Online Appendices for Designing Random Allocation Mechanisms

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A Proofs of Theorems 1 and 11

In this appendix we provide a fuller self-contained proof of Theorems 1 and 11. Since Theorem 1 is a special case of Theorem 11, we prove the latter.

A matrix is **totally unimodular** if the determinant of every square submatrix is 0 or -1 or $+1$. We make use of the following result.

Lemma A.1. *(Hoffman and Kruskal (1956)) If a matrix* $\mathbf{A}$ *is totally unimodular, then the vertices of the polyhedron defined by linear integral constraints are integer valued.*

The proof strategy for Theorem 11 proceeds in two steps. First we show that if a constraint structure forms a bihierarchy, then the incidence matrix of the constraint structure is totally unimodular. Second we apply Lemma A.1 to show that the constraint structure is universally implementable.

After an earlier draft was circulated, we were informed that Edmonds (1970) has previously shown that the incidence matrix of a bihierarchical constraint structure is totally unimodular. We include our proof for completeness below. We utilize the following result for our proof.

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Lemma A.2. *(Ghouila-Houri (1962)) A $\{0,1\}$ incidence matrix is totally unimodular if and only if each subcollection of its columns can be partitioned into red and blue columns such that for every row of that collection, the sum of entries in the red columns differs by at most one from the sum of the entries in the blue columns.*

Proof of Theorem 11. Suppose first $\mathcal{H}$ forms a bihierarchy, with $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}$, $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$ and both $\mathcal{H}_1$ and $\mathcal{H}_2$ are hierarchies. Let $\mathbf{A}$ be the associated incidence matrix. Take any collection of columns of $\mathbf{A}$, corresponding to a subcollection E of $\mathcal{H}$. We shall partition E into two sets, B and R. First, for each $i = 1, 2$, we partition $E \cap \mathcal{H}_i$ into nonempty sets $E_i^1, E_i^2, \ldots, E_i^{k_i}$ defined recursively as follows: Set $E_i^0 \equiv \emptyset$ and, for each $j = 1, \ldots,$ we let

$$E_i^j := \{S \in (E \cap \mathcal{H}_i) \setminus (\bigcup_{j'=1}^{j-1} E_i^{j'}) \,|\, \nexists S' \in (E \cap \mathcal{H}_i) \setminus (\bigcup_{j'=1}^{j-1} E_i^{j'} \cup \{S\}) \text{ such that } S' \supset S\}.$$

(The non-emptiness requirment means that once all sets in $E \cap \mathcal{H}_i$ are accounted for, the recursive definition stops, which it does at a finite $j = k_i$.) Since $\mathcal{H}_i$ is a hierarchy, any two sets in E_i^j must be disjoint, for each $j = 1, \ldots, k_i$. Hence, any element of Ω can belong to at most one set in each E_i^j. Observe next for $j < l$, $\bigcup_{S \in E_i^l} S \subset \bigcup_{S \in E_i^j} S$. In other words, if an element of Ω belongs to a set in E_i^l, it must also belong to a set in E_i^j for each $j < l$.

We now define sets B and R that partition E:

$$B := \{S \in E | S \in E_i^j, i + j \text{ is an even number }\},$$

and

$$R := \{S \in E | S \in E_i^j, i + j \text{ is an odd number }\}.$$

We call the elements of B "blue" sets, and call the elements of R "red" sets.

Fix any $\omega \in \Omega$. If ω belongs to any set in $E \cap \mathcal{H}_1$, then it must belong to exactly one set $S_1^j \in E_1^j$, for each $j = 1, \ldots, l$ for some $l \leq k_1$. These sets alternate in colors in $j = 1, 2, \ldots,$ starting with blue: S_1^1 is blue, S_1^2 is red, S_1^3 is blue, and so forth. Hence, the number of blue sets in $E \cap \mathcal{H}_1$ containing ω either equals or exceeds by one the number of red sets in $E \cap \mathcal{H}_1$ containing ω. By the same reasoning, if ω belongs to any set in $E \cap \mathcal{H}_2$, then it must belong to one set $S_2^j \in E_2^j$, for each $j = 1, \ldots, m$ for some $m \leq k_2$. These sets alternate in colors in $j = 1, 2, \ldots,$ starting with red: S_2^1 is red, S_2^2 is blue, S_2^3 is red, and so forth. Hence, the number of blue sets in $E \cap \mathcal{H}_2$ containing ω is less by one than or equal to the number of red sets in $E \cap \mathcal{H}_2$ containing ω. In sum, the number of blue sets in E containing ω differs at most by one from the number of red sets in E containing ω. Thus $\mathbf{A}$ is totally unimodular by Lemma A.2.

Choose an arbitrary expected assignment $\mathbf{X}$ and consider the set

$$\{\mathbf{X}' | \lfloor x_S \rfloor \leq x'_S \leq \lceil x_S \rceil, \forall S \in \mathcal{H}\}. \tag{1}$$

By Lemma A.1, every vertex of the set (1) is integer valued. Since (1) is a convex polyhedron, any point of it (including $\mathbf{X}$) can be written as a convex combination of its vertices. Since we chose $\mathbf{X}$ arbitrarily, the constraint structure $\mathcal{H}$ is universally implementable. *Q.E.D.*

B Algorithm for Implementing Expected Assignments

This appendix provides a constructive algorithm for implementing expected assignments. The algorithm also serves as a constructive proof for Theorem 11 (and hence Theorem 1). For ease of understanding, we first illustrate the algorithm, using an example. We then formally define the algorithm.

Consider $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\mathcal{H} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, S_1, S_2\}$, where $S_1 := \{\omega_2, \omega_3\}$ and $S_2 := \{\omega_3, \omega_4\}$. Observe that $\mathcal{H}$ is a bihierarchy consisting of two hierarchies, $\mathcal{H}_1 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, S_1\}$ and $\mathcal{H}_2 = \{S_2\}$. Suppose we wish to implement an expected assignment $\mathbf{X}$ with $x_{\{\omega_1\}} = 0.3, x_{\{\omega_2\}} = 0.7, x_{\{\omega_3\}} = 0.3$ and $x_{\{\omega_4\}} = 0.7$. We represent the given expected assignment $\mathbf{X}$ as a network flow. The particular way in which the flow network is constructed is crucial for the algorithm, and we first illustrate the construction informally based on the example (depicted in Figure B.1).

Intuitively, we view the total assignment as flows that travel from source s to sink s' of a network (s and s' can be interpreted as corresponding to the entire set Ω). First, the flows travel through the sets in one hierarchy $\mathcal{H}_1$, arranged in "descending" order of set-inclusion; the flows move from bigger to smaller sets along the directed edges representing the set-inclusion tree, reaching at last the singleton sets. This accounts for the left side of the flow network in Figure B.1, where the numbers on the edges depict the flows. From then on, the flows travel through the sets in the other hierarchy $\mathcal{H}_2$ which is augmented, without loss, to include the singleton sets and the entire set Ω, with primes attached for notational clarity. These sets are now arranged in "ascending" order of set-inclusion; the flows travel from smaller to bigger sets along the directed edges representing the reverse set-inclusion tree, reaching at the end the total set s', or the sink.

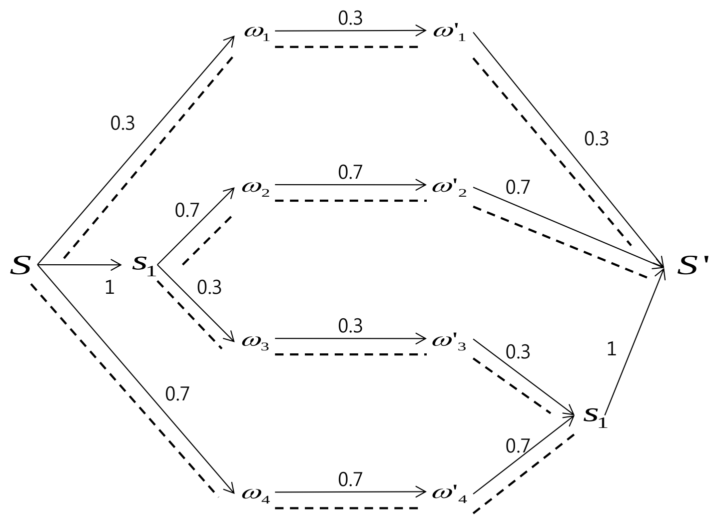

Figure B.1 – A network flow representation of the example **X**.

Notice that the flow associated with each edge reflects the expected assignment for the corresponding set. For instance, the flow from ω_2 to ω_2' is the expected assignment $x_{\{\omega_2\}} = 0.7$ for set ω_2, and likewise the flow from ω_3 to ω_3' is $x_{\{\omega_3\}} = 0.3$. The flow from s to S_1 represents the expected assignment $\omega_{S_1} = 1$ for set S_1. Naturally, the latter flow must be the sum of the two former flows. More generally, the additive structure of the expected assignment is translated into the "law of conservation": *the flow reaching each vertex except for s and s' must equal the flow leaving that vertex.*

Given the flow network, the algorithm identifies a cycle of agent-object pairs with fractional assignments. Starting with any edge with fractional flow, say (ω_2, ω_2'), we find another edge with a fractional flow that is adjacent to ω_2'. Such an edge, (ω_2', s'), exists due to the law of conservation: if all neighboring flows were integer we would have a contradiction. We keep adding new edges with fractional flows in this fashion, the ability to do so ensured by the law of conservation, until we create a cycle. In this case, the cycle of vertices is $\omega_2 - \omega_2' - s' - \omega_1' - \omega_1 - s - \omega_4 - \omega_4' - S_2 - \omega_3' - \omega_3 - S_1 - \omega_2$. This cycle is denoted by the dotted lines in Figure B.1.

We next modify the flows of the edges in the cycle. First, we raise the flow of each forward edge and reduce the flow of each backward edge at the same rate until at least one flow reaches an integer value. In our example, the flows along all the forward edges rise from 0.7 to 1 and the flows along all the backward edges fall from 0.3 to 0. Importantly, this process preserves the law of conservation, meaning that the operation maintains the feasibility of the new expected assignment. The resulting network flow then gives rise to an expected assignment $\mathbf{X}'$ where $x'_{\{\omega_1\}} = 0, x'_{\{\omega_2\}} = 1, x'_{\{\omega_3\}} = 0$, and $x'_{\{\omega_4\}} = 1$. Next, we readjust the flows of the edges in the cycle in the reverse direction, raising those with backward edges and reducing those with forward edges in an analogous manner, which gives rises to another expected assignment $\mathbf{X}''$ where $x''_{\{\omega_1\}} = 1, x''_{\{\omega_2\}} = 0, x''_{\{\omega_3\}} = 1$, and $x''_{\{\omega_4\}} = 0$. We can now decompose $\mathbf{X}$ into these two matrices, i.e., $\mathbf{X} = 0.7\mathbf{X}' + 0.3\mathbf{X}''$.

The random algorithm then selects $\mathbf{X}'$ with probability 0.7 and $\mathbf{X}''$ with probability 0.3. Since in this particular example both $\mathbf{X}'$ and $\mathbf{X}''$ are integer valued, there is no need to re-iterate the decomposition process. In general, each step in the algorithm reduces the number of fractional flows in the network, converting at least one to an integer. The total number of steps in the random algorithm is therefore limited to the number of fractional flows. Also, each step visits each remaining fractional flow at most once, so the total number of visits grows at most as the square of the number of fractional flows. Thus, the run time of the algorithm is polynomial in $|\mathcal{H}|$.

We now define the algorithm formally. Let $\mathcal{H}$ be a constraint structure associated with a set Ω and assume that $\mathcal{H}$ is a bihierarchy, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are hierarchies such that $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$. Let $\mathbf{X} = [x_\omega]$ be an expected assignment whose entries sum up to an integer (the generalization to the case with a fractional sum is straightforward). We construct a flow network as follows. The set of vertices is composed of the source s and the sink s', two vertices v_ω and $v_{\omega'}$ for each element $\omega \in \Omega$, and v_S for each $S \in \mathcal{H} \setminus [(\bigcup_{\omega\in\Omega}\{\omega\}) \cup (N \times O)]$. We place (directed) edges according to the following rule.[1]

1. For each $\omega \in \Omega$, an edge $e = (v_\omega, v_{\omega'})$ is placed from v_ω to $v_{\omega'}$.

2. An edge $e = (v_S, v_{S'})$ is placed from S to $S' \neq S$ where $S, S' \in \mathcal{H}_1$, if $S' \subset S$ and there is no $S'' \in \mathcal{H}_1$ where $S' \subset S'' \subset S$.[2]

3. An edge $e = (v_S, v_{S'})$ is placed from S to $S' \neq S$ where $S, S' \in \mathcal{H}_2$, if $S \subset S'$ and there is no $S'' \in \mathcal{H}_2$ where $S \subset S'' \subset S'$.

4. An edge $e = (s, v_S)$ is placed from the source s to v_S if $S \in \mathcal{H}_1$ and there is no $S' \in \mathcal{H}_1$ where $S \subset S'$.

[1] An edge is defined as an ordered pair of verticies. All edges in this paper are directed, so we omit the adjective "directed."

[2] For the purpose of placing edges, we regard v_ω as a vertex corresonding to a singleton set $\{\omega\} \in H_1$, and v'_ω as a vertex corresonding to a singleton set $\{\omega\} \in H_2$.

5. An edge $e = (v_S, s')$ is placed from v_S to the sink s' if $S \in \mathcal{H}_2$ and there is no $S' \in \mathcal{H}_2$ where $S \subset S'$.

We associate flow with each edge as follows. For each $e = (v_\omega, v_{\omega'})$, we associate flow $x_e = x_\omega$. For each e that is not of the form $(v_\omega, v_{\omega'})$ for some $\omega \in \Omega$, the flow x_e is (uniquely) set to satisfy the flow conservation, that is, for each vertex v different from s and s', the sum of flows into v is equal to the sum of flows from v. Observe that the construction of the network (specifically items (2)-(5) above) utilizes the fact that $\mathcal{H}$ is a bihierarchy.

We define the **degree of integrality** of $\mathbf{X}$ with respect to $\mathcal{H}$:

$$\deg[\mathbf{X}(\mathcal{H})] := \#\{S \in \mathcal{H} | x_S \in \mathbb{Z}\}.$$

Lemma B.1. *(Decomposition) Suppose a constraint structure $\mathcal{H}$ forms a bihierarchy. Then, for any $\mathbf{X}$ such that $\deg[\mathbf{X}(\mathcal{H})] < |\mathcal{H}|$, there exist $\mathbf{X}^1$ and $\mathbf{X}^2$ and $\gamma \in (0,1)$ such that*

(i) $\mathbf{X} = \gamma \mathbf{X}^1 + (1-\gamma)\mathbf{X}^2$*:*

(ii) $x_S^1, x_S^2 \in [\lfloor x_S \rfloor, \lceil x_S \rceil], \forall S \in \mathcal{H}$*.*

(iii) $\deg[\mathbf{X}^i(\mathcal{H})] > \deg[\mathbf{X}(\mathcal{H})]$ *for* $i = 1, 2$*.*

The following algorithm gives a constructive proof of Lemma B.1 and hence the Theorem. Let $\mathbf{X}$ be an expected assignment on a bihierarchy $\mathcal{H}$ with $\deg[\mathbf{X}(\mathcal{H})] < |\mathcal{H}|$.

□ **Decomposition Algorithm**

1. **Cycle-Finding Procedure**

   (a) **Step 0:** Since $\deg[\mathbf{X}(\mathcal{H})] < |\mathcal{H}|$ by assumption, there exists an edge $e_1 = (v_1, v_1')$ such that its associated flow x_{e_1} is fractional. Define an edge $f_1 = (v_1, v_1')$ from v_1 to v_1'.

   (b) **Step t=1,...:** Consider the vertex v_t' that is the destination of edge f_t.

      i. If v_t' is the origin of some edge $f_{t'} \in \{f_1, \dots, f_{t-1}\}$, then stop.[3] The procedure has formed a cycle $(f_{t'}, f_{t'+1}, \dots, f_t)$ composed of edges in $\{f_1, \dots, f_t\}$. Proceed to **Termination - Cycle**.

      ii. Otherwise, since the flow associated with f_t is fractional by construction and the flow conservation holds at v_t', there exists an edge $e_{t+1} = (u_{t+1}, u_{t+1}') \neq e_t$ with

[3] Since there are a finite number of vertices, this procedure terminates in a finite number of steps.

fractional flow such that v'_t is either its origin or destination. Draw an edge f_{t+1} by $f_{t+1} = e_{t+1}$ if v'_t is the origin of e_{t+1} and $f_{t+1} = (u'_{t+1}, u_{t+1})$ otherwise. Denote $f_{t+1} = (v_{t+1}, v'_{t+1})$.

2. **Termination - Cycle**

   (a) Construct a set of flows associated with edges (x^1_e) which is the same as (x_e), except for flows $(x_{e_\tau})_{t' \leq \tau \leq t}$, that is, flows associated with edges that are involved in the cycle from the last step. For each edge e_τ such that $f_\tau = e_\tau$, set $x^1_{e_\tau} = x_{e_\tau} + \alpha$, and each edge e_τ such that $f_\tau \neq e_\tau$, set $x^1_{e_\tau} = x_{e_\tau} - \alpha$, where $\alpha > 0$ is the largest number such that the induced expected assignment $\mathbf{X}^1 = (x^1_\omega)_{\omega \in \Omega}$ still satisfies all constraints in $\mathcal{H}$. By construction, $x^1_S = x_S$ if x_S is an integer, and there is at least one constraint set $S \in \mathcal{H}$ such that x^1_S is an integer while x_S is not. Thus $\deg[\mathbf{X}^1(\mathcal{H})] > \deg[\mathbf{X}(\mathcal{H})]$.

   (b) Construct a set of flows associated with edges (x^2_e) which is the same as (x_e), except for flows $(x_{e_\tau})_{t' \leq \tau \leq t}$, that is, flows associated with edges that are involved in the cycle from the last step. For each edge e_τ such that $f_\tau = e_\tau$, set $x^1_{e_\tau} = x_{e_\tau} - \beta$, and each edge e_τ such that $f_\tau \neq e_\tau$, set $x^1_{e_\tau} = x_{e_\tau} + \beta$, where $\beta > 0$ is the largest number such that the induced expected assignment $\mathbf{X}^2 = (x^2_\omega)_{\omega \in \Omega}$ still satisfies all constraints in $\mathcal{H}$. By construction, $x^2_S = x_S$ if x_S is an integer, and there is at least one constraint set $S \in \mathcal{H}$ such that x^2_S is an integer while x_S is not. Thus $\deg[\mathbf{X}^2(\mathcal{H})] > \deg[\mathbf{X}(\mathcal{H})]$.

   (c) Set γ by $\gamma\alpha + (1-\gamma)(-\beta) = 0$, i.e., $\gamma = \frac{\beta}{\alpha+\beta}$.

   (d) The decomposition of $\mathbf{X}$ into $\mathbf{X} = \gamma\mathbf{X}^1 + (1-\gamma)\mathbf{X}^2$ satisfies the requirements of the Lemma by construction.

C Proofs of Lemma 1 and Theorem 2

Since Theorem 2 uses the necessity result from Lemma 1, we first provide its proof. The proof will be given, however, in the general framework of Section VI that does not refer to the two-sided assignment structure.

Proof of Lemma 1. Suppose for contradiction that $\mathcal{H}$ is universally implementable and contains an odd cycle $S_1, \ldots, S_l$, with $\omega_i \in S_i \cap S_{i+1}$, $i = 1, \ldots, l-1$ and $\omega_l \in S_l \cap S_1$. Consider an expected assignment $\mathbf{X}$ specified by

$$x_\omega = \begin{cases} \frac{1}{2} & \text{if } \omega \in \{\omega_1, \ldots, \omega_l\}, \\ 0 & \text{otherwise,} \end{cases}$$

where x_ω is the entry corresponding to $\omega \in N \times O$. By definition of an odd cycle, $x_{S_i} = 1$ for all $i \in \{1, \ldots, k\}$. Since $\mathcal{H}$ is universally implementable, there exist $\mathbf{X}^1, \mathbf{X}^2, \ldots, \mathbf{X}^K$ and $\lambda^1, \lambda^2, \ldots, \lambda^K$ such that

1. $\mathbf{X} = \sum\limits_{k=1}^{K} \lambda^k \mathbf{X}^k$,
2. $\lambda^k \in (0, 1]$ for all k and $\sum_{k=1}^{K} \lambda^k = 1$,
3. $x_S^k \in \{\lfloor x_S \rfloor, \lceil x_S \rceil\}$ for all $k \in \{1, \ldots, K\}$ and $S \in \mathcal{H}$.

In particular, it follows that $x^k_{S_i} = 1$ for each i and k. Thus there exists k such that $x^k_{\omega_1} = 1$. Since $x^k_{S_2} = 1$, it follows that $x^k_{\omega_2} = 0$. The latter equality and the assumption that $x^k_{S_3} = 1$ imply $x^k_{\omega_3} = 1$. Arguing inductively, it follows that $x^k_{\omega_i} = 0$ if i is even and $x^k_{\omega_i} = 1$ if i is odd. In particular, we obtain $x^k_{\omega_l} = 1$ since l is odd by assumption. Thus $x^k_{S_l} = x^k_{\omega_l} + x^k_{\omega_1} = 2$, contradicting $x^k_{S_l} = 1$. *Q.E.D.*

Proof of Theorem 2. In order to prove the Theorem, we study several cases.

- Assume there is $S \in \mathcal{H}$ such that $S = N' \times O'$ where $2 \leq |N'| < |N|$ and $2 \leq |O'| < |O|$. Let $\{i, j\} \times \{a, b\} \subseteq S$, $k \notin N'$ and $c \notin O'$ (observe that such $i, j, k \in N$ and $a, b, c \in O$ exist by the assumption of this case). Then the sequence of constraint sets

  $$S_1 = S, S_2 = \{i\} \times O, S_3 = N \times \{c\}, S_4 = \{k\} \times O, S_5 = N \times \{b\},$$

  is an odd cycle together with

  $$\omega_1 = (i, a), \omega_2 = (i, c), \omega_3 = (k, c), \omega_4 = (k, b), \omega_5 = (j, b).$$

  Therefore, by Lemma 1, $\mathcal{H}$ is not universally implementable.

- Assume there is $S \in \mathcal{H}$ such that, for some $i, j \in N$ and $a, b \in O$, we have $(i, a), (j, b) \in S$ with $i \neq j$ and $a \neq b$, and $(i, b) \notin S$. Then the sequence of constraint sets

  $$S_1 = S, S_2 = \{i\} \times O, S_3 = N \times \{b\},$$

is an odd cycle together with

$$\omega_1 = (i, a), \omega_2 = (i, b), \omega_3 = (j, b).$$

Thus, by Lemma 1, $\mathcal{H}$ is not universally implementable.

By the above arguments, it suffices to consider cases where all constraint sets in $\mathcal{H}$ have one of the following forms.

1. $\{i\} \times O'$ where $i \in N$ and $O' \subseteq O$,
2. $N' \times O$ where $N' \subseteq N$,
3. $N' \times \{a\}$ where $a \in O$ and $N' \subseteq N$,
4. $N \times O'$ where $O' \subseteq O$.

Therefore it suffices to consider the following cases.

1. Assume that there are $S', S'' \in \mathcal{H}$ such that $S' = \{i\} \times O'$ and $S'' = \{i\} \times O''$ for some $i \in N$ and some $O', O'' \subset O$, $S' \cap S'' \neq \emptyset$ and S' is neither a subset nor a superset of S''. Then we can find $a, b, c \in O$ such that $a \in O' \setminus O''$, $b \in O' \cap O''$ and $c \in O'' \setminus O'$. Fix $j \neq i$, who exists by assumption $|N| \geq 2$. Then the sequence of constraint sets

   $$S_1 = S', S_2 = S'', S_3 = N \times \{c\}, S_4 = \{j\} \times O, S_5 = N \times \{a\},$$

   is an odd cycle together with

   $$\omega_1 = (i, a), \omega_2 = (i, b), \omega_3 = (i, c), \omega_4 = (j, c), \omega_5 = (j, a).$$

   Therefore, by Lemma 1, $\mathcal{H}$ is not universally implementable.

2. Assume that there are $S', S'' \in \mathcal{H}$ such that $S' = N' \times O$ and $S'' = N'' \times O$ for some $N', N'' \subset N$, $S' \cap S'' \neq \emptyset$ and S' is neither a subset nor a superset of S''. In such a case, we can find $i, j, k \in N$ such that $i \in N' \setminus N''$, $j \in N' \cap N''$ and $k \in N'' \setminus N'$. Fix $a, b \in O$. The sequence of constraint sets

   $$S_1 = S', S_2 = S'', S_3 = N \times \{b\},$$

is an odd cycle together with

$$\omega_1 = (j, a), \omega_2 = (k, b), \omega_3 = (i, b).$$

Hence, by Lemma 1, $\mathcal{H}$ is not universally implementable.

3. Assume that there are $S', S'' \in \mathcal{H}$ such that $S' = N' \times \{a\}$ and $S'' = N'' \times \{a\}$ for some $a \in O$ and some $N', N'' \subset N$, $S' \cap S'' \neq \emptyset$ and S' is neither a subset nor a superset of S''. This is a symmetric situation with Case 1, so an analogous argument as before goes through.

4. Assume that there are $S', S'' \in \mathcal{H}$ such that $S' = N \times O'$ and $S'' = N \times O''$ for some $O', O'' \subset O$, $S' \cap S'' \neq \emptyset$ and S' is neither a subset nor a superset of S''. This is a symmetric situation with Case 2, so an analogous argument as before goes through.

Q.E.D.

D Extension of The Generalized Probabilistic Serial Mechanism to The Full Preference Domain

This section provides an extension of our generalized probabilistic serial mechanism to the full preference domain. That is, we now allow for preferences of agents to be weak, so that different objects can be equally preferred by an agent.

Katta and Sethuraman (2006) generalize the probabilistic serial mechanism to the full preference domain for the constraint structure of Bogomolnaia and Moulin (2001) based on the tools of network flow. Fortunately their method can be incorporated into our generalization in a natural manner. In what follows, we use notation used by Katta and Sethuraman (2006) unless explicitly noted otherwise. Definitions of concepts in network flow can be found in their paper as well.

Let $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ be a bihierarchy, where $\mathcal{H}_1$ is composed of all rows and $\mathcal{H}_2$ is a hierarchy that includes all columns. The assignment is determined through an iterative algorithm. For each (i, a), let $H_{(i,a)} := \{S \in \mathcal{H}_2 | (i, a) \in S\}$ be the all subsets in $\mathcal{H}_2$ that contain (i, a). Then, for each $\mathcal{H}' \subset \mathcal{H}_2$, we define $A(i, \mathcal{H}') = \{(i, a) | H_{(i,a)} \subset \mathcal{H}'\}$ and $H(i, \mathcal{H}') = \{(i, a) \in A(i, \mathcal{H}') | a \succeq_i b, \forall (i, b) \in A(i, \mathcal{H}')\}$.

1. As initialization, let $\mathcal{H}' = \mathcal{H}_2$; $\mathbf{X}$ be the zero matrix, i.e., $x_{ia} = 0$ for all i, a; $c(i, 1) = 0$ for all $i \in N$, and $k = 1$.

2. Construct a flow network as follows, with $\lambda \geq 0$ a parameter. The set of vertices is composed of the source s and the sink s', a vertex v_i for each agent i, and a vertex v_S for each $S \in \mathcal{H}'$. We place (directed) edges according to the following rule.

   (a) An edge (s, v_i) is placed from the source s to v_i for each agent i, and each of these edges is endowed with capacity $c(i, k) + \lambda$.

   (b) An edge $(v_i, v_{(i,a)})$ is placed from v_i to $v_{(i,a)}$ if and only if $(i, a) \in H(i, \mathcal{H}')$, and each of these edges is endowed with capacity ∞.

   (c) An edge $(v_S, v_{S'})$ is placed from S to $S' \neq S$ where $S, S' \in \mathcal{H}'$, if $S \subset S'$ and there is no $S'' \in \mathcal{H}'$ where $S \subset S'' \subset S'$, and each of these edges is endowed with capacity $\overline{q}_S$.

   (d) An edge (v_S, s') is placed from v_S to the sink s' if $S \in \mathcal{H}'$ and there is no $S' \in \mathcal{H}'$ where $S \subset S'$, and each of these edges is endowed with capacity $\overline{q}_S$.

3. Solve the corresponding parametric max-flow problem. Let λ_k^* be the smallest break point and B be the bottleneck set of constraint sets.

4. For each agent i, if $H(i, \mathcal{H}') \subseteq B$, then update $c(i, k + 1) = 0$ and give her a total amount $c(i, k) + \lambda_k^*$ of shares from objects in $\{a \in O | (i, a) \in H(i, \mathcal{H}')\}$; otherwise update $c(i, k+1) = c(i, k) + \lambda_k^*$.

5. If $S \in \mathcal{H}' \setminus B$ and $S' \in B$ for $S' \subset S$ such that there is no $S'' \in \mathcal{H}'$ with $S' \subset S'' \subset S$, then subtract the ceiling quota of S' from the ceiling quota of S. Update $\mathcal{H}' = \mathcal{H}' \setminus B$ and $k = k + 1$. If $\mathcal{H}' \neq \emptyset$ then go to step 2. Otherwise, terminate the algorithm.

E Proof of Theorems 3, 4, and 5

E.1 Proof of Theorem 3

As with Bogomolnaia and Moulin (2001), a different characterization of ordinal efficiency proves useful. To this end, we first define the **minimal constraint set containing** (i, a):

$$\nu(i, a) := \bigcap_{S \in \mathcal{H}(i,a)} S,$$

if the set $\mathcal{H}(i,a) := \{S \in \mathcal{H}_2 : (i,a) \in S, \sum_{(j,b)\in S} x_{jb} = \overline{q}_S\}$ is nonempty. If $\mathcal{H}(i,a) = \emptyset$ (or equivalently $\sum_{(j,b)\in S} x_{jb} < \overline{q}_S$ for all $S \in \mathcal{H}_2$ containing (i,a)), then we let $\nu(i,a) = N \times O$.

We next define the following binary relations on $N \times O$ given $\mathbf{X}$ as follows:[4]

$$\begin{aligned} (j,b) \rhd_1 (i,a) &\iff i = j, b \succ_i a, \text{ and } x_{ia} > 0, \\ (j,b) \rhd_2 (i,a) &\iff \nu(j,b) \subseteq \nu(i,a). \end{aligned} \tag{2}$$

We then say

$$(j,b) \rhd (i,a) \iff (j,b) \rhd_1 (i,a) \text{ or } (j,b) \rhd_2 (i,a).$$

We say a binary relation $\rhd$ is **strongly cyclic** if there exists a finite cycle $(i_0,a_0) \rhd (i_1,a_1) \rhd \cdots \rhd (i_k,a_k) \rhd (i_0,a_0)$ such that $\rhd = \rhd_1$ for at least one relation. We next provide a characterization of ordinal efficiency.

Lemma E.1. *Expected assignment* $\mathbf{X}$ *is ordinally efficient if and only if* $\rhd$ *is not strongly cyclic given* $\mathbf{X}$.[5]

A remark is in order. In their environment, Bogomolnaia and Moulin (2001) define the binary relation $\rhd$ over the set of objects where $b \rhd a$ if there is an agent i such that $b \succ_i a$ and $x_{ia} > 0$. Bogomolnaia and Moulin show that in their environment a random assignment is ordinally efficient if and only if $\rhd$ is acyclic. Our contribution over their characterization is that we expand the domain over which the binary relation is defined to the set of agent-object pairs, in order to capture the complexity that results from a more general environment than that of BM.

Proof of Lemma E.1. **"Only if" part.** First note that the following property holds.

[4]Given that $\mathcal{H}_2$ has a hierarchical structure,

$$(j,b) \rhd_2 (i,a) \iff (j,b) \in S \text{ for any } S \in \mathcal{H}_2 \text{ such that } (i,a) \in S, x_S = \overline{q}_S.$$

[5]In Kojima and Manea (2010), ordinal efficiency is characterized by two conditions, acyclicity and non-wastefulness. We do not need non-wastefulness as a separate axiom in our current formulation since a "wasteful" random assignment (in their sense) contains a strong cycle as defined here.

Claim 1. *$\rhd_1$ and $\rhd_2$ are transitive, that is,*

$$(k,c) \rhd_1 (j,b), (j,b) \rhd_1 (i,a) \Rightarrow (k,c) \rhd_1 (i,a),$$
$$(k,c) \rhd_2 (j,b), (j,b) \rhd_2 (i,a) \Rightarrow (k,c) \rhd_2 (i,a).$$

Proof. Suppose $(k,c) \rhd_1 (j,b)$ and $(j,b) \rhd_1 (i,a)$. Then, by definition of $\rhd_1$, we have $i = j = k$ and (i) $c \succ_i b$ since $(k,c) \rhd_1 (i,b)$ and (ii) $b \succ_i a$ since $(j,b) \rhd_1 (i,a)$. Thus $c \succ_i a$. Since $(j,b) \rhd_1 (i,a)$, we have $x_{ia} > 0$. Therefore $(k,c) \rhd_1 (i,a)$ by definition of $\rhd_1$.

Suppose $(k,c) \rhd_2 (j,b)$ and $(j,b) \rhd_2 (i,a)$. Then $\nu(k,c) \subseteq \nu(j,b)$ and $\nu(j,b) \subseteq \nu(i,a)$ by property (2). Hence $\nu(k,c) \subseteq \nu(i,a)$ which is equivalent to $(k,c) \rhd_2 (i,a)$, completing the proof by property (2). *Q.E.D.*

To show the "only if" part of the Theorem, suppose $\rhd$ is strongly cyclic. By Claim 1, there exists a cycle of the form

$$(i_0,b_0) \rhd_1 (i_0,a_0) \rhd_2 (i_1,b_1) \rhd_1 (i_1,a_1) \rhd_2 (i_2,b_2) \rhd_1 (i_2,a_2) \rhd_2 \cdots \rhd_1 (i_k,a_k) \rhd_2 (i_0,b_0),$$

in which every pair (i,a) in the cycle appears exactly once except for (i_0,b_0) which appears exactly twice, namely in the beginning and in the end of the cycle. Then there exists $\delta > 0$ such that a matrix $\mathbf{Y}$ defined by

$$y_{ia} = \begin{cases} x_{ia} + \delta & \text{if} \quad (i,a) \in \{(i_0,b_0),(i_1,b_1),\ldots,(i_k,b_k)\}, \\ x_{ia} - \delta & \text{if} \quad (i,a) \in \{(i_0,a_0),(i_1,a_1),\ldots,(i_k,a_k)\}, \\ x_{ia} & \text{otherwise,} \end{cases}$$

satisfies quotas. Since $\delta > 0$ and $b_l \succ_{i_l} a_l$ for every $l \in \{0,1,\ldots,k\}$, $\mathbf{Y}$ ordinally dominates $\mathbf{X}$. Therefore $\mathbf{X}$ is not ordinally efficient.

"If" part. Suppose $\mathbf{X}$ is ordinally inefficient. Then, there exists an expected assignment $\mathbf{Y}$ which ordinally dominates $\mathbf{X}$. We then prove that $\rhd$, given $\mathbf{X}$, must be strongly cyclic.

1. **Step 1: Initiate a cycle.**

   (a)

   Claim 2. *There exist $(i_0,a_0),(i_1,a_1) \in N \times O$ such that $i_0 = i_1$, $x_{i_1a_1} < y_{i_1a_1}$ and $(i_1,a_1) \rhd_1 (i_0,a_0)$ given $\mathbf{X}$.*

Proof. Since **Y** ordinally dominates **X**, there exists $(i_1, a_1) \in N \times O$ such that $y_{i_1 a_1} > x_{i_1 a_1}$ and $y_{i_1 a} = x_{i_1 a}$ for all $a \succ_{i_1} a_1$. So there exists $a_0 \prec_{i_1} a_1$ with $x_{i_1 a_0} > y_{i_1 a_0} \geq 0$ since $x_{\{i_1\} \times N} = y_{\{i_1\} \times N}$ by assumption. Hence, we have $(i_1, a_1) \rhd_1 (i_1, a_0) = (i_0, a_0)$ given **X**. *Q.E.D.*

(b) If $(i_0, a_0) \in \nu(i_1, a_1)$, then $(i_0, a_0) \rhd_2 (i_1, a_1) \rhd_1 (i_0, a_0)$, so we have a strong cycle.

(c) Else, circle (i_1, a_1) and go to Step 2.

2. **Step $t + 1$ ($t \in \{1, 2 \ldots\}$): Consider the following cases.**

(a) Suppose (i_t, a_t) is circled.

i.

Claim 3. *There exists $(i_{t+1}, a_{t+1}) \in \nu(i_t, a_t)$ such that $x_{i_{t+1} a_{t+1}} > y_{i_{t+1} a_{t+1}}$. Hence, $(i_{t+1}, a_{t+1}) \rhd_2 \nu(i_t, a_t)$.*

Proof. Note that $\nu(i_t, a_t) \subsetneq N \times O$ since if $\nu(i_t, a_t) = N \times O$, then there exists $(i_{t'}, a_{t'})$ with $t' < t$ and $(i_{t'}, a_{t'}) \in \nu(i_t, a_t)$, so we have terminated the algorithm. Thus we have $\sum_{(i,a) \in \nu(i_t, a_t)} x_{ia} = \bar{q}_{\nu(i_t, a_t)}$. Since $x_{i_t a_t} < y_{i_t a_t}$, there exists $(i_{t+1}, a_{t+1}) \in \nu(i_t, a_t)$ such that $x_{i_{t+1} a_{t+1}} > y_{i_{t+1} a_{t+1}}$. *Q.E.D.*

ii. If $(i_{t'}, a_{t'}) \in \nu(i_{t+1}, a_{t+1})$ for $t' < t$, then we have a strong cycle, $(i_{t'}, a_{t'}) \rhd (i_{t+1}, a_{t+1}) \rhd \ldots \rhd (i_{t'}, a_{t'})$, and at least one $\rhd$ is $\rhd_1$.

iii. Else, square (i_{t+1}, a_{t+1}) and move to the next step.

(b) Case 2: Suppose (i_t, a_t) is squared.

i.

Claim 4. *There exists $(i_{t+1}, a_{t+1}) \in \nu(i_t, a_t)$ such that $i_{t+1} = i_t$, $x_{i_{t+1} a_{t+1}} < y_{i_{t+1} a_{t+1}}$, and $(i_{t+1}, a_{t+1}) \rhd_1 \nu(i_t, a_t)$.*

Proof. Since (i_t, a_t) is squared, by Claim 3, $x_{i_t a_t} > y_{i_t a_t}$. Since **Y** ordinally dominates **X**, there must be $(i_{t+1}, a_{t+1}) \in \nu(i_t, a_t)$ with $i_{t+1} = i_t$ such that $x_{i_{t+1} a_{t+1}} < y_{i_{t+1} a_{t+1}}$, and $a_{t+1} \succ_{i_t} a_t$. Since $x_{i_t a_t} > y_{i_t a_t} \geq 0$, we thus have $(i_{t+1}, a_{t+1}) \rhd_1 \nu(i_t, a_t)$. *Q.E.D.*

ii. If $(i_{t'}, a_{t'}) \in \nu(i_{t+1}, a_{t+1})$ for $t' \leq t$, then we have a strong cycle, $(i_{t'}, a_{t'}) \rhd (i_{t+1}, a_{t+1}) \rhd \ldots \rhd (i_{t'}, a_{t'})$, and at least one $\rhd$ is $\rhd_1$.

iii. Else, circle (i_{t+1}, a_{t+1}) and move to the next step.

The process must end in finite steps and, at the end we must have a strong cycle. *Q.E.D.*

Given the above lemma, we are ready to proceed to the proof of Theorem 3.

Proof of Theorem 3. We prove the claim by contradiction. Suppose that $\mathbf{PS}(\succ)$ is ordinally inefficient for some $\succ$. Then, by Lemma E.1 and Claim 1 there exists a strong cycle

$$(i_0, b_0) \rhd_1 (i_0, a_0) \rhd_2 (i_1, b_1) \rhd_1 (i_1, a_1) \rhd_2 (i_2, b_2) \rhd_1 (i_2, a_2) \rhd_2 \cdots \rhd_1 (i_k, a_k) \rhd_2 (i_0, b_0),$$

in which every pair (i, a) appears exactly once except for (i_0, b_0) which appears exactly twice, namely in the beginning and the end of the cycle. Let v^l and w^l be the steps of the symmetric simultaneous eating algorithm at which (i_l, a_l) and (i_l, b_l) become unavailable, respectively (that is, $(i_l, a_l) \in S^{v_l - 1} \setminus S^{v_l}$ and $(i_l, a_l) \in S^{w_l - 1} \setminus S^{w_l}$.) Since $(i_l, b_l) \rhd_1 (i_l, a_l)$, by the definition of the algorithm we have $w^l < v^l$ for each $l \in \{0, 1, \ldots, k\}$. Also, by $(i_l, a_l) \rhd_2 (i_{l+1}, b_{l+1})$, we have $v^l \leq w^{l+1}$ for any $l = \{0, 1, \ldots, k\}$ (with notational convention $(i_{k+1}, a_{k+1}) = (i_0, a_0)$.) Combining these inequalities we obtain $w^0 < v^0 \leq w^1 < v^1 \leq \cdots \leq w^k < v^k \leq w^{k+1} = w^0$, a contradiction. *Q.E.D.*

E.2 Proof of Theorem 4: Constrained Envy-Freeness of PS

E.2.1 Notation

An *eating function* e describes an eating schedule for each agent, $e_i : [0, 1] \to O$ for all $i \in N$; $e_i(t)$ represents the object that agent i is eating at time t. We require that e_i be right-continuous with respect to the discrete topology on O (the topology in which all subsets are open), that is,

$$\forall t \in [0, 1), \exists \varepsilon > 0 \text{ such that } e_i(t') = e_i(t),\ \forall t' \in [t, t + \varepsilon).$$

For an eating function e and constraint set S, let $n_S(t, e)$ be the number of agent-object pairs $(i, a) \in S$ such that $e_i(t) = a$ and $\rho_S(t, e)$ be the share of cumulative consumption from S by time t, i.e.,[6]

$$\begin{aligned} n_S(t, e) &= |\{(i, a) \in S | e_i(t) = a\}|, \\ \rho_S(t, e) &= \int_0^t n_S(s, e) ds. \end{aligned}$$

Note that $\rho_S(\cdot, e)$ is continuous.

[6] It can be shown that $n_S(\cdot, e)$ is Riemann integrable.

For every preference profile $\succ$, let $e^{\succ}$ denote the eating function generated by the eating algorithm when agents report $\succ$. Formally, $e_i^{\succ}(t) = a$ for $t \in [t^{v-1}, t^v)$ if $\chi(i, a, S^{v-1}) = 1$, for (S^v) and (t^v) constructed in the definition of the probabilistic serial mechanism.

E.2.2 Proof

Suppose that there exists no set $S \in \mathcal{H}_2$ that is binding in $\mathbf{PS}(\succ)$ (in the sense that $\mathbf{PS}_S(\succ) = \overline{q}_S$) such that $(i, a) \in S$ but $(j, a) \notin S$, for some $a \in O$. Then for any t and any set $S \in \mathcal{H}_2$ such that $(i, a) \in S$ and $\rho_S(t, e^{\succ}) = \overline{q}_S$, $(j, a) \in S$ as well. By definition of the simultaneous eating algorithm, this implies that, at each time t, if (i, a) has expired by time t, then (j, a) has expired by time t as well. Hence the set of objects available for eating for i at t is a superset of that for j. Therefore $e_i^{\succ}(t) \succeq_i e_j^{\succ}(t)$ for every time $t \in [0, 1]$. Thus $\mathbf{PS}_i(\succ)$ stochastically dominates $\mathbf{PS}_j(\succ)$ at preference $\succ_i$, completing the proof.

E.3 Proof of Theorem 5: Weak Strategy-Proofness of PS

The main part of the proof is to establish Lemma E.5 below. Informally, this lemma establishes that an agent's gain from misreporting her preference is limited in an appropriate sense. More specifically, the lemma establishes that, when an agent misreports her preference and consequently eats something other than her most preferred available object for the duration of δ, she can delay the cumulative consumption of her most preferred available object at most by δ. Given this lemma, the rest of the proof is relatively easy. The proof establishes that the delay of the expiration time she can accomplish by preference misreporting is insufficient for her to increase the share of her preferred objects in total.

To establish the main lemma, Lemma E.5, we proceed by showing a series of auxiliary lemmas. To do so, we continue to use notation defined in Section E.2.1 and introduce some new ones as follows. Fix a preference profile $\succ$, and denote by $\succ' = (\succ_i', \succ_{N\setminus\{i\}})$ the preference profile where agent i reports $\succ_i'$ instead of $\succ_i$. Let $\overline{e}$ be the eating function such that

$$\overline{e}_i(t) = \begin{cases} e_i^{\succ}(t) & \text{if } e_i^{\succ}(t) = e_i^{\succ'}(t) \\ \emptyset & \text{otherwise} \end{cases},$$

and at each instance, under $\overline{e}_j$ agent $j \neq i$ is eating from his most preferred object at speed 1 among the ones still available (accounting for agent i's specified eating function $\overline{e}_i$). Note that $\overline{e}_j$ may diverge from $e_j^{\succ}$ or $e_j^{\succ'}$ for $j \neq i$ since the available objects at each time may vary across

$\overline{e}$, $e^{\succ}$ and $e^{\succ'}$ due to the different eating behavior adopted by i.

Let $\delta(t)$ denote the sums of the lengths of time intervals, before time t, on which agent i's consumption in the eating algorithm is different when the reported preferences change from $\succ$ to $\succ'$. Formally,

$$\delta(t) \;=\; \int_0^t \mathbf{1}_{e_i^{\succ'}(s) \neq e_i^{\succ}(s)} ds,$$

where for any logical proposition p, $\mathbf{1}_p = 1$ if p is true and $\mathbf{1}_p = 0$ if p is false.

Equipped with the notation introduced so far, we are now ready to state the first of the lemmas that we will use for the proof.

Lemma E.2. *For all $t \in [0,1]$ and $(j,a) \in N \times O$ such that the condition*

$$a \neq \emptyset, \text{ and } \rho_S(t, e^{\succ}) < q_S \text{ for all } S \in \mathcal{H} \text{ such that } (j,a) \in S, \tag{3}$$

is satisfied, we have

$$\begin{aligned} \rho_{(j,a)}(t, e^{\succ}) \;&\geq\; \rho_{(j,a)}(t, \overline{e}) \\ \rho_{(j,a)}(t, e^{\succ'}) \;&\geq\; \rho_{(j,a)}(t, \overline{e}), \end{aligned}$$

where $\rho_{(j,a)}(\cdot,\cdot)$ is a shorthand notation for $\rho_{\{(j,a)\}}(\cdot,\cdot)$.

Proof. By symmetry, we only need to prove the first inequality. We proceed by contradiction. Assume that there exist t and (j,a) such that $\rho_{(j,a)}(t, e^{\succ}) < \rho_{(j,a)}(t, \overline{e})$ and condition (3) is satisfied. Let

$$t_0 = \inf\{t \in [0,1] | \exists (j,a) \in N \times O, \rho_{(j,a)}(t, e^{\succ}) < \rho_{(j,a)}(t, \overline{e}), \text{ condition (3) holds}\}. \tag{4}$$

By continuity of $\rho_{(j,a)}(\cdot, e^{\succ}) - \rho_{(j,a)}(\cdot, \overline{e})$, it follows that $t_0 < 1$, and

$$\rho_{(j,a)}(t_0, e^{\succ}) - \rho_{(j,a)}(t_0, \overline{e}) \geq 0, \; \forall (j,a) \in N \times O \text{ that satisfies condition (3)}. \tag{5}$$

This holds trivially if $t_0 = 0$.

One consequence of (5) is that any agent-object pair (i,a) that has not expired by time t_0 under $e^{\succ}$ cannot expire by t_0 under $\overline{e}$ either (that is, if no constraint for (i,a) has been reached by time t_0 under $e^{\succ}$, then no constraint for (i,a) has been reached by t_0 under $\overline{e}$ either). Hence the set of agent-object pairs available for eating at t_0 under $e^{\succ}$ is a subset of that under $\overline{e}$. It must be that if agent $j \in N$ is eating object $a \in O \setminus \{\o\}$ at t_0 under $\bar{e}$ and (j,a) is available at t_0 under $e^{\succ}$, then j is eating a at t_0 under $e^{\succ}$. Formally,

$$\forall j \in N,\ \bar{e}_j(t_0) = a \text{ \& condition (3) } \Rightarrow e^{\succ}_j(t_0) = a.$$

For $j = i$ the latter step follows from the definition of $\bar{e}$. Therefore,

$$\forall j \in N,\ \bar{e}_j(t_0) = a \text{ \& condition (3) } \Rightarrow n_{(j,a)}(t_0, e^{\succ}) \geq n_{(j,a)}(t_0, \bar{e}). \tag{6}$$

Given the right-continuity of $e^{\succ}$ and $\bar{e}$, for sufficiently small $\varepsilon > 0$, we have that for all $t \in [t_0, t_0 + \varepsilon)$ and (j,a) such that condition (3) is satisfied,

$$\begin{aligned} \rho_{(j,a)}(t, e^{\succ}) &= \rho_{\{(j,a)\}}(t_0, e^{\succ}) + n_{\{(j,a)\}}(t_0, e^{\succ})(t - t_0) \\ \rho_{(j,a)}(t, \bar{e}) &= \rho_{\{(j,a)\}}(t_0, \bar{e}) + n_{\{(j,a)\}}(t_0, \bar{e})(t - t_0). \end{aligned}$$

Using (5) and (6) we obtain $\rho_{(j,a)}(t, e^{\succ}) \geq \rho_{(j,a)}(t, \bar{e})$ for all $t \in [t_0, t_0 + \varepsilon)$.

By (4), $\rho_{(j,a)}(t, e^{\succ}) \geq \rho_{(j,a)}(t, \bar{e})$ for all $t \in [0, t_0)$ and (j,a) such that condition (3) is satisfied. The arguments above establish that $\rho_{(j,a)}(t, e^{\succ}) \geq \rho_{(j,a)}(t, \bar{e})$ for all $t \in [0, t_0 + \varepsilon)$ and (j,a) such that condition (3) is satisfied, which contradicts the definition of t_0. *Q.E.D.*

Lemma E.3. *For all $t \in [0,1]$,*

$$\rho_{\o}(t, e^{\succ}) - \rho_{\o}(t, \bar{e}) \geq -\delta(t),$$

where $\rho_{\o}(\cdot,\cdot)$ is a shorthand notation for $\rho_{N \times \{\o\}}(\cdot,\cdot)$

Proof. Note that

$$\rho_{\o}(t, e^{\succ}) - \rho_{\o}(t, \bar{e}) + \delta(t) = \int_0^t [n_{\o}(s, e^{\succ}) - n_{\o}(s, \bar{e}) + \mathbf{1}_{e^{\succ}_i(s) \neq e^{\succ'}_i(s)}] ds.$$

Since $\rho_{(j,a)}(t, e^{\succ}) \geq \rho_{(j,a)}(t, \bar{e})$ for all (j,a) such that condition (3) is satisfied and $t \in [0,1]$ by Lemma E.2, an argument similar to Lemma E.2 leads to

$$\begin{aligned} e^{\succ}_i(s) \neq \bar{e}_i(s) &\Rightarrow n_{\o}(s, e^{\succ}) \geq n_{\o}(s, \bar{e}) - 1 \\ e^{\succ}_i(s) = \bar{e}_i(s) &\Rightarrow n_{\o}(s, e^{\succ}) \geq n_{\o}(s, \bar{e}). \end{aligned}$$

Thus the integrand $n_{\o}(s, e^{\succ}) - n_{\o}(s, \bar{e}) + \mathbf{1}_{e^{\succ}_i(s) \neq e^{\succ'}_i(s)}$ is non-negative for all $s \in [0,t]$, which completes the proof. *Q.E.D.*

Let us introduce a condition on $S \subseteq N \times O$ and $e^{\succ}$,

$$\rho_{S'}(t, e^{\succ}) < q_{S'} \text{ for all } S' \text{ such that } S \subsetneq S'. \tag{7}$$

Lemma E.4. *For all $t \in [0, 1]$, $S \in \mathcal{H}$ such that $S \cap (N \times \{\emptyset\}) = \emptyset$ and $e^{\succ}$ for which condition (7) is satisfied,*

$$\rho_S(t, e^{\succ}) - \rho_S(t, \overline{e}) \leq \delta(t).$$

Proof. For each $(j, a) \in S' := (N \times O) \setminus (S \cup (N \times \{\emptyset\}))$, consider the maximal set S'', if any, such that $(j, a) \in S''$ and $\rho_{S''}(t, e^{\succ}) = q_{S''}$ is satisfied. By condition (7), $S \cap S'' = \emptyset$. Let $S_1, \ldots, S_k$ be all such sets, and define $S''' := S' \setminus (S_1 \cup \cdots \cup S_k)$. Note that for all $t \in [0, 1]$, we have the identity

$$\begin{aligned} \rho_S(t, e^{\succ}) - \rho_S(t, \overline{e}) + \sum_{l=1}^{k} [\rho_{S_l}(t, e^{\succ}) - \rho_{S_l}(t, \overline{e})] \\ + \sum_{(j,a) \in S'''} [\rho_{(j,a)}(t, e^{\succ}) - \rho_{(j,a)}(t, \overline{e})] + \rho_{\emptyset}(t, e^{\succ}) - \rho_{\emptyset}(t, \overline{e}) = 0. \end{aligned} \tag{8}$$

By definition of $S_1, \ldots, S_k$, the second term of the left-hand side, $\sum_{l=1}^{k} [\rho_{S_l}(t, e^{\succ}) - \rho_{S_l}(t, \overline{e})] = \sum_{l=1}^{k} [q_{S_l} - \rho_{S_l}(t, \overline{e})]$, is no less than zero. Next, note that $(j, a) \in S'''$ and $e^{\succ}$ satisfy condition (3) by the construction of S''' and the assumption that $(S, e^{\succ'})$ satisfies condition (7). Therefore, by Lemma E.2 the third term $\sum_{(j,a) \in S'''} [\rho_{(j,a)}(t, e^{\succ}) - \rho_{(j,a)}(t, \overline{e})]$ is no less than zero. Finally, by Lemma E.3, the last term $\rho_{\emptyset}(t, e^{\succ}) - \rho_{\emptyset}(t, \overline{e})$ is no less than $-\delta(t)$. Therefore, the left-hand side of the equality (8) is no less than

$$\rho_S(t, e^{\succ}) - \rho_S(t, \overline{e}) + 0 + 0 - \delta(t),$$

thus obtaining $\rho_S(t, e^{\succ}) - \rho_S(t, \overline{e}) - \delta(t) \leq 0$. Rearranging terms, we obtain the desired inequality. *Q.E.D.*

Lemma E.5. *For all $t \in [0, 1]$, $S \in \mathcal{H}$ such that $S \cap (N \times \{\emptyset\}) = \emptyset$, and $e^{\succ}$ and $e^{\succ'}$ such that both $(S, e^{\succ})$ and $(S, e^{\succ'})$ satisfy condition (7),*

$$\rho_S(t, e^{\succ}) - \rho_S(t, e^{\succ'}) \leq \delta(t).$$

Proof. For any $(j,a) \in S$, consider the maximal set S'', if any, such that $(j,a) \in S''$ and $\rho_{S''}(t, e^{\succ'}) = q_{S''}$ is satisfied. By condition (7) with respect to $\succ'$, $S'' \subseteq S$. Let $S_1, \ldots, S_k$ be such sets, and define $S''' := S \setminus (S_1 \cup \cdots \cup S_k)$. Note that for all $t \in [0,1]$, we have the following identity:

$$\begin{aligned}\rho_S(t, e^{\succ}) - \rho_S(t, e^{\succ'}) &= [\rho_S(t, e^{\succ}) - \rho_S(t, \bar{e})] - [\rho_S(t, e^{\succ'}) - \rho_S(t, \bar{e})] \\ &= [\rho_S(t, e^{\succ}) - \rho_S(t, \bar{e})] - \sum_{l=1}^{k} [\rho_{S_l}(t, e^{\succ'}) - \rho_{S_l}(t, \bar{e})] \\ &\quad - \sum_{(j,a) \in S'''} [\rho_{(j,a)}(t, e^{\succ'}) - \rho_{(j,a)}(t, \bar{e})].\end{aligned}$$

Since $(S, e^{\succ})$ satisfies condition (7), from Lemma E.4 the first term of the last expression $[\rho_S(t, e^{\succ}) - \rho_S(t, \bar{e})]$ is no larger than $\delta(t)$. By definition of $S_1, \ldots, S_k$, the second term $-\sum_{l=1}^{k} [\rho_{S_l}(t, e^{\succ'}) - \rho_{S_l}(t, \bar{e})] = -\sum_{l=1}^{k} [q_{S_l} - \rho_{S_l}(t, \bar{e})]$ is no larger than zero. Next, note that $(j,a) \in S'''$ and $e^{\succ'}$ satisfy condition (3) by the construction of S''' and the assumption that $(S, e^{\succ'})$ satisfies condition (7). Therefore, by Lemma E.2, the last term $-\sum_{(j,a) \in S'''} [\rho_{(j,a)}(t, e^{\succ'}) - \rho_{(j,a)}(t, \bar{e})]$ is no larger than zero. Therefore we obtain the desired inequality. *Q.E.D.*

Now we are ready to prove the weak strategy-proofness of the generalized PS mechanism. Write $\mathbf{X} = [x_{ia}]_{i,a} = \mathbf{PS}(\succ)$ and $\mathbf{X}' = [x'_{ia}]_{i,a} = \mathbf{PS}(\succ')$. Assume $x_{ia} \le x'_{ia}$, where $a \succ_i b$ for every $b \in O \setminus \{a\}$. Let t_{ia} be the time at which (i,a) becomes unavailable in the eating algorithm under $\succ$, and t'_{ia} be the time at which (i,a) becomes unavailable in the eating algorithm under $\succ'$. Since $x_{ia} = t_{ia}$ and $x'_{ia} \le t'_{ia}$, we obtain $t_{ia} \le t'_{ia}$. If $t_{ia} = 1$, then $x'_{ia} = 1$ by $x'_{ia} \ge x_{ia} = t_{ia} = 1$ and hence $\mathbf{x}_i = \mathbf{x}'_i$, so there is nothing to prove. Thus assume $t_{ia} < 1$. This implies that there exists $S \in \mathcal{H}$ such that $(i,a) \in S$ and $\rho_S(t, e^{\succ}) = q_S$. Let S be a maximal such set. Note that condition (7) is satisfied by S and $e^{\succ}$ by maximality of S.

If condition (7) is not satisfied by S and $e^{\succ'}$, then i cannot eat a any time after t_{ia}. Thus $x'_{ia} \le t_{ia}$ and, combined with previous relations $t_{ia} = x_{ia} \le x'_{ia}$, we conclude $t_{ia} = t'_{ia}$ and $x_{ia} = x'_{ia}$. Hence assume that condition (7) is satisfied by S and $e^{\succ'}$. Then, by Lemma E.5,

$$\begin{aligned}\rho_S(t_{ia}, e^{\succ'}) &\ge \rho_S(t_{ia}, e^{\succ}) - \delta(t_{ia}) \\ &= q_S - \delta(t_{ia}).\end{aligned}$$

Suppose that $\delta(t_{ia}) \neq 0$. Then, since $t_{ia} < 1$, there exists an agent-object pair $(j,b) \in S$ such that $e_j^{\succ'}(t_{ia}) = b$. By definition of the eating algorithm, this implies that $x'_{ia} < t_{ia} - \delta(t_{ia}) + \delta(t_{ia}) = t_{ia} = x_{ia}$, a contradiction to the assumption that $x'_{ia} \ge x_{ia}$. This implies that $\delta(t_{ia}) = 0$, which in tern implies that $t'_{ia} = t_{ia} = x_{ia} = x'_{ia}$.

Thus we have shown that $x_{ia} = x'_{ia}$ and $t_{ia} = t'_{ia}$. An inductive argument shows that $\mathbf{x}_i = \mathbf{x}'_i$, completing the proof.

F Proof of Theorem 6

First, define a price space $\mathcal{P} = [0, |N|B]^{|O|}$. Let $\mathcal{X}_i := \{\mathbf{x}_i = [x_{ia}] \in \mathbb{R}^{|O|} | 0 \leq \sum_{(i,a)\in S_i} x_{ia} \leq \overline{q}_{S_i}, \forall S_i \in \mathcal{H}_i\}$ be the set of fractional consumptions satisfying the quotas for agent i. We then define for each agent i his demand correspondence $d_i^*(\cdot)$ in the usual manner:

$$d_i^*(\mathbf{p}) := \arg\max_{\mathbf{x}_i \in \mathcal{X}_i} \left\{ u_i(\mathbf{x}_i) \text{ subject to} \sum_{a\in O} p_a^* x_{ia} \leq B \right\}.$$

That is, for each $\mathbf{p} \in \mathcal{P}$, $d_i^*(\mathbf{p})$ is the set of fractional consumption bundles that maximize the utility of agent i subject to the constraint that the total expenditure is at most B under prices $\mathbf{p}$.

Claim 1: *For each i, his demand correspondence $d_i^*(\mathbf{p})$ is nonempty- and convex-valued for all $\mathbf{p} \in \mathbb{R}_+^{|O|}$, and upper hemicontinuous in $\mathbf{p}$.*

Proof. The demand correspondence is nonempty since the feasible set of the linear program is nonempty (since zero demand is feasible for any price vector, as all floor constraints are assumed to be zero) and compact (since $0 \leq x_{ia} \leq \overline{q}_{ia}$), and since the objective function is continuous. The convexity of $d_i^*(\mathbf{p})$ is shown as follows. Suppose $\mathbf{x}, \mathbf{x}' \in d_i^*(\mathbf{p})$, and fix any $s \in (0, 1)$. Since the feasible set of the linear program is convex and since its objective function is linear, it follows that $s\mathbf{x} + (1 - s)\mathbf{x}'$ is in $d_i^*(\mathbf{p})$ as well. The upper-hemicontinuity of $d_i^*(\cdot)$ follows from Berge's theorem of the maximum. To see this point, first note that the objective function, the agent's utility function, is continuous on the domain of the function. Second, the correspondence from price vector $\mathbf{p}$ to budget set $\{\mathbf{x}_i \in \mathcal{X}_i | \sum_{a\in O} p_a x_{ia} \leq B\}$ is clearly continuous (i.e., both upper-hemicontinuous and lower-hemicontinuous). Thus Berge's theorem implies that the set of the maximizers of the objective function, which is exactly the demand correspondence $d_i^*(\cdot)$, is upper-hemicontinuous. *Q.E.D.*

Define the excess demand correspondence $z(\cdot)$ by $z(\mathbf{p}) = \sum_i d_i^*(\mathbf{p}) - \mathbf{q}$ for each $\mathbf{p} \in \mathcal{P}$. Note that this correspondence is also upper hemicontinuous and convex-valued because it is a linear sum of upper hemicontinuous and convex-valued correspondences. Introduce the following objects:[7]

1. Let $\overline{q} = \max\{\max_{a\in O} q_a, \max_{i\in N, a\in O} q_{\{(i,a)\}}\}$.
2. Define an auxiliary enlargement of the price space, $\tilde{\mathcal{P}} = [-\overline{q}, |N|B + |N|\overline{q}]^{|O|}$.
3. Define a truncation function $t : \tilde{\mathcal{P}} \to \mathcal{P}$ by $t(\mathbf{p}) = (\max\{0, \min\{p_a, |N|B\}\})_{a\in O}$.

Let a correspondence $f : \tilde{\mathcal{P}} \to \tilde{\mathcal{P}}$ be defined by $f(\mathbf{p}) = t(\mathbf{p}) + z(t(\mathbf{p}))$. We will show that we can apply Kakutani's fixed point theorem. To do so, first note that $z(t(\mathbf{p}))$ is upper hemicontinuous and convex-valued on $\tilde{\mathcal{P}}$ because $t(\cdot)$ is a continuous function and $z(\cdot)$ is an upper hemicontinuous and convex-valued correspondence. This implies that $f(\mathbf{p})$ is upper hemicontinuous and convex-valued as well. Second, note that the range of $t(\mathbf{p}) + z(t(\mathbf{p}))$ lies in $\tilde{\mathcal{P}}$ as required because, for any $\mathbf{p} \in \tilde{\mathcal{P}}$ and $a \in O$, the excess demand $z_a(\mathbf{p})$ is at least $-\overline{q}$ (because the supply of object a is $q_a \leq \overline{q}$) and at most $|N|\overline{q}$ (because the demand of object a by any agent i is at most $q_{\{(i,a)\}} \leq \overline{q}$). Thus $f(\mathbf{p})$ is an upper hemicontinuous and convex-valued correspondence defined on the compact and convex set $\tilde{\mathcal{P}}$. Thus by Kakutani's fixed point theorem, there exists a fixed point $\mathbf{p}^* \in f(\mathbf{p}^*)$.

To complete the proof, we will show that any fixed point $\mathbf{p}^*$ of $f(\cdot)$ corresponds to a competitive equilibrium; specifically, $t(\mathbf{p}^*)$ is a competitive equilibrium price vector. To show this claim, suppose that $\mathbf{p}^*$ is a fixed point of $f(\cdot)$. By the definition of a fixed point and correspondence $f(\cdot)$, this means that there exists $\mathbf{z}^* = (z^*_a)_{a\in O} \in z(t(\mathbf{p}^*))$ such that $\mathbf{p}^* = t(\mathbf{p}^*) + \mathbf{z}^*$, or equivalently $p^*_a = t_a(\mathbf{p}^*) + z^*_a$ for all $a \in O$. First suppose that $p^*_a \in [0, |N|B]$. The truncation does not bite for such an object a, that is, $t_a(\mathbf{p}^*) = p^*_a$. Then $p^*_a = t_a(\mathbf{p}^*) + z^*_a$ implies $z^*_a = 0$ (i.e., the demand and supply for object a exactly clear at $t(\mathbf{p}^*)$). Second, suppose that $p^*_a < 0$. Then $t_a(\mathbf{p}^*) = 0$ and hence $p^*_a = t_a(\mathbf{p}^*) + z^*_a$ implies $z^*_a = p^*_a < 0$.[8] Lastly, suppose that $p^*_a > |N|B$. Then $t_a(\mathbf{p}^*) = |N|B$ and hence $p^*_a = t_a(\mathbf{p}^*) + z^*_a$ implies that $z^*_a = p^*_a - |N|B > 0$ (i.e., object a is in excess demand at $t(\mathbf{p}^*)$). But this is impossible because $t_a(\mathbf{p}^*) = |N|B$, so even if all agents spend their entire budget on object a at price vector $t(\mathbf{p}^*)$, total demand is less than or equal to one (which is weakly less than supply by assumption). These arguments complete the proof.

[7]These objects prove useful in handling some boundary issues that arise because we allow objects to be in excess supply at price zero (and preferences are satiable, so prices of zero may actually arise).

[8]This means that object a is in excess supply at $t(\mathbf{p}^*)$. Note that this excess supply does not cause a problem because $t_a(\mathbf{p}^*) = 0$, which is allowed by the "complementary slackness" condition in the definition of the mechanism.

Remark 1. In their model, HZ assume positive floor constraints, specifically each agent needs to consume *exactly* one object. This constraint makes their proof quite involved, and they overcome the associated difficulties through a number of techniques as illustrated below.

1. The first problem is that the set of feasible consumption bundles could be empty: If the budget is too small relative to the prices, then it is simply infeasible to buy an expected consumption bundle that satisfies the unit-consumption constraint. To address this problem, HZ restrict attention to price vectors such that the price of at least one object is zero, so that some consumption bundle is always feasible.

2. However, the set of price vectors such that the price of at least one object is zero obviously violates convexity, which is needed to apply Kakutani's fixed point theorem.[9] In order to cope with this problem, HZ consider the space

$$S = \{s \in \mathbb{R}^{|O|} | \sum_{a \in O} s_a = 0\},$$

   and a function f such that $f_a(s) = s_a - \min\{s_b | b \in O\}$ for each $a \in O$. The set S is convex, and given any $s \in S$, $f(s)$ provides a desired price vector. Based on this idea, HZ proceed to find a fixed point in S.

Unfortunately, their method of proof cannot be generalized to multi-unit demand cases with floor constraints. To see this point, assume that there are two agents 1 and 2 and three objects a, b, c. Assume that 1 and 2 can consume multiple objects, but need to consume exactly one object from $\{a, b\}$ and $\{b, c\}$, respectively. In order to guarantee feasibility, the price space should be such that at least one price from $\{p_a, p_b\}$ is zero and at least one price from $\{p_b, p_c\}$ is zero. Then the counterpart of the set S above should have the additional property that at least one component from $\{s_a, s_b\}$ and at least one component from $\{s_b, s_c\}$ attain the minimum of all the coordinates of $s = (s_a, s_b, s_c)$. Consider two vectors $s = (1/2, -1/4, -1/4)$ and $s' = (-1/4, 1/2, -1/4)$. Both vectors satisfy the restrictions, but $\frac{1}{2}s + \frac{1}{2}s' = (1/8, 1/8, -1/4)$ does not, thus violating the desired convexity of the set.

Remark 2. In our analysis, it is assumed that each agent has a unique bliss point. To see why this assumption is needed, let there be two agents, i and j, and three objects, a, b, and c. Each object has one unit of supply. Agent i has single-unit demand while j has demand for at most two units, and utilities for individual objects are given by $v_{ia} = v_{ib} = v_{ic} = 1$, $v_{ja} = 2$, and

[9] For instance, price vectors $(1, 0)$ and $(0, 1)$ both satisfy the requirement, but $\frac{1}{2}(1, 0) + \frac{1}{2}(0, 1) = (\frac{1}{2}, \frac{1}{2})$ does not.

$v_{jb} = v_{jc} = 1$. In this problem, the price vector $\mathbf{p}^* = (p_a, p_b, p_c) = (2, 1, 1)$ and an allocation where i gets a and j gets b and c is a competitive equilibrium under the generalized pseudo-market mechanism (associated with budget $B = 2$). However, this allocation is not ex-ante Pareto efficient, because it is a Pareto improvement for i to get b and j to get a and c. Notice that the non-unique bliss points here arise from preference indifferences that are not generic (for instance in the sense that the set of vNM values giving rise to the required indifference has zero Lebesque measure).

As illustrated by the above example, if an agent's bliss point is not unique, our mechanism may not produce an ex-ante Pareto efficient allocation. This problem can be solved by modifying the mechanism to select an expenditure-minimizing bundle whenever multiple bliss points are feasible for an agent. Specifically, for a given vector (v_{ia}) of vNM values reported by agent i and a price vector $\mathbf{p}$, let the mechanism find a consumption bundle that maximizes $u_i(\mathbf{x}_i) - \epsilon[B - \sum_a p_a x_{ia}]$ subject to the budget constraint, for a small $\epsilon > 0$, inducing demand correspondence $d_i^*(\mathbf{p}, \epsilon)$. A competitive equilibrium $(\mathbf{p}_\epsilon^*, \mathbf{X}_\epsilon^*)$ exists in this modified economy for any small $\epsilon > 0$ by an argument analogous to the proof of Theorem 9. Consider an arbitrary sequence $(\mathbf{p}_{\epsilon_n}^*, \mathbf{X}_{\epsilon_n}^*)_{n=1}^\infty$ where $\epsilon_n > 0$ for all n and $\lim_{n\to\infty} \epsilon_n = 0$. Because both the set of feasible expected consumption bundles and the set of price vectors $\mathcal{P}$ are closed and bounded subsets of Euclidean spaces, there exists a subsequence $(\mathbf{p}_{\epsilon_{n_k}}^*, \mathbf{X}_{\epsilon_{n_k}}^*)_{k=1}^\infty$ of $(\mathbf{p}_{\epsilon_n}^*, \mathbf{X}_{\epsilon_n}^*)_{n=1}^\infty$ that converges to a pair $(\mathbf{p}^*, \mathbf{X}^*)$, where $\mathbf{p}^*$ is a price vector and $\mathbf{X}^*$ is a feasible expected consumption bundle $\mathbf{X}^*$. We define $\mathbf{X}^*$ to be the output of our modified mechanism. Note that $(\mathbf{p}^*, \mathbf{X}^*)$ is a competitive equilibrium. To see this, note first that for each i the demand correspondence $d_i^*(\mathbf{p}, \epsilon)$ is upper hemicontinuous, so $\mathbf{x}_i^* \in d_i^*(\mathbf{p}^*, 0)$. Also note that $\sum_{i\in N} x_{ia}^* \le q_a$ for all $a \in O$, with a strict inequality only if $p_a^* = 0$ since the corresponding inequalities hold for each equilibrium associated with ϵ_{n_k}.

Now we shall show that $\mathbf{X}^*$ is Pareto efficient. For contradiction, suppose that there exists $\tilde{\mathbf{X}}$ that Pareto improves upon $\mathbf{X}^*$. If $u_i(\tilde{\mathbf{x}}_i) > u_i(\mathbf{x}_i^*)$, then revealed preference implies $\mathbf{p}^* \cdot \tilde{\mathbf{x}}_i > \mathbf{p}^* \cdot \mathbf{x}_i^*$. Suppose $u_i(\tilde{\mathbf{x}}_i) = u_i(\mathbf{x}_i^*)$. If $\mathbf{x}_i^*$ is not a bliss point, then by an argument similar to the one in Theorem 7, we obtain $\mathbf{p}^* \cdot \tilde{\mathbf{x}}_i \ge \mathbf{p}^* \cdot \mathbf{x}_i^*$. Thus suppose that $\mathbf{x}_i^*$ is a bliss point. We claim $\mathbf{p}^* \cdot \tilde{\mathbf{x}}_i \ge \mathbf{p}^* \cdot \mathbf{x}_i^*$ in this case as well. To show this suppose for contradiction that $\mathbf{p}^* \cdot \tilde{\mathbf{x}}_i < \mathbf{p}^* \cdot \mathbf{x}_i^*$. Then, since $(\mathbf{p}^*, \mathbf{X}^*)$ is a limit of $(\mathbf{p}_{\epsilon_{n_k}}^*, \mathbf{X}_{\epsilon_{n_k}}^*)_{k=1}^\infty$, for any sufficiently large k we obtain $\mathbf{p}_{\epsilon_{n_k}}^* \cdot \tilde{\mathbf{x}}_i < \mathbf{p}_{\epsilon_{n_k}}^* \cdot \mathbf{x}_i^*$, a contradiction to the definition of $d_i^*(\mathbf{p}_{\epsilon_{n_k}}^*, \epsilon_{n_k})$. From the above and the assumption that $\tilde{\mathbf{X}}$ is a Pareto improvement on $\mathbf{X}^*$, we have established that $\mathbf{p}^* \cdot \tilde{\mathbf{x}}_i \ge \mathbf{p}^* \cdot \mathbf{x}_i^*$ for all i with at least one strict. Therefore, an argument analogous to the last part of the proof of Theorem 9 leads to a contradiction, showing that $\mathbf{X}^*$ is Pareto efficient.

G Proof of Theorem 9

Proof. For each $i \in N$, let $(a_i^1, a_i^2, \ldots, a_i^{|O|})$ be a sequence of objects in decreasing order of i's preferences so that $v_{ia_i^1} \geq v_{ia_i^2} \geq \ldots, v_{ia_i^{|O|}}$. Define the class of sets $\mathcal{H}' = \mathcal{H}'_1 \cup \mathcal{H}'_2$ by

$$\mathcal{H}'_1 = \mathcal{H}_1 \cup \left(\bigcup_{\substack{i \in N, \\ k \in \{1,\ldots,|O|\}}} \{i\} \times \{a_i^1, \ldots, a_i^k\} \right),$$
$$\mathcal{H}'_2 = \mathcal{H}_2.$$

By inspection, $\mathcal{H}'$ is a bihierarchy. Therefore, by Theorem 1, there exists a convex decomposition such that

$$\sum_{(i,a)\in S} x'_{ia}, \sum_{(i,a)\in S} x''_{ia} \in \left\{ \left\lfloor \sum_{(i,a)\in S} x_{ia} \right\rfloor, \left\lceil \sum_{(i,a)\in S} x_{ia} \right\rceil \right\} \quad \text{for all } S \in \mathcal{H}', \tag{9}$$

for any integer-valued matrices $\mathbf{X}'$ and $\mathbf{X}''$ that are part of the decomposition. In particular, property (9) holds for each $\{(i,a)\} \in \mathcal{H}'_1$ and $\{i\} \times \{a_i^1, \ldots, a_i^k\} \in \mathcal{H}'_1$. This means that

- **Observation 1:** For any i and k, $x'_{ia_i^k} - x''_{ia_i^k} \in \{-1, 0, 1\}$. This follows from the fact that $|x'_{ia_i^k} - x''_{ia_i^k}| \leq \lceil x_{ia_i^k} \rceil - \lfloor x_{ia_i^k} \rfloor \leq 1$ and that $x'_{ia_i^k}$ and $x''_{ia_i^k}$ are integer valued.

- **Observation 2:** By the same logic as for Observation 1, it follows that $\sum_{j=1}^{k}(x'_{ia_i^j} - x''_{ia_i^j}) \in \{-1, 0, 1\}$ for any i and k.

- **Observation 3:** Let $(a_i^{k_l})_{l=1}^{\bar{l}}$ be the (largest) subsequence of $(a_i^1, \ldots, a_i^k)$ such that $x'_{ia_i^{k_l}} \neq x''_{ia_i^{k_l}}$ for all l. Then, (i) $x_{ia_i^{k_l}} \notin \mathbb{Z}$ for all l, and (ii) $x'_{ia_i^{k_{2l'}}} - x''_{ia_i^{k_{2l'}}} = -(x'_{ia_i^{k_{2l'-1}}} - x''_{ia_i^{k_{2l'-1}}})$ for any $l' = 1, \ldots, \bar{l}/2$.

  Observation 3 (ii) can be shown as follows. First, the result must hold for $l' = 1$, or else $\sum_{j=1}^{k_2}(x'_{ia_i^j} - x''_{ia_i^j}) = x'_{ia_i^{k_1}} - x''_{ia_i^{k_1}} + x'_{ia_i^{k_2}} - x''_{ia_i^{k_2}} \in \{-2, 2\}$, which violates Observation 2. Now, working inductively, suppose the statement holds for all $l' = 1, \ldots, m-1$ for $m \leq \bar{l}/2$.

Then the statement must hold for $l' = m$, or else

$$\begin{aligned}
&\sum_{j=1}^{k_{2m}}(x'_{ia_i^j} - x''_{ia_i^j}) \\
= &\sum_{l'=1}^{m-1}\left(x'_{ia_i^{k_{2l'-1}}} - x''_{ia_i^{k_{2l'-1}}} + x'_{ia_i^{k_{2l'}}} - x''_{ia_i^{k_{2l'}}}\right) + x'_{ia_i^{k_{2m-1}}} - x''_{ia_i^{k_{2m-1}}} + x'_{ia_i^{k_{2m}}} - x''_{ia_i^{k_{2m}}} \\
= &\; x'_{ia_i^{k_{2m-1}}} - x''_{ia_i^{k_{2m-1}}} + x'_{ia_i^{k_{2m}}} - x''_{ia_i^{k_{2m}}}
\end{aligned}$$

must be either -2 or 2, which again violates Observation 2.

These observations imply that

$$\begin{aligned}
\sum_{a\in O} x'_{ia}v_{ia} - \sum_{a\in O} x''_{ia}v_{ia} &= \sum_{k=1}^{|O|}(x'_{ia_i^k} - x''_{ia_i^k})v_{ia_i^k} \\
&= \sum_{l=1}^{\bar{l}}(x'_{ia_i^{k_l}} - x''_{ia_i^{k_l}})v_{ia_i^{k_l}} \\
&\leq \sum_{l'=1}^{\bar{l}/2} v_{ia_i^{k_{2l'-1}}} - v_{ia_i^{k_{2l'}}} \\
&\leq v_{ia_i^{k_1}} - v_{ia_i^{k_{\bar{l}}}} \\
&\leq \Delta_i,
\end{aligned}$$

where the first inequality follows from $v_{ia_i^k} \geq v_{ia_i^{k'}}$ for $k < k'$ and Observations 1 and 3-(ii), the second inequality follows from $v_{ia_i^k} \geq v_{ia_i^{k'}}$ for $k < k'$, and the last inequality follows from the definition of Δ_i and Observation 3-(i). Therefore, we obtain property (1) of the theorem. Property (2) of the theorem follows immediately from property (1) of the theorem. *Q.E.D.*

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