# Consensus building: How to persuade a group

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## Additional materials

### Proof of Lemma C1

Referring to Appendix C and using feasibility constraints, note that a mechanism is alternatively given by $(\gamma, \theta_i, \mu_i, \bar{\mu}_i, \nu_0, \nu_3)$ and:

$$\eta = 1 - \gamma - \bar{\mu}_1 - \bar{\mu}_2 - \nu_0 \geq 0, \tag{1}$$

$$\nu_1 = \nu_0 - \theta_1 - \mu_1 + \bar{\mu}_1 \geq 0, \tag{2}$$

$$\nu_2 = \nu_0 - \theta_2 - \mu_2 + \bar{\mu}_2 \geq 0, \tag{3}$$

$$\xi_3 = \nu_0 - \theta_1 - \theta_2 - \mu_1 - \mu_2 + \bar{\mu}_1 + \bar{\mu}_2 - \nu_3 \geq 0. \tag{4}$$

Using measurability and individual rationality, the expected probability that the project is implemented is given by:

$$Q = \gamma + p_1\theta_1 + p_2\theta_2 + P\xi_3.$$

Plugging in the value of $\xi_3$ from (4), we find:

$$Q = \gamma + (p_1 - P)\theta_1 + (p_2 - P)\theta_2 + P\nu_0 - P\mu_1 - P\mu_2 + P\bar{\mu}_1 + P\bar{\mu}_2 - P\nu_3. \tag{5}$$

We now write incentive constraints using measurability, individual rationality and feasibility constraints. Equation (C1) in Appendix C can be written as:

$$\begin{aligned} & \theta_i p_i(G-c) - (\mu_i p_i + \bar{\mu}_i(1-p_i))c + \xi_3 P(G-c) \\ & -(\nu_0(1+P-p_1-p_2) + \nu_1(p_1-P) + \nu_2(p_2-P) + \nu_3 P)c \\ \geq\ & \theta_i p_i G - \theta_i(1-p_i)L + \xi_3 PG - \xi_3(p_j-P)L. \end{aligned} \tag{6}$$

Given previous results and using the expressions for $\xi_3$ and $\nu_i$, this constraint can be written as: for $i = 1, 2$ and $j \neq i$,

$$\begin{aligned} & \theta_i(1-p_i)L + (\nu_0 - \theta_1 - \theta_2 - \mu_1 - \mu_2 + \bar{\mu}_1 + \bar{\mu}_2 - \nu_3)(p_j - P)L \\ \geq\ & c[\nu_0 + \bar{\mu}_i + p_j(\bar{\mu}_j - \theta_j - \mu_j)]. \end{aligned} \tag{7}$$

Equation (C2) in Appendix C can be written as:

$$\begin{aligned} 0 \leq\ & \theta_i p_i(G-c) - (\mu_i p_i + \bar{\mu}_i(1-p_i))c + \xi_3 P(G-c) \\ & -(\nu_0(1+P-p_1-p_2) + \nu_1(p_1-P) + \nu_2(p_2-P) + \nu_3 P)c. \end{aligned} \tag{8}$$

Using the same manipulations as above, the latter inequality becomes: for $i = 1, 2$ and $j \neq i$,

$$\begin{aligned} & \theta_i p_i G + (\nu_0 - \theta_1 - \theta_2 - \mu_1 - \mu_2 + \bar{\mu}_1 + \bar{\mu}_2 - \nu_3) PG \\ \geq\ & c[\nu_0 + \bar{\mu}_i + p_j(\bar{\mu}_j - \theta_j - \mu_j)]. \end{aligned} \tag{9}$$

Finally, we write Equation (C3) in Appendix C as follows: for $i = 1, 2$ and $j \neq i$,

$$\gamma u^R(p_i) + \theta_j p_j u^R(\hat{p}_i) \geq 0. \tag{10}$$

The program is to maximize (5) under the constraints (1)-(2)-(3)-(4), (7), (9) and (10).

It is first immediate that $\nu_3 = 0$ at the optimum. With $A_i$, $B_i$ and $C_i$ the multipliers associated with constraints (7), (9) and (10), and $D$, $E_1$, $E_2$ and $F$ the multipliers associated with (1)-(2)-(3)-(4), one can compute the derivatives of the Lagrangian with respect to $(\mu_1, \mu_2, \bar{\mu}_1, \bar{\mu}_2, \nu_0)$ (omitting the constraints that each of these must lie within $[0, 1]$):

$$\begin{aligned}\frac{\partial \mathfrak{L}}{\partial \mu_i} &= -P - A_i(p_j - P)L - A_j(p_i - P)L + A_j c p_i \\ &\quad -B_i PG - B_j PG + B_j c p_i - (D + E_i)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathfrak{L}}{\partial \bar{\mu}_i} &= P + A_i(p_j - P)L + A_j(p_i - P)L - A_j c p_i \\ &\quad +B_i PG + B_j PG - B_j c p_i - c(A_i + B_i) + (D + E_i) - F\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathfrak{L}}{\partial \nu_0} &= P + A_1(p_2 - P)L - cA_1 + A_2(p_1 - P)L - cA_2 \\ &\quad +B_1 PG - cB_1 + B_2 PG - cB_2 + (D + E_1 + E_2) - F\end{aligned}$$

Note that if $(A_j + B_j) = 0$, then $\frac{\partial \mathfrak{L}}{\partial \mu_i} < 0$ and so, $\mu_i = 0$.

From the derivatives of the Lagrangian, one can derive useful relationships:

$$\frac{\partial \mathfrak{L}}{\partial \mu_i} + \frac{\partial \mathfrak{L}}{\partial \bar{\mu}_i} = -F - c(A_i + B_i) \leq 0, \tag{11}$$

$$\frac{\partial \mathfrak{L}}{\partial \bar{\mu}_i} + E_j = \frac{\partial \mathfrak{L}}{\partial \nu_0} + c(1 - p_i)(A_j + B_j). \tag{12}$$

**Claim 1.** *The optimum cannot be such that $\nu_0 > 0$, $\mu_1 > 0$ and $\mu_2 > 0$.*

**Proof:** If $\nu_0 > 0$, $\mu_i > 0$ for $i = 1, 2$, it follows that $\frac{\partial \mathfrak{L}}{\partial \nu_0} \geq 0$, $\frac{\partial \mathfrak{L}}{\partial \mu_i} \geq 0$. $A_1$, $A_2$, $B_1$ and $B_2$ must be strictly positive so that $\frac{\partial \mathfrak{L}}{\partial \mu_i} + \frac{\partial \mathfrak{L}}{\partial \bar{\mu}_i} < 0$. Hence, $\frac{\partial \mathfrak{L}}{\partial \bar{\mu}_i} < 0$ and $\bar{\mu}_i = 0$ from (11).

Moreover, (12) implies that $E_j > 0$, which implies $\nu_j = 0$ and so, summing (2) and (3), $\xi_3 = -\nu_0 < 0$, a contradiction.

**Claim 2.** *The optimum is without loss of generality such that for $i = 1, 2$, $\mu_i \bar{\mu}_i = 0$.*

**Proof:** Fix $\bar{\mu}_i - \mu_i$. A simple examination of $Q$ and of all the constraints reveals that decreasing $\bar{\mu}_i$ only relaxes (1) and (7)-(9). Therefore, if $\bar{\mu}_i - \mu_i \geq 0$, the optimum can be chosen so that $\mu_i = 0$ and if $\bar{\mu}_i - \mu_i \leq 0$, the optimum can be chosen so that $\bar{\mu}_i = 0$.

Therefore, we will now focus on optima that satisfy Claim 2.

**Claim 3.** *An optimum satisfying Claim 2 cannot be such that* $\nu_0 = 0$ *and* $\mu_i > 0$ *for some* $i$.

**Proof:** Suppose that $\nu_0 = 0$ and there exists $i$ such that $\mu_i > 0$. From Claim 2, the optimum is such that $\bar{\mu}_i = 0$. Then, the constraint that $\nu_i \geq 0$ is violated.

**Claim 4.** *An optimum satisfying Claim 2 cannot be such that* $\nu_0 > 0$, $\mu_1 > 0$ *and* $\mu_2 = 0$.

**Proof:** Suppose $\nu_0 > 0$ and $\mu_1 > 0 = \mu_2 = \bar{\mu}_1$. It must be that $\frac{\partial \mathfrak{L}}{\partial \nu_0} \geq 0$, $\frac{\partial \mathfrak{L}}{\partial \mu_1} \geq 0$, $\frac{\partial \mathfrak{L}}{\partial \bar{\mu}_1} \leq 0$ and $A_2 + B_2 > 0$. As in the proof of Claim 1, it follows that $E_2 > 0$, which implies that $\nu_2 = 0$. So, we have:

$$\begin{aligned} 0 &\leq \xi_3 = \nu_0 - \theta_1 - \theta_2 - \mu_1 - \mu_2 + \bar{\mu}_1 + \bar{\mu}_2 \\ &= \nu_2 - \theta_1 - \mu_1 + \bar{\mu}_1 = -\theta_1 - \mu_1 < 0, \end{aligned}$$

a contradiction.

**Claim 5.** *If* $\mu_1 = \mu_2 = 0$, *the optimum is without loss of generality such that* $\nu_0 = 0$.

**Proof:** Suppose $\mu_1 = \mu_2 = 0 < \nu_0$, then $\frac{\partial \mathfrak{L}}{\partial \nu_0} \geq 0$.

Note first that if there exists $i$ such that $\frac{\partial \mathfrak{L}}{\partial \bar{\mu}_i} > 0$, then $\bar{\mu}_i = 1$ and then $\eta < 0$, a contradiction. So, for $i = 1, 2$, $\frac{\partial \mathfrak{L}}{\partial \bar{\mu}_i} \leq 0$.

Note also that if $E_i > 0$, then $\nu_i = 0$ so that $\nu_j = \xi_3 + \nu_0 > 0$ and therefore $E_i = 0$. With the previous remark, using (12), this implies that $\frac{\partial \mathfrak{L}}{\partial \nu_0} = 0$ and for some $i$, $A_i = B_i = 0$.

Suppose $A_1 = B_1 = 0 < A_2 + B_2$ and $E_2 > 0 = E_1$. Consider the simplified program where the constraints corresponding to $A_1$, $B_1$ and $E_1$ are omitted. In this program, $\nu_0$ and $\bar{\mu}_2$ enter only through $(\nu_0 + \bar{\mu}_2)$ within $(0, 1]$; and so, there is no loss of generality in looking for the optimum with $\nu_0 = 0$.

The last possibility is such that $A_i = B_i = E_i = 0$ for $i = 1, 2$. Then, the simplified program where all corresponding constraints are omitted only depends upon $\nu_0 + \bar{\mu}_1 + \bar{\mu}_2$, and again, one can set $\nu_0 = 0$ without loss of generality.

To summarize, the optimal mechanism is without loss of generality such that $\nu_0 = \mu_1 = \mu_2 = 0$. It is fully characterized by $(\gamma, \theta_1, \theta_2, \bar{\mu}_1, \bar{\mu}_2)$, or, defining $\lambda_i = \bar{\mu}_i - \theta_i$, as in Lemma C1. This completes the proof of Lemma C1.

## Proof of Proposition C2

In the symmetric setting, feasibility requires: $\gamma + \theta_1 + \theta_2 + \lambda_1 + \lambda_2 = 1$. Incentive constraints (7), (9) and (10) now become:

$$\theta_i(1-p)L + (\lambda_1 + \lambda_2)(p - P)L \geq c[\theta_i + \lambda_i + \lambda_j p], \tag{13}$$

$$\theta_i pG + (\lambda_1 + \lambda_2)PG \geq c[\theta_i + \lambda_i + \lambda_j p], \tag{14}$$

$$\gamma(pG - (1-p)L) + \theta_i p(\hat{p}G - (1-\hat{p})L) \geq 0. \tag{15}$$

The sponsor maximizes $Q = \gamma + (\theta_1 + \theta_2)p + (\lambda_1 + \lambda_2)P$ subject to these constraints.

If $(\gamma, \theta_1, \theta_2, \lambda_1, \lambda_2)$ is an optimal mechanism, $(\gamma, \frac{\theta_1+\theta_2}{2}, \frac{\theta_1+\theta_2}{2}, \frac{\lambda_1+\lambda_2}{2}, \frac{\lambda_1+\lambda_2}{2})$ is a symmetric mechanism that satisfies the feasibility constraints, the incentive constraints, obtained by summing over $i = 1$ and 2 the constraints (13), (14) and (15), and that yields the same $Q$. We will therefore focus wlog on symmetric mechanisms.

For a symmetric mechanism $(\gamma, \theta, \lambda)$, feasibility requires $\gamma + 2\theta + 2\lambda = 1$ and incentive constraints become:

$$\theta(p_+ - p) + \lambda\{p_+(1+p) - (1-p) - 2P\} \geq 0, \tag{16}$$

$$\theta(p - p_-) + \lambda(2P - (1+p)p_-) \geq 0, \tag{17}$$

$$\gamma(p - p_0) + \theta p(\hat{p} - p_0) \geq 0. \tag{18}$$

The sponsor maximizes $Q = \gamma + 2\theta p + 2\lambda P$ subject to these constraints. Since for $p \geq p_0$, the unconstrained optimum ($\gamma = 1$) is implementable, we focus on the case where $p < p_0$.

**First case:** $\hat{p} < p_0$. (18) implies that $\gamma = \theta = 0$. The situation is the symmetric stochastic version of the deterministic situation in which both committee members investigate sequentially. If $P \geq p_-$, the optimum is $\lambda = \frac{1}{2}$ and $Q = P$, while $Q = 0$ if $P < p_-$.

**Second case:** $p_- \leq p < p_0 < \hat{p}$. Consider the relaxed program where (16) and (17) are omitted:

$$\begin{aligned} & \max_{\theta,\lambda \geq 0} \{-2\theta(1-p) - 2\lambda(1-P)\} \\ \text{s.t. } 0 & \leq 1 - 2\theta - 2\lambda \\ 1 & \leq 2\lambda + \theta \frac{[2(p_0 - p) + p(\hat{p} - p_0)]}{p_0 - p}. \end{aligned}$$

It is immediate that the solution is $\lambda = 0$ and $\theta = \frac{1-\gamma}{2} = \theta^* \equiv \frac{p_0 - p}{2(p_0-p)+p(\hat{p}-p_0)}$. Moreover, since $p - p_- \geq 0$ and $p_+ - p > 0$, this solution satisfies also (16) and (17). Hence, it is the optimal mechanism in this range of parameters.

**Third case:** $p < p_- < p_0 \leq \hat{p}$. As in the previous case, we use variables $(\theta, \lambda) \geq 0$ such that $\gamma = 1 - 2\theta - 2\lambda \geq 0$. The constraints can be written as follows:

$$\lambda \frac{[(1-p) + 2P - p_+(1+p)]}{p_+ - p} \equiv X\lambda \leq \theta, \tag{19}$$

$$\theta \leq \lambda \frac{(2P-(1+p)p_-)}{p_- - p} \equiv Y\lambda, \tag{20}$$

$$1 \leq 2\lambda + \frac{\theta}{\theta^*}. \tag{21}$$

Note first that if $Y \leq 0$, then $\theta = \lambda = 0$ necessarily and the set of constraints is empty. Hence $Q = 0$. Suppose now that $Y > 0$. Again, if $X > Y$, then the set of constraints is empty and $Q = 0$. The project can then be implemented with positive probability only if $Y \geq X$ and $Y > 0$. In this last case, consider the relaxed program where the sole constraints are $\theta \geq 0$, $\lambda \geq 0$, (20) and (21):

$$\begin{aligned} &\max_{\theta,\lambda \geq 0} \{-2\theta(1-p) - 2\lambda(1-P)\} \\ \text{s.t. } \theta &\leq Y\lambda \\ 1 &\leq 2\lambda + \frac{\theta}{\theta^*}. \end{aligned}$$

The constraint (21) must necessarily be binding, since otherwise the optimum would be $\theta = \lambda = 0$ which would violate (21). The constraint (20) must also be binding, since otherwise, the optimum would be $\lambda = 0$, $\theta = \theta^*$ and this would violate (20). Hence, the solution is: $\theta = Y\lambda = \theta^{**} \equiv \left(\frac{2}{Y} + \frac{1}{\theta^*}\right)^{-1}$. Moreover, since (20) is binding and $Y \geq X$, (19) is satisfied. For these values,

$$\gamma = 1 - 2\theta^{**}(1 + \frac{1}{Y}) = \frac{\frac{1}{Y} + \frac{1}{\theta^*} - 1}{\frac{2}{Y} + \frac{1}{\theta^*}};$$

since $\theta^* \leq \frac{1}{2}$ and $Y > 0$, $\gamma > 0$.

Therefore, in the range $p < p_- < p_0 \leq \hat{p}$, there exists a stochastic mechanism that yields a positive probability $Q$ if and only if:

$$\begin{aligned} 2P - (1+p)p_- &> 0 \text{ and} \\ \frac{2P-(1+p)p_-}{p_- - p} &\geq \frac{(1-p) + 2P - p_+(1+p)}{p_+ - p} \end{aligned}$$

that is, if and only if:

$$\begin{aligned} 2P &> (1+p)p_- \text{ and} \\ 2P &\geq (1+p)p + (1-p)\frac{p_- - p}{p_+ - p_-}. \end{aligned}$$

The condition for $Q > 0$ is therefore:

$$\hat{p} \geq \max\{\frac{(1+p)p_-}{2p}, \frac{(1+p)}{2} + \frac{(1-p)(p_- - p)}{2p(p_+ - p_-)}\}.$$

In a left neighborhood of $p_-$, both terms in the supremum tends to $\frac{1+p_-}{2} < 1$; therefore, the domain for which $Q > 0$ is not empty.