Appendix S1: The Consumption-Savings Model

I. We first give a result for general per-period utility functions in the simple savings model. Consider the problem of maximizing

$$U(\bar{a}) = \sum_{t=1}^{\infty} \delta^{t-1} [(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t)]$$

over all feasible plans $\bar{a}$, i.e. plans that satisfy $a_t \in [0, 1]$ and the wealth equation $y_t = Ra_{t-1}y_{t-1}$. We suppose that $u$ is non-decreasing and continuous on $(0, \infty)$; we do not require continuity on $[0, \infty)$ because we want to allow for the logarithmic case where $u(0) = \lim_{c \to 0} u(c) = -\infty$. Let $\bar{U}$ be the supremum in this problem.

**Proposition S1.1:** Suppose $R > 1$ and

$$\sum_{t=1}^{\infty} \delta^{t-1} u(R^{t-1}y_t) < \infty.$$  \hspace{1cm} (A.1)

Then: (i) For any feasible plan the sum defining $U$ has a well defined value in the sense that either the sum converges absolutely or converges to $-\infty$.

(ii) The supremum $\bar{U}$ of the feasible values satisfies $-\infty < \bar{U} < \infty$

(iii) If feasible $\bar{a}^n \to \bar{a}^*$ in the product topology then $\bar{a}^*$ is feasible. If in addition $U(\bar{a}^n) \to \bar{U}$ then $U(\bar{a}^*) = \bar{U}$.

(iv) An optimal plan exists. That is, there is a feasible plan that attains $\bar{U}$. 
Proof: (i) Define $\chi_+(x) = \max\{x, 0\}$ and $\chi_-(x) = -\chi_+(-x)$.

We can write any finite sum as the sum of negative and positive parts, so for any sequence $(\tilde{a}, \tilde{y})$ we have

$$
\sum_{t=1}^{T} \delta^{t-1} \left[ (1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t) \right] = \\
\sum_{t=1}^{T} \delta^{t-1} \chi_+ \left[ (1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t) \right] + \\
\sum_{t=1}^{T} \delta^{t-1} \chi_- \left[ (1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t) \right]
$$

The positive part of the sum is summable from (A.1), since

$$(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t) \leq u((1 - a_t)y_t) \leq u(R^{t-1}y_t).$$

The negative part is monotone decreasing in $T$, so it either converges absolutely or converges to $-\infty$. In the former case the entire sum converges absolutely; in the latter case the sum converges to $-\infty$.

(ii) Part (i) already shows that $\overline{U} < \infty$. To see that $\overline{U} > -\infty$, note that it is feasible to set $a_t = 1 / R$ for all $t$, and that for $R > 1$ this plan yields a finite value.

(iii) Consider a sequence of feasible plans $\tilde{a}^n \to \tilde{a}^*$. Because the constraints are period by period and closed, it is clear that $\tilde{a}^*$ satisfies the constraints, so it is feasible.

Now suppose in addition that $U(\tilde{a}^n) \to \overline{U}$. Choose any $\varepsilon > 0$ and pick $N$ large enough that $\overline{U} - U(\tilde{a}^n) < \varepsilon / 2$ for all $n > N$. If we now pick $T$ that
\[
\sum_{t=1}^{\infty} \delta^{t-1} |u(R^{t-1}y_1)| < \varepsilon / 2 ,
\]

we know that

\[
\bar{U} - \sum_{t=1}^{\tau} \delta^{t-1} [(1 + \gamma)u((1 - a^n_i)y^n_i) - \gamma u(y^n_i)] \leq \varepsilon
\]

for all \( n > N \) and \( \tau > T \). Since per-period payoffs are continuous at any \((a, y)\) with \( a > 0, \ a^n \to a^* \), and \( y^n \to y^* \), it follows that

\[
\bar{U} - \sum_{t=1}^{\tau} \delta^{t-1} [(1 + \gamma)u((1 - a_i^*)y^*_i) - \gamma u(y^*_i)] \leq \varepsilon \text{ for all } \tau > T .
\]

Since this is true for any \( \varepsilon > 0 \) and we know that \( U(a^*) \leq \bar{U} \), we conclude that

\[ U(a^*) = \bar{U} . \]

(iv) Now consider a feasible sequence \((\tilde{a}^n, \tilde{y}^n)\) with \( U(\tilde{a}^n) \to \bar{U} \). Each savings rate \( a_i \) must lie in the compact interval \([0, 1]\) and each \( y_i \) must lie in the compact interval \([0, R^{t-1}y_i]\), so the sequence \((\tilde{a}^n, \tilde{y}^n)\) has an accumulation point \((\tilde{a}^*, \tilde{y}^*)\) in the product topology. This accumulation point is a maximum by part (iii).

\[
(\tilde{a}^*, \tilde{y}^*)
\]

II. Now we specialize to the CRRA utility functions

\[
u(c) = \frac{c^{1-\rho} - 1}{1 - \rho}
\]

and \( u(c) = \ln(c) \), which corresponds to the case \( \rho = 1 \). Assuming \( \delta < R^{\rho-1} \) implies
\[
\sum_{t=1}^{\infty} \delta^{t-1} u(R^{t-1}y_t) < \infty.
\]

It follows from Proposition S1.1 that an optimum \( \bar{a}^* \) exists.

**Proposition S1.2:** With CRRA utility a stationary optimum with \( a_t = a \) exists.

Proof: Suppose that \( \bar{a}^* \) is an optimal plan. By homogeneity of the objective function, and the fact that plans are defined in terms of savings rates, \( \bar{a}^* \) is also an optimal plan starting in period 2 (for any initial condition). Note that the plan \( \bar{a}^{*2} = (a_1^*, a_1^*, a_2^*, a_3^*, \ldots) \) yields wealth in period 2 of \( a_1^* R y_1 \), and let \( \bar{U}(y_1) \) denote the maximized utility when starting in the second period with wealth \( y_1 \). Then

\[
U(\bar{a}^{*2}) = (1 + \gamma)u((1 - a_1^*)y_1) - \gamma u(y_1) + \delta \bar{U}(a_1^* R y_1) = \bar{U}
\]

where the first equality follows because \( \bar{a}^* \) is optimal from period 2 on, and the second equality because \( \bar{a}^* \) is optimal from the first period. Proceeding in this way we can construct sequence of feasible plans \( \bar{a}^n = (a_1^*, a_1^*, \ldots, a_1^*, a_2^*, a_3^*, \ldots) \) that play \( a_1^* \) for the first \( n \) periods such that \( U(\bar{a}^n) = U(a_1^*) = \bar{U} \). Clearly \( \bar{a}^n \) converges in the product topology to the plan of choosing the fixed savings rate \( a_1^* \). From Proposition A.1 (iii), this limiting plan is feasible and gives utility \( \bar{U} \); that is, it is optimal.

III. We have shown that it is sufficient to compute the present value utility from a fixed savings rate \( a \), and maximize over savings rates. We have present value utility
\[
U = \frac{y_1^{1-\rho}}{1-\rho} \sum_{t=1}^{\infty} \delta(Ra)^{(1-\rho)}\left[ (1 + \gamma)(1 - a)^{(1-\rho)} - \gamma \right] - \frac{1}{(1-\delta)(1-\rho)}
= \frac{y_1^{1-\rho}}{1-\rho} \frac{[(1 + \gamma)(1 - a)^{(1-\rho)} - \gamma]}{1 - \delta(Ra)^{(1-\rho)}} - \frac{1}{(1-\delta)(1-\rho)}
\]

Since the optimal savings rate cannot be 0 or 1, we may differentiate with respect to the saving rate to find

\[
dU / da \propto [(1 + \gamma)(1 - a) - \gamma(1 - a)^\rho] \delta R(a)^{-\rho} - (1 + \gamma)(1 - \delta(Ra)^{-\rho})
\]

which gives the first-order condition for an optimum

\[
(1 + \gamma)a_\gamma^{\ast\rho} = R^{-\rho}\delta((1 + \gamma) - \gamma(1 - a_\gamma^{\ast\rho})).
\]

When \(\gamma = 0\) we get the usual solution \(a^\ast = R^{(1-\rho)/\rho} \delta^{1/\rho}\). Thus we can rewrite the first order condition as

\[
(a_\gamma^\ast / a^\ast)^\rho = ((1 + \gamma) - \gamma(1 - a_\gamma^\ast)^\rho) / (1 + \gamma).
\]

IV: Turning to the simple banking model, utility starting in the second period is the \(\gamma = 0\) solution

\[
U_2(y_2) = \frac{y_2^{1-\rho}}{1-\rho} \frac{(1 - a^\ast)^{1-\rho}}{1 - \delta(Ra^\ast)^{(1-\rho)}} - \frac{1}{(1-\delta)(1-\rho)}
= \frac{y_2^{1-\rho}}{1-\rho} \frac{1}{(1 - a^\ast)^\rho} - \frac{1}{(1-\delta)(1-\rho)}
= \frac{y_2^{1-\rho}}{1-\rho} \frac{1}{(1 - \delta^{1/\rho} R^{(1-\rho)/\rho})^\rho} - \frac{1}{(1-\delta)(1-\rho)}
\]
The utility of both selves in the first period is

\[
(1 + \gamma) \frac{c_1^{1-\rho} - 1}{1 - \rho} - \gamma \frac{(x_1 + z_1)^{1-\rho} - 1}{1 - \rho},
\]

and so the overall objective of the long-run self is to maximize

\[
(1 + \gamma) \frac{c_1^{1-\rho} - 1}{1 - \rho} - \gamma \frac{(x_1 + z_1)^{1-\rho} - 1}{1 - \rho} + \frac{(R(y_1 + z_1 - c_1))^{1-\rho}}{1 - \rho} \frac{\delta}{(1 - \delta^{1/\rho} R^{(1-\rho)/\rho})} - \frac{\delta}{(1 - \delta)(1 - \rho)}
\]

The first order condition for optimal consumption is

\[
\frac{c_1^*}{R(y_1 + z_1 - c_1)} = \frac{(1 + \gamma)^{1/\rho} (1 - \delta^{1/\rho} R^{(1-\rho)/\rho})}{(\delta R)^{1/\rho}}.
\]

If there are one or more solutions that satisfy the constraint \( c_1^* \leq x_1 + z_1 \) then one of them represents the optimum; otherwise the optimum is to consume all pocket cash.

Note that \( x_1 \) is the solution for \( \gamma = 0 \), so it satisfies

\[
\frac{x_1}{R(y_1 - x_1)} = \frac{(1 - \delta^{1/\rho} R^{(1-\rho)/\rho})}{R^{1/\rho}}.
\]

Thus we can write the first order condition as

\[
\frac{c_1^*}{y_1 + z_1 - c_1^*} = (1 + \gamma)^{1/\rho} \frac{x_1}{y_1 - x_1}.
\]
or

\[ c_1^* = \frac{(1 + \gamma)^{1/\rho} x_1}{y_1 - x_1 + (1 + \gamma)^{1/\rho} x_1} (y_1 + z_1) \]

\[ = \frac{1 + \left[ (1 + \gamma)^{1/\rho} - 1 \right]}{1 + \left[ (1 + \gamma)^{1/\rho} - 1 \right] (1 - a^*)} (1 - a^*) (y_1 + z_1) \]

\[ = B(1 - a^*) (y_1 + z_1). \]
Appendix S2: Hyperbolic Procrastination and Delay

Theorem 4:

a) If $v > \mu$ and $\bar{x} > \delta \beta v$, the “sophisticated quasi-hyperbolic model” has a stationary equilibrium with $\underline{x} < x^{**} < \bar{x}$.

b) There is an open set of parameters that satisfies the restrictions of part a) and for which there are other equilibria.

proof: Recall equations (10) into (11) from the paper:

$$W^{**} = \frac{P(x^{**}) (\delta V) + (1 - P(x^{**})) E(x \mid x > x^{**})}{1 - \delta + \delta P(x^{**})}. \quad (10)$$

$$x^{**} = \delta \beta (V - W^{**}) \quad (11).$$

Substituting (10) into (11) we see that

$$x^{**} = \delta \beta \left( \frac{V - \delta V + \delta P(x^{**}) V - P(x^{**}) (\delta V) - (1 - P(x^{**})) E(x \mid x > x^{**})}{1 - \delta + \delta P(x^{**})} \right)$$

$$= \delta \beta \left( \frac{v - (1 - P(x^{**})) E(x \mid x > x^{**})}{1 - \delta + \delta P(x^{**})} \right).$$

Let $F(x^{**}) = x^{**} - \delta \beta \left( \frac{v - (1 - P(x^{**})) E(x \mid x > x^{**})}{1 - \delta + \delta P(x^{**})} \right)$. 
To prove the theorem it suffices to show that there is an $x^{**}$ where $F(x^{**}) = 0$. The assumption that $v > \mu$ implies that $F(0) = 0 - \delta \beta \left( \frac{v - \mu}{1 - \delta} \right) < 0$, and the assumption that $x > \delta \beta v$ implies that $F(x) = x - \delta \beta v > 0$.

b) Suppose that $x^1 = x$ and $x^2 = x$ so that the odd-numbered agents never act an even ones always act. Then the equilibrium payoff of an even-numbered agent is $\delta \beta V$, and the payoff of an even-numbered agent with cost $x_t$ who chooses to wait is $x_t + \delta \beta \mu + \delta^3 \beta V$, so the even agents’ strategy is a best response for all $x_t$ if $\delta \beta (1 - \delta^2) > x + \delta \beta \mu$, or equivalently

$$\delta \beta (v + \delta^2) > x.$$

The payoff of the odd-numbered agents who wait is $x_t + \delta^2 \beta V$, and the payoff to acting is $\delta \beta V$, so waiting is better for all $x_t$ if

$$x > \delta \beta (1 - \delta) V = \delta \beta v.$$

To complete the proof we must show that there is an open set of parameters such that (S2.1) and (S2.2) and the restrictions $v > \mu$ and $x > \delta \beta v$. To do this, fix $v > 0$ and $\delta, \beta \in (0,1)$, and some $\varepsilon > 0$. Set $x = (1 + \varepsilon) \delta \beta v$, $\mu = (1 + 2\varepsilon) \delta \beta v$, and $x = (1 + 3\varepsilon) \delta \beta v$. (Note that these conditions are consistent with a range of distributions, including the uniform.) By construction this satisfies $x > \delta \beta v$ and (S2.2).
If \( \varepsilon < \frac{1 - \delta \beta}{2\delta \beta} \) then \( \mu = (1 + 2\varepsilon)\delta \beta v < (1 +\frac{1 - \delta \beta}{\delta \beta})\delta \beta v = v \), and for

\[
\varepsilon < \frac{\delta(1 - \beta)}{(3 + 2\delta \beta)}
\]

we compute that

\[
\delta \beta(\mu(1 + \delta) - \mu) = \delta \beta(\mu(1 + \delta) - (1 + 2\varepsilon)\delta \beta v) \\
= \delta \beta v(1 + \delta (1 - \beta) - 2\varepsilon \delta \beta) \\
> \delta \beta v (1 + 3\varepsilon) \\
= \bar{x}.
\]

This shows that there is a ‘2-cycle equilibrium” whenever \( \varepsilon \) is sufficiently small. Since the inequalities in (12) and (13) hold strictly for the specified relationship between the parameters, they hold for an open set of \( \beta, \delta, V, \mu \); the inequalities also hold for a range of distributions with the given mean and endpoints.  

\[\square\]

1 We do not know if the first-order condition has a unique solution, except in the logarithmic case.