Intermediary Asset Pricing: Online Appendix

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I. Verification of optimality

In this section we take the equilibrium Price/Dividend ratio \( F(y) \) as given, and verify that the specialist’s consumption policy \( c = D_t (1 + l - y_t) \) is optimal subject to his budget constraint. Our argument is a variant of the standard one: it uses the strict concavity of \( u(\cdot) \) and the specialist’s budget constraint to show that the specialist’s Euler equation is necessary and sufficient for the optimality of his consumption plan.

Specifically, fixing \( t = 0 \) and the starting state \((y_0, D_0)\), define the pricing kernel as

\[
\xi_t \equiv e^{-\rho t} c_t^{-\gamma} = e^{-\rho t} D_t^{-\gamma} (1 + l - \rho y_t)^{-\gamma}.
\]

Consider another consumption profile \( \tilde{c} \) which satisfies the budget constraint \( E \left[ \int_0^\infty \tilde{c}_t \xi_t dt \right] \leq \xi_0 D_0 (F_0 - y_0) \) (recall that the specialist’s wealth is \( D_0 (F_0 - y_0) \); here we require that the specialist’s feasible trading strategies be well-behaved, e.g., his wealth process remains non-negative). Then we have

\[
E \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right] \geq E \left[ \int_0^\infty e^{-\rho t} u(\tilde{c}_t) dt \right] + E \left[ \int_0^\infty e^{-\rho t} u'(c_t) (c_t - \tilde{c}_t) dt \right]
\]

This implies

\[
E \left[ \int_0^\infty \xi_t c_t dt \right] = \xi_0 D_0 (F_0 - y_0),
\]

then the result follows. Somewhat surprisingly, for our model this seemingly obvious claim requires an involved argument because of the singularity at \( y^b = \frac{1 + l}{\rho} \).

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One can easily check that, for $\forall T > 0$, we have

\[ \xi_0 D_0 (F_0 - y_0) = \int_0^T c_t \xi_t dt + \int_0^T \sigma (D_t, y_t) dZ_t + \xi_T D_T (F_T - y_T), \]

where $\sigma (D_t, y_t)$ corresponds to the specialist’s equilibrium trading strategy (which involves terms such as $(1 + l - \rho y)^{-\gamma - 1}$ and is NOT uniformly bounded as $y \to y^b$). Our goal in the following steps is to show that in expectation, the latter two terms vanishes when $T \to \infty$.

**Step 1: Limiting Behavior of $y$ at $y^b$**

The critical observation regarding the evolution of $y$ is that when $y$ approaches $y^b$, it approximately follows a Bessel process with a dimension $\delta = \gamma + 2 > 2$. (Given a $\delta$-dimensional Brownian motion $Z$, a Bessel process with a dimension $\delta$ is the evolution of $\|Z\| = \sqrt{\sum_{i=1}^\delta Z_i^2}$, which is the Euclidean distance between $Z$ and the origin.) According to standard results on Bessel processes, $y^b$ is an entrance-no-exit point, and is not reachable if the starting value $y_0 < y^b$ (if $\delta > 2$). Intuitively, when $y$ is close to $y^b$, the dominating part of $\mu_y$ is proportional to $\frac{1}{y - y^b} < 0$, while the volatility $\sigma_y$ is bounded—therefore a drift that diverges to negative infinity keeps $y$ away from the singular point $y^b$. This result implies that our economy never hits $y^b$.

To show that for $y$ close to $y^b$, $y$’s evolution can be approximated by a Bessel Process, one can easily check that when $y \to y^b$,

\[ r \simeq -\frac{(\gamma + 1) \sigma^2}{2} \frac{\rho^b \hat{\theta} b G}{1 + l - \rho^b y}, \mu_y \simeq \frac{(\gamma + 1) \sigma^2}{2} \frac{\rho^b \hat{\theta}^2 b^2 G^2}{1 + l - \rho^b y}, \sigma_y = -G \sigma \hat{\theta} b; \]

and therefore

\[ dy = -\frac{(\gamma + 1) \sigma^2}{2} \frac{\rho^b \hat{\theta}^2 b^2 G^2}{1 + l - \rho^b y} dt - G \sigma \hat{\theta} b dZ_t. \]

Utilizing the result $F'(y^b) = 1$ established in Section ??, we know that when $y \to y^b$, $\hat{\theta}_b \simeq F - \theta_s y \simeq \frac{1}{1 + m} y^b = \frac{1 + l}{1 + m} \frac{1}{\rho}$, and $G \simeq 1 + m$. Let

\[ z_t = 1 + l - \rho y_t; \]

then it is easy to show that $q = \frac{s}{G \sigma \hat{\theta}_b \rho} = \frac{s}{\sigma (1 + l)}$ evolves approximately according to

\[ dq_t = -\frac{1}{G \sigma \hat{\theta}_b} dy_t = \frac{(\gamma + 1)}{2q_t} dt + dZ_t, \]

which is a standard Bessel process with a dimension $\delta = \gamma + 2$. Therefore, $z$ is also a scaled version of a Bessel process, and can never reach 0 (or, $y$ cannot reach $y^b$). In the following analysis, we focus on the limiting behavior of $z$. 
Step 2: Localization

Note that in (1), due to the singularity at \( y = y^b \), both the local martingale part \( \int_0^T \sigma(D_t, y_t) dZ_t \) and the terminal wealth part \( \xi_T D_T (F_T - y_T) \) are not well-behaved. To show our claim, we have to localize our economy, i.e., stop the economy once either \( y \) is sufficiently close to \( y^b \) or \( D \) is sufficiently close to 0. Specifically, we define

\[
T_n = \inf \left\{ t : \text{either } z_t = n^{-1} \text{ or } D_t = n^{-h} \right\}
\]

where \( h \) is a positive constant (as we will see, the choice of \( h \), which is around 1, gives some flexibility for \( \gamma \) other than 2). Here, because \( y \) and \( z \) have a one-to-one relation \( (z = 1 + l - \rho y) \), for simplicity we localize \( z \) instead.

Clearly this localization technique ensures that the local martingale part \( \int_0^{T_n} \sigma(D_t, y_t) dZ_t \) is a martingale (one can check that \( \sigma(D_t, y_t) \) is continuous in \( D_t \) and \( y_t \); therefore \( \sigma(D_t, y_t) \) is bounded). As \( T_n \to \infty \) when \( n \to \infty \), for our claim we need to show

\[
\lim_{n \to \infty} E \left[ \xi_{T_n} D_{T_n} (F_{T_n} - y_{T_n}) \right] = 0
\]

We substitute from the definition of \( \xi \):

\[
E \left[ e^{-\rho T_n} D_{T_n}^{1-\gamma} z_{T_n}^{-\gamma} (F (y_{T_n}) - y_{T_n}) \right] \leq E \left[ e^{-\rho T_n} n^{h(\gamma-1)} z_{T_n}^{-\gamma} (F (y_{T_n}) - y_{T_n}) \right].
\]

Since the analysis will be obvious if \( z^{-\gamma} (F (y) - y) \) is uniformly bounded (notice here \( z = 1 + l - \rho y \), it is sufficient to consider \( z_{T_n} = \frac{1}{n} \) (or, \( y_{T_n} = y^b - \frac{1}{np} \)). Because \( F(y^b) = y^b \) and \( F'(y^b) = 1 \), by Taylor expansion we know that \( F(y - \frac{1}{np}) - (y - \frac{1}{np}) \) can be written as \( \psi(n) \frac{1}{n} \) when \( n \) is sufficiently large, and \( \psi(n) \to 0 \) as \( n \to \infty \). Therefore we have to show that, as \( n \to \infty \),

\[
E \left[ e^{-\rho T_n} n^{(\gamma-1)(1+h)} \right] \psi(n) \to 0
\]

and a sufficient condition is that there exists some constant \( M \) so that

\[
E \left[ e^{-\rho T_n} \right] n^{(\gamma-1)(1+h)} \to M.
\]

We apply existing analytical results in the literature to show our claim. To do so, we have to separate our two state variables. We define

\[
T_n^D = \inf \left\{ t : D_t = n^{-h} \right\}, \quad T_n^{x} = \inf \left\{ t : z_t = n^{-1} \right\}.
\]
We want to bound \( E[e^{-\rho T_n}] \) by the sum of \( E[e^{-\rho T^D_n}] \) and \( E[e^{-\rho T^z_n}] \). The Laplace transform of \( T_n \) is simply

\[
E[e^{-\rho T_n}] = \int_0^\infty e^{-\rho T} dF(T) = \rho \int_0^\infty e^{-\rho T} F(T) \, dT,
\]

where the bold \( F \) denotes the distribution function of \( T_n \). The similar relation also holds for \( T^D_n \) or \( T^z_n \). Denote \( F^D(\cdot) \) (or \( F^z(\cdot) \)) as the distribution function for \( T^D_n \) (or \( T^z_n \)), and notice that

\[
1 - F(T) = \Pr(T_n > T) = \Pr(T^D_n > T, T^z_n > T) > \Pr(T^D_n > T) \Pr(T^z_n > T)
\]

where \( \{ T^D_n > T \} \) and \( \{ T^z_n > T \} \) are positively correlated (both take the value 1 when the Brownian \( Z \) is high). Therefore \( F(T) < F^D(T) + F^z(T) \), or

\[
E[e^{-\rho T_n}] n^{(\gamma-1)(1+h)} < E[e^{-\rho T^D_n}] n^{(\gamma-1)(1+h)} + E[e^{-\rho T^z_n}] n^{(\gamma-1)(1+h)}.
\]

Our goal is to show the right hand side of the above inequality goes to zero when \( n \to \infty \).

There are two terms in the right hand side of the above inequality. For the first term, we can use the standard result of the Laplace transform of the first-hitting time distribution for a GBM process (e.g., Borodin and Salminen (2002), page 622):

\[
E[e^{-\rho T^D_n}] = n^{-\frac{h}{\sigma^2}} \left( \sqrt{2\rho \sigma^2 + (g - 0.5\sigma^2)^2} + g - 0.5\sigma^2 \right).
\]

Thus, by choosing some appropriate \( h \) so that

\[
\frac{h}{\sigma^2} \left( \sqrt{2\rho \sigma^2 + (g - 0.5\sigma^2)^2} + g - 0.5\sigma^2 \right) > (\gamma - 1)(1 + h),
\]

the first term \( E[e^{-\rho T^D_n}] n^{(\gamma-1)(1+h)} \) vanishes as \( n \to \infty \). For instance, this condition holds when \( h = 0.9 \) under our parameterization. The next step is for the second term.

**Step 3: Regulated Bessel Process**

For the second term \( E[e^{-\rho T^z_n}] n^{(\gamma-1)(1+h)} \), because our economy (i.e., evolution...
of $z$) differs from the evolution of a Bessel process when $z$ is far away from 0, an extra care needs to be taken. We consider a regulated Bessel process which is reflected at some positive constant $\bar{x}$. Intuitively, by doing so, we are putting an upper bound for $E \left[ e^{-\rho T_n} \right]$, as the reflection makes $z_t$ to hit $n^{-1}$ more likely (therefore, a larger $F_z$). Also, for a sufficiently small $\bar{x} > 0$, when $z \in (0, \bar{x}]$, $z$ follows a Bessel process with a dimension $\gamma + 2 - \epsilon$. Therefore, $F_z$ must be bounded by the first-hitting time distribution of a Bessel process with a dimension $\delta$, where $\delta$ takes value from $\gamma + 2 - \epsilon$ to $\gamma + 2$, where $\epsilon$ is sufficiently small. Finally, note that by considering a Bessel process we are neglecting certain drift for $z$. However, one can easily check that when $z$ is close to 0, the adjustment term for $\mu_y$ is $-\frac{1}{\rho} \gamma \sigma^2 < 0$. This implies that we are neglecting a positive drift for $z$—which potentially makes hitting less likely—thereby yielding an upper-bound estimate.

We have the following Lemma from the Bessel process.

**LEMMA 1:** Consider a Bessel process $x$ with $\delta > 2$ which is reflected at $\bar{x} > 0$. Let $\nu = \frac{\delta}{\nu} - 1$. Starting from $x_0 \leq \bar{x}$, we consider the hitting time $T_n^x = \inf \{ t : x_t = \frac{1}{n} \}$. Then we have

$$E \left[ e^{-\rho T_n^x} \right] \propto n^{-2\nu} \text{ as } n \to \infty$$

**PROOF:**

Due to the standard results in Bessel process and the Laplace transform of the hitting time (e.g., see Borodin and Salminen (1996), Chapter 2), we have

$$E \left[ e^{-\rho T_n^x} \right] = \varphi (z_0) / \varphi (n^{-1}) ,$$

where

$$\varphi (u) = c_1 u^{-\nu} I_v \left( \sqrt{2\rho u} \right) + c_2 u^{-\nu} K_v \left( \sqrt{2\rho u} \right) ,$$

and $I_v (\cdot)$ (and $K_v (\cdot)$) is modified Bessel function of the first (and second) kind of order $v$. Because $R$ is a reflecting barrier, the boundary condition is

$$\varphi' (\bar{x}) = 0 ,$$

which pins down the constants $c_1$ and $c_2$ (up to a constant multiplication; notice that this does not affect the value of $E \left[ e^{-\rho T_n^x} \right]$). Therefore the growth rate of $E \left[ e^{-\rho T_n^x} \right]$ is determined by $n^\nu K_v \left( \sqrt{2\rho n}^{-1} \right)$ as $K_v$ dominates $I_v$ near 0. Since $K_v (z)$ has a growth rate $z^{-\nu}$ when $z \to 0$, the result is established.

For any $y_0$, redefine starting point as $z_0 = \min (1 + l - y_0, \bar{x})$; clearly this leads to an upper-bound estimate for $E \left[ e^{-\rho T_n^x} \right]$. However, since for all $\delta \in [\gamma + 2 - \epsilon, \gamma + 2]$, the above Lemma tells us that for any $\epsilon \in \left[ 0, \epsilon \right]$, given $\gamma = 2$ and $h = 0.9$, when $n \to \infty$, for some sufficiently small $\epsilon > 0$ we have

$$n^{(\gamma - 1)(1 + h)} E \left[ e^{-\rho T_n^x} \right] \propto n^{(\gamma - 1)(1 + h)} n^{-2\nu} = n^{(\gamma - 1)(1 + h) - \gamma + \epsilon} \to 0 \text{ uniformly} .$$
Therefore we obtain our desirable result. Finally $c_t \xi_t > 0$ implies that $\int_0^\infty c_t \xi_t dt$ converges monotonically, and therefore the specialist’s budget equation $\lim_{T \to \infty} E \left[ \int_0^T \xi c_t dt \right] = \xi_0 D_0 (F_0 - y_0)$ holds for all stopping times that converge to infinity. Q.E.D.

II. Appendix for Section 6

A. Borrowing Subsidy

We have the same ODE as in Appendix A. The only difference is that

$$\mu_y = \frac{1}{1 - \theta_s F} \left( \theta_s + l + (r + \sigma^2 - g) \hat{\theta}_b - \hat{\theta}_b \Delta r - \rho y + \frac{1}{2} \theta_s F' \sigma_y^2 \right).$$

B. Direct Asset Purchase

In this case, the intermediaries hold $1 - s$ of the risky asset (where $s$ is a function of $(y, D)$). In the unconstrained region, $\alpha^h = 1$, and

$$\frac{\alpha^I (w + \alpha^h (1 - \lambda) w^h)}{P} = 1 - s$$

which implies that $\alpha^I = \frac{(1 - s) F}{F - \lambda y}$. Therefore the households’ holding of the risky asset through intermediaries is

$$\theta^I_s = \frac{(1 - s) (1 - \lambda) y}{F - \lambda y},$$

and the total holding is $\theta_s = \theta^I_s + s = \frac{(1 - s)(1 - \lambda) y}{F - \lambda y} + s (y, D)$.

In the constrained region, $\alpha^h = \frac{m (F - y)}{(1 - \lambda) y}$ and $\alpha^I = \frac{1}{1 + m} \frac{(1 - s) F}{F - y}$. So

$$\theta^I_s = m \frac{(F - y)}{(1 - \lambda) y} \frac{1}{1 + m} \frac{(1 - s) F}{F - y} \frac{(1 - \lambda) y}{F} = m \frac{1}{1 + m} (1 - s)$$

and the total holding is

$$\theta_s = \frac{m}{1 + m} (1 - s) + s = \frac{m + s}{1 + m}.$$

The same constraint cutoff applies $y^c = \frac{m}{1 - \lambda + m} F^c$.

Finally, the expressions for the case of capital infusion (i.e., changing $m$) is isomorphic to the case of $s > 0$. This is because given $s$ we can find some appropriate $m'(s) = \frac{s + m}{1 - s}$ such that $\frac{m'}{1 + m'} = \frac{m + s}{1 + m}$.