}{p^2 \ln[\beta]} \left[ 1 - \frac{(1 - p\beta)\theta}{\theta(1 - p) + p(1 - \beta)} \right]
\]

\[
= -\frac{1}{p \ln[\beta]} \left[ \frac{(1 - \theta)(1 - \beta)}{\theta(1 - p) + p(1 - \beta)} \right].
\]

Combining these inequalities, it follows that \( \partial \Phi\{E[c]\}/\partial p < 0 \).

Reference