Online Appendix for When Should Sellers Use Auctions?

James W. Roberts[*] Andrew Sweeting[†]

October 2, 2012

A Propositions and Proofs

This Appendix shows that the entry thresholds, beliefs of potential entrants and jump bidding functions defined in the text form an equilibrium and is the only equilibrium consistent with our refinement assumptions. For clearer exposition, we begin with a two period game and show the there exists a unique equilibrium under the D1 refinement. We extend the result to games with more than two rounds by showing how a recursive application of the same arguments leads to the uniqueness of bidding and entry rules in earlier rounds.

A.1 Two Round Game

In a two round game, the equilibrium consists of strategies for potential entrants in both rounds, a jump bidding rule for a first round entrant and the beliefs of the potential entrant about the value of the first-round potential entrant given a jump bid. As explained in the text, we assume that both firms would bid up to their values in a second-round knockout auction.

The main proposition that we prove below is that there exists a unique equilibrium to this game under the D1 refinement. To show this, we establish the following three lemmas which immediately yield the proposition and characterize the nature of the unique equilibrium.

Lemma 1. *The expected post-entry profits of the potential entrant in round 2 are strictly increasing in its signal, S_2.*

[*]Department of Economics, Duke University and NBER. Contact: j.roberts@duke.edu.
[†]Department of Economics, Duke University and NBER. Contact: atsweet@duke.edu.

Proof. Given the knockout bidding assumption, the expected profit of the potential entrant who enters with signal S_2 will be

$$\int_{\widehat{b_2^{pre}}}^{\overline{V}} \int_{\widetilde{v}}^{\overline{V}} (v - \widetilde{v}) f_{\tau(2)}^{V}(v|S_2) g_2(\widetilde{v}) dv d\widetilde{v} \tag{1}$$

when his belief is that the value of the incumbent has pdf $g_2(\widetilde{v})$. If there is no incumbent, then the reserve price R can be viewed as an incumbent with known value R. The expression in (1) is weakly increasing in v and since $F_{\tau(2)}^{V}(v|S)$ is strictly decreasing in S, the entire expression is strictly increasing in S_2. □

Since the expected post-entry profits are monotonic in the second potential entrant's signal we get the following corollary.

Corollary 1. *If the expected post-entry profits are less than K for all S, then the second round potential entrant does not enter. Otherwise he enters if and only if his signal exceeds a threshold $S_{2,\tau(2)}^{\prime *}$ uniquely given by the solution to $\int_{\widehat{b_2^{pre}}}^{\overline{V}} \int_{\widetilde{v}}^{\overline{V}} (v-\widetilde{v}) f_{\tau(2)}^{V}(v|S_{2,\tau(2)}^{\prime *}) g_2(\widetilde{v}) dv d\widetilde{v} - K = 0$.*

Lemma 2. *There exists a unique equilibrium jump bidding function for a new entrant in period one under the D1 refinement which can be described as:*
(i) strictly increasing for incumbent values on $[R, \overline{V} - K]$ and characterized by the differential equation in (5) of the manuscript and the lower boundary condition $\beta_{1,\tau(1)}(R, R) = R$;
(ii) equal to $\beta_{1,\tau(1)}(\overline{V} - K, R)$ for incumbent values greater than $\overline{V} - K$;
(iii) and submitting no bid for incumbent values $< R$.

Proof. We begin by showing (i). Theorems 2 and 4 of Mailath and von Thadden (2011), generalizing Mailath (1987), provide sufficient conditions under which there is a unique separating equilibrium signal function $\beta_{1,\tau(1)}(v, R)$, determined by the differential equation (5) of the manuscript and the initial condition $\beta_{1,\tau(1)}(R, R) = R$. We now list these conditions (a)-(f) in our setting and show that each holds.

(a) *The possible value of the incumbent and its action space are compact intervals.* This is true in our model given our assumptions that values lie on $[0, \overline{V}]$ and possible bids lie on $[0, \overline{B}]$, $\overline{V} < \overline{B}$.

(b) *If the final round potential entrant observed the value of the incumbent, the jump bidding problem of the incumbent would have a unique solution.* The optimal bid would be equal to R. This is because it cannot be optimal for the incumbent to submit a bid above its value. Further, no bid below its value affects the potential entrant's entry decision but will reduce the incumbent's profit, relative to submitting a bid equal to R, if the potential entrant stays out.

(c) $\Pi_{1,\tau(1)}(v, v', b)$ *is continuous and differentiable in each argument.* This is true in the model since the exact form of $\Pi_{1,\tau(1)}(v, v', b)$ is

$$\Pi_{1,\tau(1)}(v, v', b) = [v - b] \overline{F}_1(b|v') + \int_b^v (v - x)\overline{f}_1(x|v')dx, \tag{2}$$

$$\text{where } \overline{F}_1(x|v') = \left[\int_0^x f^V_{\tau(2)}(y)dy + \int_x^{\overline{V}} F_{S,\tau(2)}(S'^*_{2,\tau(2)}(v')|y) f^V_{\tau(2)}(y)dy\right] \tag{3}$$

$$\text{and } \overline{f}_1(x|v') = \frac{\partial \overline{F}_1(x|v')}{\partial x}. \tag{4}$$

These profits will be continuous and differentiable in each argument as all of the pdfs and cdfs in these functions are continuous and differentiable and $S'^*_{2,\tau(2)}(v')$, determined by the threshold rule described above, will be continuous and differentiable in v'. Below we will make use of the fact that:

$$\frac{\partial \overline{F}_1(w|z)}{\partial z} = -\int_w^{\overline{V}} \frac{\partial \overline{f}_1(y|z)}{\partial z} dy$$

and that

$$\frac{\partial \overline{f}_1(y|v')}{\partial v'} = -\frac{\partial F_{S,\tau(2)}(S'^*_{2,\tau(2)}(v')|y)}{\partial S'^*_{2,\tau(2)}(v')} \frac{\partial S'^*_{2,\tau(2)}(v')}{\partial v'} f^V_{\tau(2)}(y) < 0$$

since $S'^*_{2,\tau(2)}$ is increasing in the potential entrant's perception of the incumbent's value because the incumbent will have a higher dropout point in a knockout auction.

(d) $\frac{\partial \Pi_{1,\tau(1)}(v,v',b)}{\partial v'} > 0$ *for all* (v, v'). After some algebra we have:

$$\begin{aligned}
\frac{\partial \Pi_{1,\tau(1)}(v, v', b)}{\partial v'} &= -[v - b] \int_b^{\overline{V}} \frac{\partial \overline{f}_1(y|v')}{\partial v'} dy + \int_b^v (v - x) \frac{\partial \overline{f}_1(y|v')}{\partial v'} dy \\
&= -\left[[v - b] \int_v^{\overline{V}} \frac{\partial \overline{f}_1(y|v')}{\partial v'} dy + [v - b] \int_b^v \frac{\partial \overline{f}_1(y|v')}{\partial v'} dy - \int_b^v (v - x) \frac{\partial \overline{f}_1(y|v')}{\partial v'} dy\right] \\
&> -\left[[v - b] \int_v^{\overline{V}} \frac{\partial \overline{f}_1(y|v')}{\partial v'} dy\right] > 0
\end{aligned}$$

(e) $\frac{\partial \Pi_{1,\tau(1)}(v,v',b)}{\partial b} \neq 0$ *for all* b. This is immediate since $\frac{\partial \Pi_{1,\tau(1)}(v,v',b)}{\partial b} = -\overline{F}_1(b|v') < 0$.

(f) $\frac{\partial \Pi_{1,\tau(1)}(v,v',b)}{\partial b} / \frac{\partial \Pi_{1,\tau(1)}(v,v',b)}{\partial v'}$ *is monotonic in* v *for all* (v', b). We can prove this directly. Alternatively we can define the profit function in terms of entry thresholds instead of beliefs about the incumbent's value: $\pi_{1,\tau(1)}(v, S'^*_{2,\tau(2)}, b)$ and show single crossing in terms of signal threshold: $\frac{\partial \pi_{1,\tau(1)}(v,S'^*_{2,\tau(2)},b)}{\partial b} / \frac{\partial \pi_{1,\tau(1)}(v,S'^*_{2,\tau(2)},b)}{\partial S'^*_{2,\tau(2)}}$ is monotonic in v for all $(S'^*_{2,\tau(2)}, b)$. Roddie (2011) shows (his fact 2) that when $S'^*_{2,\tau(2)}(v')$ is monotonically increasing in v', which was shown above, that this signal-threshold version of single crossing implies $\frac{\partial \Pi_{1,\tau(1)}(v,v',b)}{\partial b} / \frac{\partial \Pi_{1,\tau(1)}(v,v',b)}{\partial v'}$ is monotonic in v for all (v', b). As it will be useful to have a single crossing condition written in terms of the potential entrant's signal threshold for proving that no pooling equilibria exist below, we take this second route by establishing $\frac{\partial \pi_{1,\tau(1)}(v, S'^*_{2,\tau(2)}, b)}{\partial b} / \frac{\partial \pi_{1,\tau(1)}(v, S'^*_{2,\tau(2)}, b)}{\partial S'^*_{2,\tau(2)}}$ is monotonic in v for all $(S'^*_{2,\tau(2)}, b)$.

We prove this by showing that the derivative of this expression with respect to v is always positive. Differentiating this expression with respect to v yields (using superscripts to denote partial derivatives) $\pi^{13}_{1,\tau(1)} \left[\pi^2_{1,\tau(1)}\right]^{-1} - \pi^3_{1,\tau(1)} \pi^{12}_{1,\tau(1)} \left[\pi^2_{1,\tau(1)}\right]^{-2}$, which is equal to:

$$\overline{F}_1(b|S'^*_{2,\tau(2)}) \left[- \int_v^{\overline{V}} \frac{\partial \overline{f}_1(y|S'^*_{2,\tau(2)})}{\partial S'^*_{2,\tau(2)}} dy \right] \left[-[v-b] \int_b^{\overline{V}} \frac{\partial \overline{f}_1(y|S'^*_{2,\tau(2)})}{\partial S'^*_{2,\tau(2)}} dy + \int_b^v (v-x) \frac{\partial \overline{f}_1(y|S'^*_{2,\tau(2)})}{\partial S'^*_{2,\tau(2)}} dy \right]^{-2}$$

This expression is always positive since all three terms being multiplied are positive.

This establishes the form and the uniqueness of the separating equilibrium bid function on the interval $[R, \overline{V} - K]$. We now show that no pooling equilibria exist over this interval. Theorem 3 of Ramey (1996) shows that if the incumbent does not want to submit the maximum possible bid and $\frac{\partial \pi_{1,\tau(1)}(v, S'^*_{2,\tau(2)}, b)}{\partial b} / \frac{\partial \pi_{1,\tau(1)}(v, S'^*_{2,\tau(2)}, b)}{\partial S'^*_{2,\tau(2)}}$ is monotonic in v for all $(S'^*_{2,\tau(2)}, b)$, then no pooling equilibria can exist under D1. We just established the second condition and we know that the first condition holds since our assumption that $\overline{B} > \overline{V}$ implies that even the highest incumbent type will not submit the maximum possible bid.

We now show part (ii) of the lemma. A potential entrant who believes that the incumbent's value is $\overline{V} - K$ will not enter whatever his signal as the signal technology implies that there is some probability that the entrant's value will be less than $\overline{V}$. Given this, the expected benefit of entering the mechanism is less than the entry cost K. Therefore, considering only bids greater than or equal to $\beta_{1,\tau(1)}(\overline{V} - K, R)$, the strictly dominant strategy will be to bid $\beta_{1,\tau(1)}(\overline{V} - K, R)$. The single crossing condition implies that if $\beta_{1,\tau(1)}(\overline{V} - K, R)$ is preferred to a lower bid by the incumbent with value $\overline{V} - K$ then it is also preferred by an incumbent with a value greater than $\overline{V} - K$.

Part (iii) of the lemma is immediate since an incumbent should not bid more than his value as he may have to pay this bid if the potential entrant stays out or comes in with a value less than the incumbent. □

Lemma 3. *The expected post-entry profits of the potential entrant in round 1 are strictly increasing in S_1.*

Proof. In the first round, the expected post-entry profit of a potential entrant if it enters with signal S_1 is

$$\int_R^{\overline{V}} \Pi_{1,\tau(1)}(v, v, \beta_{1,\tau(1)}(v, R)) f^V_{\tau(1)}(v|S_1) dv \tag{5}$$

where $\beta_{1,\tau(1)}(v,R)$ is the equilibrium jump bidding strategy, characterized above, for the firm if it enters and has a value above the reserve (if it has a value less than the reserve it does not submit a bid after entering). As long as the expression in (5) is weakly increasing in v, it will be strictly increasing in S_1 since $F^V_{\tau(1)}(v|S)$ is strictly decreasing in S. We now show that the expression in (5) is weakly increasing in v.

To do this we must establish that $\Pi_{1,\tau(1)}(v,v,\beta_{1,\tau(1)}(v,R))$ is increasing in v for $v>R$. Consider any v on $[R,\overline{V}-K]$, where we know from above that the jump bidding schedule is separating. Incentive compatibility of the jump bidding strategy implies that

$$\Pi_{1,\tau(1)}(v,v,\beta_{1,\tau(1)}(v,R)) \geq \Pi_{1,\tau(1)}(v,\widehat{v},\beta_{1,\tau(1)}(\widehat{v},R)) \text{ for any } \widehat{v}<v$$

and, as the payoff of a v incumbent will be higher than a $\widehat{v}$ incumbent if he wins without having to compete in a knockout auction when both use a bid of $\beta_{1,\tau(1)}(\widehat{v},R)$, we also know that

$$\Pi_{1,\tau(1)}(v,\widehat{v},\beta_{1,\tau(1)}(\widehat{v},R)) > \Pi_{1,\tau(1)}(\widehat{v},\widehat{v},\beta_{1,\tau(1)}(\widehat{v},R)) \text{ for any } \widehat{v}<v$$

and thus $\Pi_{1,\tau(1)}(v,v,\beta_{1,\tau(1)}(v,R)) > \Pi_{1,\tau(1)}(\widehat{v},\widehat{v},\beta_{1,\tau(1)}(\widehat{v},R))$ for any $\widehat{v}<v$, as required. For any v greater than $\overline{V}-K$, equilibrium payoffs will also be increasing in v as $\Pi_{1,\tau(1)}(v,v,\beta_{1,\tau(1)}(v,R))$ $= v-\beta_{1,\tau(1)}(\overline{V}-K,R)$. □

Since the expected post-entry profits are monotonic in the first round potential entrant's signal we get the following corollary.

Corollary 2. *If the expected post-entry profits are less than K for all S, then the first round potential entrant does not enter. Otherwise he enters if and only if his signal exceeds a threshold $S'^*_{1,\tau(1)}$ uniquely given by the solution to $\int_R^{\overline{V}} \Pi_{1,\tau(1)}(v,v,\beta_{1,\tau(1)}(v,R)) f^V_{\tau(1)}(v|S'^*_{1,\tau(1)})dv - K = 0$.*

The above lemmas immediately imply that the following:

Proposition 1. *There exists a unique equilibrium bid function and entry thresholds in the two round sequential mechanism with pre-entry signals under the D1 refinement.*

A.2 Three or More Round Games

We now explain how the above proposition's existence and uniqueness results can be extended to sequential mechanisms with three or more rounds. To do so, we use the same recursive arguments that were used in the two round game. Consider a three round game. The proofs for the equilibrium strategies in the penultimate and final rounds are exactly the same as above, except that the incumbent in the penultimate round may be bidding from a standing bid determined by the value of a previous incumbent rather than the reserve price, and the penultimate round entry threshold will depend on the agent's beliefs about the value of the incumbent if there is one. Following the arguments above, this threshold, $S'^{*}_{2,\tau(2)}$, is uniquely determined by the zero profit condition $\int_{\widehat{b_2^{pre}}}^{\overline{V}} \int_{\widetilde{v}}^{\overline{V}} \Pi_{2,\tau(2)}(v, v, \beta_{2,\tau(2)}(v, \widetilde{v})) f^{V}_{\tau(2)}(v|S'^{*}_{2,\tau(2)}) dv d\widetilde{v} - K = 0$.

We need to characterize the jump bidding function for an incumbent in the first round. After this, extending the arguments to four or more round games is straightforward as again the proofs for the equilibrium strategies in the last three rounds of a four round game would be exactly the same as in a three round game (except that the incumbent in the second round may be bidding from a standing bid determined by the value of a previous incumbent rather than the reserve price, and the second round entry threshold will depend on the agent's beliefs about the value of the incumbent if there is one).

To characterize the first round jump bidding function requires establishing the three-plus-round versions of properties (a)-(f) listed in the proof of part (i) of Lemma 2. Properties (a)-(c) and (e) are immediate. For property (d) to hold, so that the incumbent in the first round is better off being perceived as having a higher value, we must show that the entry thresholds of the subsequent potential entrants are increasing in their beliefs about his value since this implies that they are less likely to enter for any potential entrant value. We know from above that this will be the case for the final round potential entrant. As the following lemma illustrates, it is also true for the second round potential entrant and so property (d) holds.

Lemma 4. *The second round potential entrant's entry threshold is increasing in its beliefs about a round one incumbent's value v'.*

Proof. This requires showing that $\Pi_{2,\tau(2)}(v, v, \beta_{2,\tau(2)}(v, v'))$ decreases in v', the standing bid at the end of a knockout that the potential entrant wins. This will be the case because $\beta_{2,\tau(2)}(v, v')$ increases in v' (since, by standard arguments, two bid functions defined by the same differential equation, but with different initial conditions, cannot cross) and since the final round potential entrant's entry decision depends only the second round entrant's value if he wins the knockout (since the bid function is fully revealing), then this jump bid will only serve to increase the price paid by the second round entrant in the event the final round entrant stays out. $\square$

The final property needed to show the existence and uniqueness of a separating equilibrium bid function, and that there are no pooling equilibria, in the first round is the three-plus-round version of single crossing, property (f) above. With three rounds, this can be more compactly proved by using the non-derivative form of single crossing.

Lemma 5. *Consider any two possible bid and entry threshold combinations* $(S_A^{2\prime}, S_A^{3\prime}, b_A)$ *and* $(S_B^{2\prime}, S_B^{3\prime}, b_B)$ *where* $b_B > b_A$. *For* $v^H > v^L$, *if* $\Pi_{1,\tau(1)}(v^L, S_B^{2\prime}, S_B^{3\prime}, b_B) \geq \Pi_{1,\tau(1)}(v^L, S_A^{2\prime}, S_A^{3\prime}, b_A)$, *then* $\Pi_{1,\tau(1)}(v^H, S_B^{2\prime}, S_B^{3\prime}, b_B) > \Pi_{1,\tau(1)}(v^H, S_A^{2\prime}, S_A^{3\prime}, b_A)$.

Proof. Consider all possible combinations of values and signals of the second and third round potential entrants. The required implication will hold if the profit gain to $(S_B^{2\prime}, S_B^{3\prime}, b_B)$ is not lower for the incumbent with value v^H than the incumbent with type v^L for any combination, and it is strictly greater for some combination (all combinations are possible). In the following we will use $v_{2:3,A}^{\max}$ as the maximum value of an entrant under $(S_A^{2\prime}, S_A^{3\prime}, b_A)$ conditional on the incumbent still being the incumbent after round 2.

If the switch to $(S_B^{2\prime}, S_B^{3\prime}, b_B)$ has no effect on the entry of this entrant, then the payoffs of either incumbent are only affected if $b_B \geq v_{2:3,A}^{\max}$, in which case there is a cost to both incumbents of $b_B - \max\{v_{2:3,A}^{\max}, b_A\}$, which is independent of v.

If the switch to $(S_B^{2\prime}, S_B^{3\prime}, b_B)$ causes this entrant not to enter, which will happen with positive probability for any $v_{2:3,A}^{\max}$, then label the maximum value of the highest entrant $v_{2:3,B}^{\max}$, which could be equal to zero and will be less than $v_{2:3,A}^{\max}$. If $b_A \geq v_{2:3,A}^{\max}$ then the cost to both incumbents is $b_B - b_A$ and so is independent of v. If $b_B \geq v_{2:3,A}^{\max} \geq b_A$, the cost to both incumbents is $b_B - v_{2:3,A}^{\max}$, which is independent of v. If $v^H > v^L \geq v_{2:3,A}^{\max} \geq b_B$ there is a gain to both incumbents of $v_{2:3,A}^{\max} - \max\{b_B, v_{2:3,B}^{\max}\}$ and so is independent of v. If $v_{2:3,B}^{\max} \geq v^H$ there is no impact on either incumbent's profits. In the remaining cases the H incumbent will gain strictly more than the L incumbent. This can happen when $v^H > v_{2:3,A}^{\max} > v^L > v_{2:3,B}^{\max} > b_B$ in which case the gain to the H incumbent is $v_{2:3,A}^{\max} - v_{2:3,B}^{\max}$ which exceeds the gain of $v^L - v_{2:3,B}^{\max}$ for the L incumbent. It can happen when $v^H > v_{2:3,A}^{\max} > v_{2:3,B}^{\max} > v^L > b_B$ in which case gain to the H incumbent is $v_{2:3,A}^{\max} - v_{2:3,B}^{\max}$ which exceeds no gain for the L incumbent. It can happen when $v^H > v_{2:3,A}^{\max} > v^L > b_B > v_{2:3,B}^{\max}$ in which case gain to the H incumbent is $v_{2:3,A}^{\max} - b_B$ which exceeds the gain of $v^L - b_B$ for the L incumbent. It can happen when $v_{2:3,A}^{\max} > v^H > v^L > b_B > v_{2:3,B}^{\max}$ in which case gain to the H incumbent is $v^H - b_B$ which exceeds the gain of $v^L - b_B$ for the L incumbent. It can also happen when $v_{2:3,A}^{\max} > v^H > v_{2:3,B}^{\max} > v^L > b_B$ in which case gain to the H incumbent is $v^H - v_{2:3,B}^{\max}$ which exceeds no gain for the L incumbent. Finally it can happen when $v_{2:3,A}^{\max} > v^H > v^L > v_{2:3,B}^{\max} > b_B$ in which case gain to the H incumbent is $v^H - v_{2:3,B}^{\max}$ which exceeds the gain of $v^L - v_{2:3,B}^{\max}$ for the L incumbent. □

The arguments easily extend to more than three rounds leading to the following proposition.

Proposition 2. *There exists a unique equilibrium for entry and bidding behavior in the sequential mechanism with pre-entry signals in which:*

1. *A type $\tau(n)$ potential entrant in round n will enter if and only if it receives a signal above a threshold $S'^{*}_{n,\tau(n)}$ defined by the zero profit condition given by equation (2) of the manuscript for $n < N$ and by equation (4) of the manuscript for $n = N$;*

2. *Any entrant participating in a knockout auction bids up to its value;*

3. *Any incumbent placing a jump bid in round n when either the reserve or the standing bid at the end of the previous knockout is $\widehat{b}_n$ bids according to a bid function $\beta_{n,\tau(n)}(v, \widehat{b}_n)$ that is unique and:*

   (a) when $v \in \left[\widehat{b}_n, \overline{V} - K\right]$ is determined by the solution to the differential equation:

$$\frac{d\beta_{n,\tau(n)}(v, \widehat{b}_n)}{dv} = -\frac{\Pi^2_{n,\tau(n)}(v, v, \beta_{n,\tau(n)}(v, \widehat{b}_n))}{\Pi^3_{n,\tau(n)}(v, v, \beta_{n,\tau(n)}(v, \widehat{b}_n))}$$

   with lower boundary condition: $\beta_{n,\tau(n)}(\widehat{b}_n, \widehat{b}_n) = \widehat{b}_n$; and

   (b) when $v \in (\overline{V} - K, \overline{V}]$ is $\beta_{n,\tau(n)}(\overline{V} - K, \widehat{b}_n)$.

Off-the-equilibrium-path beliefs of potential entrants are not unique. While a potential entrant in round n that observes a jump bid x in an earlier round m between $\left[\widehat{b}_m, \beta_{m,\tau(m)}(\overline{V} - K, \widehat{b}_m)\right]$ will believe that the value of this incumbent is $\beta^{-1}_{m,\tau(m)}(x, \widehat{b}_m)$, the density of potential entrants' beliefs of the incumbent's type over the interval $(\overline{V} - K, \overline{V}]$ upon observing a bid greater than $\beta_{m,\tau(m)}(\overline{V} - K, \widehat{b}_m)$ is not pinned down. However, for all such beliefs, the equilibria have the common feature that entry will cease once this bid is placed.

B Details of Estimation Method

This appendix describes our estimation procedure based on Ackerberg (2009)'s method of simulated maximum likelihood with importance sampling.

This method involves solving a large number of games with different parameters once, calculating the likelihoods of the observed data for each of these games, and then re-weighting these likelihoods during the estimation of the distributions for the structural parameters. This method is attractive when it is believed that the parameters of the model are heterogeneous across auctions and it would be computationally prohibitive to re-solve the model many times (in order to integrate out the heterogeneity) each time one of the parameters changes.[1]

[1] Bajari, Hong and Ryan (2010) use a related method to analyze entry into a complete information entry game with no selection.

To apply the method, we assume that the parameters are distributed across auctions according to the specification given in Section 4.3. These specifications reflect our assumptions that σ_V, α and K are the same for mills and loggers within any particular auction, even though they may differ across auctions. The lower bound on σ_{Va} is set slightly above zero simply to avoid computational problems that were sometimes encountered when there was almost no dispersion of values. Our estimated specifications also assume that the various parameters are distributed independently across auctions. This assumption could be relaxed, although introducing a full covariance matrix would significantly increase the number of parameters to be estimated and, when we have tried to estimate these parameters, we have not found these coefficients to be consistently significant across specifications. The set of parameters to be estimated are $\Gamma = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \omega^2_{\mu,\text{logger}}, \omega^2_{\mu,\text{diff}}, \omega^2_{\sigma_V}, \omega^2_{\alpha}, \omega^2_{K}\}$, and a particular draw of the parameters $\{\mu_{a,\text{logger}}, \mu_{a,\text{mill}}, \sigma_{Va}, \alpha_a, K_a\}$ is denoted θ.

Denoting the outcome for an observed auction by y_a, the log-likelihood function for a sample of A auctions is

$$\sum_{a=1}^{A} \log \left(\int L_a(y_a|\theta)\phi(\theta|X_a, \Gamma) d\theta \right) \tag{6}$$

where $L_a(y_a|\theta)$ is the likelihood of the outcome y in auction a given structural parameters θ, $\phi(\theta|X_a, \Gamma)$ is the pdf of the parameter draw θ given Γ, our distributional assumptions, the unique equilibrium strategies implied by our equilibrium concept and auction characteristics including the number of potential entrants, the reserve price and observed characteristics X_a.

Unfortunately, the integral in (6) is multi-dimensional and cannot be calculated exactly. We follow Ackerberg by recognizing that

$$\int L_a(y_a|\theta)\phi(\theta|X_a, \Gamma) d\theta = \int L_a(y_a|\theta) \frac{\phi(\theta|X_a, \Gamma)}{g(\theta|X_a)} g(\theta|X_a) d\theta \tag{7}$$

where $g(\theta|X_a)$ is the importance sampling density whose support does not depend on Γ, which is true in our case because the truncation points are not functions of the parameters to be estimated. This can be approximated by simulation using

$$\frac{1}{S} \sum_{s} L_a(y_a|\theta_s) \frac{\phi(\theta_s|X_a, \Gamma)}{g(\theta_s|X_a)} \tag{8}$$

where θ_s is one of S draws from $g(\theta|X_a)$. Critically, this means that we can calculate $L_a(y_a|\theta_s)$ for a given set of S draws that do not vary during estimation, and simply change the weights $\frac{\phi(\theta_s|X_a,\Gamma)}{g(\theta_s|X_a)}$, which only involves calculating a pdf when we change the value of Γ rather than re-solving the game.

This simulation estimator will only be accurate if a large number of θ_s draws are in the range where $\phi(\theta_s|X_a,\Gamma)$ is relatively high, and, as is well known, simulated maximum likelihood estimators are only consistent when the number of simulations grows fast enough relative to the sample size. We therefore proceed in two stages. First, we estimate an initial guess of Γ using $S = 2,500$ draws, where $g(\cdot)$ is a multivariate uniform distribution over a large range of parameters which includes all of the parameter values that are plausible. Second, we use these estimates $\widehat{\Gamma}$ to repeat the estimation using a new importance sampling density $g(\theta|X_a) = \phi(\theta_s|X_a,\widehat{\Gamma})$ with $S = 500$ per auction. Roberts and Sweeting (2011) provide Monte Carlo evidence that the estimation procedure works well even for smaller values of S.

To apply the estimator, we also need to define the likelihood function $L_a(y_a|\theta)$ based on the data we observe about the auction's outcome, which includes the number of potential entrants of each type, the winning bidder and the highest bids announced during the open outcry auction by the set of firms that indicated that they were willing to meet the reserve price. Two problems arise when interpreting these data. First, a bidder's highest announced bid in an open outcry auction may be below its value, and it is not obvious which mechanism leads to the bids that are announced (Haile and Tamer (2003)). Second, if a firm does not know its value when taking the entry decision, it may learn (after paying the entry cost) that its value is less than the reserve price and so not submit a bid.

We therefore make the following assumptions (Roberts and Sweeting (2011) present estimates based on alternative assumptions about the data generating process that deliver similar results) that are intended to be conservative interpretations of the information that is in the data: (i) the second highest observed bid (assuming one is observed above the reserve price) is equal to the value of the second-highest bidder;[2] (ii) the winning bidder has a value greater than the second highest bid; (iii) both the winner and the second highest bidder entered and paid K_a; (iv) other firms that indicated that they would meet the reserve price or announced bids entered and paid K_a and had values between the reserve price and the second highest bid; and, (v) all other potential entrants may have entered (paid K_a) and found out that they had values less than the reserve, or they did not enter (did not pay K_a). If a firm wins at the reserve price we assume that the winner's value is above the reserve price.

[2] Alternative assumptions could be made. For example, we might assume that the second highest bidder has a value equal to the winning bid, or that the second highest bidder's value is some explicit function of his bid and the winning bid. In practice, 96% of second highest bids are within 1% of the high bid, so that any of these alternative assumptions give similar results. We have computed some estimates using the winning bid as the second highest value and the coefficient estimates are indeed similar.

C Details and Robustness of Numerical Procedure for Solving Sequential Mechanism with Pre-Entry Signals

This appendix details the recursive numerical procedure used to solve for equilibrium in the sequential mechanism.

We start with the final potential entrant, who believes that he will win if his value is greater than the incumbent's. For every possible value v' of the incumbent that this final potential entrant faces, we solve for the equilibrium entry threshold $S'^{*}_{N,\tau(N)}(v')$ on a fine grid of evenly spaced possible values $[0,\overline{V}]$. For example, the comparisons of mechanisms in Figure **??** are based on a grid with unit spacing, but we have experimented with 1/10th unit spacing with little effect on our results but substantial increases in the time needed to solve the game. Since the final price, if the final potential entrant wins, will be the value of the incumbent, the entry threshold of the final potential entrant is given by:

$$K = \int_{v'}^{\overline{V}} (x - v') f^{V}_{\tau(N)}(x|S'^{*}_{N,\tau(N)})dx \tag{9}$$

The integral in Equation (9) is approximated using the trapezoidal rule. Since the right hand side of Equation (9) is monotonic in $S'_{N,\tau(N)}$, we use the method of bisection to calculate $S'^{*}_{N,\tau(N)}(v')$ at every v' on $[0,\overline{V}]$. Our default tolerance for solving for signal thresholds is 10^{-6}.

Next we solve for the jump bid functions of the previous potential entrant were he to enter and win any knockout auction. The differential equation that defines the bid function (the definition of the individual terms appears in the body of the text) is given:

$$\frac{d\beta(\cdot)}{dv} = \frac{\overbrace{[v-\beta(\cdot)]\left[\frac{d\Pi_{k=n+1}^{N} F_{S,\tau(k)}(S'^{*}_{k,\tau(k)}(v))}{dv} + \frac{\partial \overline{F}_{n,\tau(n)}(\beta(\cdot)|v)}{\partial v}\right] + \int_{\beta(\cdot)}^{v} (v-\widehat{v})\,\frac{\partial \overline{f}_{n,\tau(n)}(\widehat{v}|v)}{\partial v} d\widehat{v}}^{(b)}}{\underbrace{\Pi_{k=n+1}^{N} F_{S,\tau(k)}(S'^{*}_{k,\tau(k)}(v))}_{(a)} + \underbrace{\overline{F}_{n,\tau(n)}(\beta(\cdot)|v)}_{(c)}} \tag{10}$$

Term (a) can be calculated directly given our parametric assumptions. The derivatives that appear in (b) are solved using numerical differentiation as we do not have analytical expressions for these terms. The integrals that appear in term (b) and (c) are approximated using the trapezoidal rule, although other methods, like Simpson's Rule, did not meaningfully change the results. All of terms (a), (b) and (c) are stored as arrays on the grid of values $[\underline{v},\overline{v}]$

and our solver uses MATLAB's `interp1` and `interp2` to read data from them and linearly interpolating functions across the grid. Using cubic interpolation does not materially affect our results.

We solve Equation (10) using MATLAB's `ode113` solver but alternative solvers, such as MATLAB's `ode45` and `ode23`, do not materially affect our results.[3] To give an example, Table 1 (below) displays summary statistics for the absolute differences in equilibrium bid functions when different differential equation solvers are used. The baseline bid function is based on `ode113` (the solver used in the paper). Each row of the table represents differences from this baseline when alternative differential equation solvers are used. These summary statistics pertain to the bid function for a potential entrant in the penultimate round when the current incumbent has a value of 90 and firms are symmetric with values distributed LN(4.5,0.2) and $K = 1$ and $\alpha = 0.5$.

	Absolute Difference in Solved Bid Function from `ode113`					
ODE Solver	Mean	Min	25th-tile	Median	75th-tile	Max
`ode23`	1.0698e-05	0	0	1.6532e-06	2.1766e-05	3.1375e-05
`ode45`	1.0731e-05	0	0	1.6879e-06	2.1872e-05	3.1420e-05

Table 1: Example of robustness of equilibrium bid function to different differential equation solvers. Details for the table's construction are found in the accompanying text.

We have also tested our bid functions using a "best-response-like" check. This involves numerically simulating the expected benefit to a bidder, say with value v_{true}, from deviating and pretending as if his value is v_{fake} by submitting a bid $b(v_{fake})$. This check is analogous to that used in Gayle and Richard (2008) to check numerical solutions to equilibrium bid functions in an asymmetric first price auction when there is no entry margin.

Take as an example the case of a potential entrant in the penultimate round who faces an incumbent with a value of 90 when firms are symmetric with values distributed $LN(4.5, 0.2)$ and $K = 1$ and $\alpha = 0.5$ (this is the same as in the example above). In this case we can compute the optimal best bid deviation as just described using 100,000 simulations and compare it to the bid function that we solved for. The average absolute difference in the two bid functions is 0.09. The 25th percentile of the absolute differences is 0, the 75th percentile is 0.07 and the maximum absolute difference is 1.09. Moreover, the change in expected profits from deviating from the equilibrium bid function for this potential entrant is a negligible 0.0014.

[3]The `ode113` is a variable order Adams-Bashforth-Moulton PECE solver. The `ode45` and `ode23` solvers are based on explicit Runge-Kutta methods using the Dormand-Prince and Bogacki-Shampine pairs, respectively. It has been shown that solvers such as MATLAB's `ode113` can be more efficient that basic Runge-Kutta methods when the function is expensive to compute (Shampine and Reichelt (1997)).

Finally, the entry threshold $S'^{*}_{n,\tau(n)}(v')$ for $n < N$ is set so that the expected profit from entering, conditional on the threshold, is zero. Using the notation from the paper, we have that $S'^{*}_{n,\tau(n)}(v')$ must satisfy

$$\int_{v'}^{\overline{v}} \left\{ [v - \beta(v,v',n)] \left[\overbrace{\prod_{k=n+1}^{N} F_{S,\tau(k)}(S'^{*}_{k,\tau(k)}(v'))}^{(a)} + \overbrace{\overline{F}_{n,\tau(n)}(\beta(v,v',n)|v')}^{(b)} \right] \right.$$
$$\left. + \int_{\beta(v,v',n)}^{v} (v - x) \underbrace{\overline{f}_{n,\tau(n)}(x|v)}_{(c)} \; dx \right\} f^{V}_{\tau(n)}(x|S'^{*}_{n,\tau(n)}(v')) \; dv = K. \quad (11)$$

As before, term (a) is easy to compute for a given distribution of signals. Term (b) is calculated via numeric integration via the trapezoidal rule, and term (c) is calculated via numeric differentiation of term (b). As in Equation (9), the left hand size is monotonic in $S'_{n,\tau(n)}(v')$, and the method of bisection can be used to determine a solution.

We also perform a check on these entry thresholds as well as the entry thresholds in the last round. We do this by numerically simulating the value of expected profits from entry at S'^{*}_{n}. We always find that the value of the simulated profits is very close to zero. For example, continuing with the example from above used to illustrate the bid check, the penultimate round potential entrant's equilibrium entry threshold is 71.344. The expected profit from entering with a signal equal to this threshold is 0.009.

At times in the paper (e.g. Figure 2, Figure 3 and Table 4 of the manuscript) we calculate optimal reserve prices for the sequential mechanism and the auction. We briefly describe how this is done in Footnote 22 of the manuscript. Here we give greater detail.

When bidders are asymmetric, or entry is endogenous and/or selective, expected revenues and optimal reserve prices must be calculated numerically. To calculate expected revenues given a particular reserve price in the simultaneous auction, we first solve the model and then calculate expected revenues using 5,000,000 sets of simulation draws of the values and signals of each potential entrant. Holding these simulation draws fixed, we can calculate expected revenues for different reserve prices, re-solving the game each time. With this number of simulation draws, expected revenues are essentially smooth in the reserve price and we are able to perform a one-dimensional maximization to find the optimal reserve price. However, we note that we find almost identical optimal reserves using a grid search.

For the sequential mechanism it is more expensive to solve the game, especially when the number of players is large. One reason for this is that the calculation of expected revenues in the sequential mechanism is based on interpolation using our solution to the differential equation, though we have checked that expected revenues are almost identical using 100,000 and 400,000 simulations. So we do not want to re-solve the game for many different reserves. Instead we exploit the fact that the expected revenue in an N player game with a reserve price of R is equal to the expected revenues from the last N players in an $N+1$ player game, where the first entrant enters and has a value of R. We therefore solve an $N+1$ player game once, which gives us later strategies for all possible values of the first round entrant. Then we simulate forward from the second round of this game to compute expected revenues. In this case we use 200,000 revenues and consider a grid (with unit spacing) of possible reserve prices. In this way we may slightly under predict expected revenues with an optimal reserve in the sequential mechanism.

References

Ackerberg, Daniel. 2009. "A New Use of Importance Sampling to Reduce Computational Burden in Simulation Estimation." *Quantitative Marketing and Economics*, 7(4): 343–376.

Bajari, Patrick, Han Hong, and Stephen Ryan. 2010. "Identification and Estimation of a Discrete Game of Complete Information." *Econometrica*, 78(5): 1529–1568.

Gayle, Wayne-Roy, and Jean-Francois Richard. 2008. "Numerical Solutions of Aysmmetric, First-Price, Independent Private Value Auctions." *Computational Economics*, 32: 245–278.

Haile, Philip A., and Elie Tamer. 2003. "Inference with an Incomplete Model of English Auctions." *Journal of Political Economy*, 111(1): 1–51.

Mailath, George J. 1987. "Incentive Compatibility in Signaling Games with a Continuum of Types." *Econometrica*, 55(6): 1349–1365.

Mailath, George J., and Ernst-Ludwig von Thadden. 2011. "Incentive Compatibility and Differentiability: New Results and Classic Applications." *Working Paper.*

Ramey, Garey. 1996. "D1 Signaling Equilibria with Multiple Signals and a Continuum of Types." *Journal of Economic Theory*, 69: 508–531.

Roberts, James W., and Andrew Sweeting. 2011. "Competition versus Auction Design." *Working Paper, Duke University.*

Roddie, Charles. 2011. "Theory of Signaling Games." *Working Paper.*

Shampine, Lawrence F., and Mark K. Reichelt. 1997. "The Matlab ODE Suite." *SIAM Journal on Scientific Computing*, 18(1): 1–22.