Information and Industry Dynamics

Emin M. Dinlersoz and Mehmet Yorukoglu

Web Appendix

Abstract

This Web Appendix contains additional material accompanying the article "Information and Industry Dynamics" by Emin M. Dinlersoz and Mehmet Yorukoglu. Equation numbers in this Web Appendix continue the numbering in the article.

A Proofs

For notational convenience, define $R^* \equiv R^*(m) = \beta \frac{\partial}{\partial m} \int V^*(m,c)h(c)dc$ as the derivative of the continuation value of a firm with respect to its size m at the beginning of the next period. Given $A \subseteq \mathbb{R}^n$ $(n \geq 1)$, $B \subseteq \mathbb{R}$, and any two functions $f, g : A \to B$, $f \preceq g$ means $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in A$. Similarly, $f \prec g$ means $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in A$ and $f(\mathbf{x}) < g(\mathbf{x})$ for some $\mathbf{x} \in A$. Whenever there is no ambiguity, the arguments of a function are suppressed.

Proof of Lemma 1. z_t is defined by (4), and the demand ω_t of a consumer who did not purchase from firm (p, m^o) in period $t-1$ is

$$\omega_t(p, m^o) = \sum_{i=0}^{\infty} \Omega_t(i) \widetilde{G}_t^i(W_t(p, m^o)), \tag{19}$$

where $\Omega_t(i)$ is the measure of consumers informed of $i \geq 0$ firms in addition to firm (p, m^o), and $\widetilde{G}_t^i$ is the probability that the consumer chooses to buy from firm (p, m^o). Let $\delta_t < 1$ denote the measure of consumers who purchased in period $t-1$ from a firm that exited at the beginning of period t. Let $\theta_t = \lambda + (1-\lambda)\delta_t < 1$ be the sum of the mass of new consumers and the mass of surviving consumers whose firms exited at the beginning of period t. Label all of these consumers as 'unattached', since they have no information before they receive any ads. Label the rest of the consumers as 'attached', as they are informed of their most-recently-visited firms before they receive any ads. Then,

$$\Omega_t(i) = \begin{cases} \theta_t \Psi_t(0) & \text{for } i = 0, \\ \theta_t \Psi_t(i) + (1-\theta_t)\Psi_t(i-1) & \text{for } i > 0. \end{cases} \tag{20}$$

In (20), the term $\theta_t \Psi_t(0)$ gives the mass of unattached consumers who receive zero ads. The term for $i > 0$ is the mass of unattached consumers who receive i ads, plus the mass of attached consumers who receive $(i-1)$ ads. Next, let $\alpha_t \equiv \alpha_t(p, m^o)$ be the probability that a consumer prefers firm (p, m^o) to the firm he purchased from in period $t-1$. Note that $G_t^0 = \widetilde{G}_t^0 = 1$. For $i \geq 1$, (20) implies

$$\widetilde{G}_t^i = \frac{\theta_t \Psi_t(i)}{\Omega_t(i)} G_t^i + \frac{(1-\theta_t)\Psi_t(i-1)}{\Omega_t(i)} G_t^{i-1} \alpha_t, \tag{21}$$

where the first term in (21) is the probability that a consumer is unattached and has i new ads conditional on having information about i other firms, and the second term is the probability that a consumer is attached and has $i-1$ new ads conditional on having information about i other firms. Using (19) and (21), one can then write

$$\omega_t(p,m^o) = \theta_t \sum_{i=0}^{\infty} \Psi_t(i) G_t^i + (1-\theta_t) \sum_{i=1}^{\infty} \Psi_t(i-1) G_t^{i-1} \alpha_t = [\theta_t + (1-\theta_t)\alpha_t] z_t(p,m^o). \quad (22)$$

Because $\alpha_t \le 1$ and $\theta_t < 1$, it follows that $\omega_t(p,m^o) \le z_t(p,m^o)$. Next, for $\alpha_t(p,m^o) < 1$, using (22) and differentiating with respect to price and rearranging yields

$$\varepsilon_{\omega_t}(p,m^o) = \frac{(1-\theta_t)\alpha_t(p,m^o)}{\theta_t + (1-\theta_t)\alpha_t(p,m^o)} \varepsilon_{\alpha_t}(p,m^o) + \varepsilon_{z_t}(p,m^o),$$

which implies $\varepsilon_{\omega_t}(p,m^o) > \varepsilon_{z_t}(p,m^o)$ because $\varepsilon_{\alpha_t}(p,m^o) > 0$ and $(1-\theta_t)\alpha_t(p,m^o) > 0$. For $\alpha_t(p,m^o) = 1$, (22) implies $\varepsilon_{\omega_t}(p,m^o) = \varepsilon_{z_t}(p,m^o)$. Consequently, $\varepsilon_{\omega_t}(p,m^o) \ge \varepsilon_{z_t}(p,m^o)$. ■

Proof of Proposition 1. Part (*i*). Let $Q^*(p|m^o)$ denote the equilibrium probability that an ad contains a price of at least p, conditional on m^o. The cumulative distribution function, $1 - Q^*(p|m^o)$, cannot have a mass at any p in its support, otherwise any firm charging p could reduce its price slightly and steal a positive mass of consumers from other firms charging p, leading to a discrete gain in firm value. Also, $1 - Q^*(p|m^o)$ cannot be flat over some interval (p_1, p_2), otherwise any firm charging p_1 could increase its price to p_2 without a reduction in its probability of sale. Thus, $1 - Q^*(p|m^o)$ is strictly increasing in p over the interior of its support $[\underline{p}^*(m^o), \overline{p}^*(m^o)]$, where $\underline{p}^*(m^o)$ and $\overline{p}^*(m^o)$ are the minimum and maximum prices observed for firms with size m^o. Therefore, $\frac{\partial Q^*(p|m^o)}{\partial p} < 0$. Let $z^*(p|m^o) = \sum_{i=0}^{\infty} \Psi^*(i) \left[Q^*(p|m^o)\right]^i$ be the demand function for a consumer conditional on m^o. It follows that $\frac{\partial z^*(p|m^o)}{\partial p} = \sum_{i=0}^{\infty} \Psi^*(i) i Q^{*i-1} \frac{\partial Q^*}{\partial p} < 0$, which also implies $\frac{\partial z^*(p,m^o)}{\partial p} < 0$. Similar arguments apply to $\omega^*(p,m^o)$.

Parts (*ii*), (*iii*) and (*iv*). By the envelope theorem, $\frac{dV^*}{dc} = \frac{\partial V^*}{\partial c} = -m^* < 0$. Next, observe that

$$\frac{\partial z^*}{\partial p} = \left(\sum_{i=0}^{\infty} \Psi^*(i) i G^{*i-1} \right) G^{*\prime} \frac{\partial W^*}{\partial p}. \quad (23)$$

Because $\frac{\partial z^*}{\partial p} < 0$ by Part (i) and $\left(\sum_{i=0}^{\infty} \Psi^*(i) i G^{*i-1}\right) G^{*\prime} > 0$, it follows that $\frac{\partial W^*}{\partial p} < 0$. Next, we argue that z^*, ω^*, V^*, and W^* must be strictly increasing in m^o. Differentiation of z^* yields

$$\frac{\partial z^*}{\partial m^o} = \left(\sum_{i=0}^{\infty} \Psi^*(i) i G^{*i-1} \right) G^{*\prime} \frac{\partial W^*}{\partial m^o}. \quad (24)$$

Thus, $\frac{\partial z^*}{\partial m^o}$ and $\frac{\partial W^*}{\partial m^o}$ have the same sign. Similarly, $\frac{\partial \omega^*}{\partial m^o}$ and $\frac{\partial W^*}{\partial m^o}$ have the same sign, and therefore so do $\frac{\partial \omega^*}{\partial m^o}$ and $\frac{\partial z^*}{\partial m^o}$. Furthermore, the envelope theorem implies

$$\frac{dV^*}{dm^o} = \frac{\partial V^*}{\partial m^o} = \frac{\partial m^*}{\partial m^o}\left(p^* - c + R^*\right). \tag{25}$$

For (11) to hold, the term in parentheses in (25) must be positive. Consequently, $\frac{dV^*}{dm^o}$ and $\frac{\partial m^*}{\partial m^o}$ have the same sign. Next, note that

$$\frac{\partial m^*}{\partial m^o} = (1-\lambda)\, z^* + \left(m^o(1-\lambda)\frac{\partial z^*}{\partial m^o} + a^*\frac{\partial \omega^*}{\partial m^o}\right).$$

To obtain a contradiction, suppose that $\frac{\partial z^*}{\partial m^o} \leq 0$ and $\frac{\partial \omega^*}{\partial m^o} \leq 0$. There are then two possibilities: $\frac{\partial m^*}{\partial m^o} \leq 0$ or $\frac{\partial m^*}{\partial m^o} > 0$. If $\frac{\partial m^*}{\partial m^o} \leq 0$, (25) implies $\frac{dV^*}{dm^o} \leq 0$. But then firm value is non-increasing in firm size, and a firm has no incentive to increase its size, because exit probability increases as firm size increases. Because the only way a firm can grow is to send ads, there is then no advertising. Therefore, $\frac{\partial z^*}{\partial m^o} \leq 0$, $\frac{\partial \omega^*}{\partial m^o} \leq 0$, and $\frac{\partial m^*}{\partial m^o} \leq 0$ cannot hold in an equilibrium with positive advertising.

Suppose now that $\frac{\partial m^*}{\partial m^o} > 0$. Then, by (25), $\frac{dV^*}{dm^o} > 0$. Now, rewrite the consumer surplus in (1) in stationary equilibrium as

$$W^*(p, m^o) = U(p) + \beta(1-\lambda)\sum_{i=0}^{\infty}\Psi^*(i)\{(1 - X^*(p, m^o))W_1^*(p^1, m^1) + X^*(p, m^o)W_2^*]\},$$

where X^* is the probability of exit, (p^1, m^1) denotes the next period price and size, and

$$W_1^*(p^1, m^1) = \begin{cases} E\left[\int \max\{W, W^*(p^1, m^1)\} dG^{*i}(W)\right] & \text{if } i > 0 \\ E[W(p', m')] & \text{if } i = 0 \end{cases},$$

$$W_2^* = \begin{cases} E\left[\int W dG^{*i}(W)\right] & \text{if } i > 0 \\ W^{*\varnothing} & \text{if } i = 0 \end{cases}.$$

Then

$$\frac{\partial W^*}{\partial m^o} = \beta(1-\lambda)\sum_{i=0}^{\infty}\Psi^*(i)\left(\frac{\partial X^*}{\partial m^o}(W_2^* - W_1^*(p^1, m^1)) + (1 - X^*)\frac{\partial W_1^*}{\partial m^1}\right), \tag{26}$$

and

$$\frac{\partial W_1^*}{\partial m^1} = \beta(1-\lambda)\sum_{i=0}^{\infty}\Psi^*(i)\left(\frac{\partial X^*}{\partial m^1}(W_2^* - W_1^*(p^2, m^2)) + (1 - X^*)\frac{\partial W_1^*}{\partial m^2}\right)\frac{\partial m^1}{\partial m^o}, \tag{27}$$

where (p^2, m^2) is the two-period ahead price and size, and m^1 is the next period's size. One can thus substitute $\frac{\partial W_1^*}{\partial m^1}$ in (26) using (27). Continuing with similar substitution for all future periods, one obtains

$$\frac{\partial W^*}{\partial m^o} = \sum_{t=1}^{\infty} \beta^t (1-\lambda)^t \left(\prod_{j=1}^{t-1} (1 - X^*(p^t, m^t)) \right) \frac{\partial X^*}{\partial m^t} \sum_{i=0}^{\infty} \Psi^*(i)(W_2^* - W_1^*(p^t, m^t)) \left(\prod_{j=1}^{t-1} \frac{\partial m^j}{\partial m^{j-1}} \right), \tag{28}$$

where (p^t, m^t) is the t-period ahead price and size, and m^j is the size in period j. Because $\frac{dV^*}{dm^o} > 0$ and exit probability declines with firm value, exit probability also declines with firm size, i.e. $\frac{\partial X^*}{\partial m^t} < 0$ for all t. In addition, it was assumed that $\frac{\partial m^j}{\partial m^{j-1}} > 0$. Furthermore, $W_2^* - W_1^*(p^1, m^1) < 0$, by the definitions of W_1^* and W_2^*. Consequently, (28) implies $\frac{\partial W^*}{\partial m^o} > 0$. But since $\frac{\partial z^*}{\partial m^o} \leq 0$, (24) implies $\frac{\partial W^*}{\partial m^o} \leq 0$, a contradiction. Therefore, $\frac{\partial z^*}{\partial m^o} \leq 0$ and $\frac{\partial \omega^*}{\partial m^o} \leq 0$ cannot hold. Thus, the only configuration consistent with positive advertising in equilibrium is $\frac{\partial z^*}{\partial m^o} > 0$, $\frac{\partial \omega^*}{\partial m^o} > 0$, $\frac{\partial V^*}{\partial m^o} > 0$ and $\frac{\partial W^*}{\partial m^o} > 0$. ■

Proof of Proposition 2. Part (i). Proposition 1(i) established that equilibrium entails a continuous (atomless) price distribution. The firm type that offers the least surplus to consumers cannot steal any consumers away from other firms and can do no better than charging its monopoly price. For other firm types, optimal price must satisfy (11). To see this claim, note that, by Proposition 1, the individual demand functions, $z^*(p, m^o)$ and $\omega^*(p, m^o)$, and hence a firm's residual demand $m^*(p, a; m^o)$, are continuous in p, implying the continuity of the profit function and the value function in p for surviving firms. Furthermore, feasible prices for a firm type can be restricted to a compact set $[\underline{p}, p^m]$, where $p^m \geq s$ is the monopoly price for the corresponding firm type, and a lower bound $\underline{p}$ can be imposed because, as price declines, $p - c + R^*$ becomes negative eventually as R^* is bounded from above. Therefore, for firms that offer more than the lowest surplus, a value maximizing price lies in the set $(\underline{p}, p^m)$. Given the continuity of m^* in p, lowering price entails a trade-off: a decline in profit per consumer (including a firm's captive consumers who are only informed of a single firm) versus an increase in profits due to a higher probability of sale to consumers who have ads from other firms. The first order condition (11) balances these two trade-offs. To see uniqueness, suppose that the value function for firm (m^o, c) admits two or more countable interior global maximizers. Take any two such maximizers, (p_1, a_1) and

(p_2, a_2), such that $p_1 \neq p_2$. Since (11) holds at both (p_1, a_1) and (p_2, a_2), by the generalized intermediate value theorem (see, e.g. Munkres (1975) p. 154) there must exist a pair (p', a') $\neq (p_1, a_1)$, (p_2, a_2) such that

$$[m^o(1-\lambda)z^*(p', m^o)+a'\omega^*(p', m^o)]+\left[m^o(1-\lambda)\frac{\partial z^*(p', m^o)}{\partial p}+a'\frac{\partial \omega^*(p', m^o)}{\partial p}\right](p'-c+R^*) = 0. \tag{29}$$

Because the distribution of price and advertising pairs is continuous conditional on m^o, there must exist some firm (m^o, c') with $c' \neq c$ for which (p', a') is optimal. For this firm, (p', a') satisfies

$$[m^o(1-\lambda)z^*(p', m^o)+a'\omega^*(p', m^o)]+\left[m^o(1-\lambda)\frac{\partial z^*(p', m^o)}{\partial p}+a'\frac{\partial \omega^*(p', m^o)}{\partial p}\right](p'-c'+R^*) = 0. \tag{30}$$

But because $c \neq c'$ and all other terms are identical in (29) and (30), the two equalities cannot hold simultaneously. Thus, there cannot be more than one discrete global interior maximizer. The remaining possibility is that there exists a continuum of maximizers for firm (m^o, c) that form a connected set S of (p, a) pairs, and no other firm type has a value-maximizing price and advertising pair within this set. Suppose that is the case. Now consider any two value-maximizing pairs (p_1, a_1) and (p_2, a_2) in this set such that $p_1 < p_2$. Because it is assumed that there is no other firm type for which some pair $(p, a) \in S$ is a value maximizer, firm (m^o, c) can raise its price from p_1 to p_2 and still send a_1 ads without a reduction in its residual demand, and thereby increase its value. This contradicts with (p_1, a_1) being part of the set S.

Taking the total derivative of (11) and (13) with respect to c leads to

$$\frac{\partial p^*}{\partial c} = -\frac{1}{\Delta}\left[\Phi''\left(m^o(1-\lambda)\frac{\partial z^*}{\partial p}+a^*\frac{\partial \omega^*}{\partial p}\right)+\left(\omega^*+\frac{\partial \omega^*}{\partial p}(p^*-c+R^*)\right)\right], \tag{31}$$

where

$$\begin{aligned}\Delta \;=\; & \left(\omega^*\tfrac{dR^*}{da}-\Phi''\right)\left[\left(m^o(1-\lambda)\tfrac{\partial z^*}{\partial p}+a^*\tfrac{\partial \omega^*}{\partial p}\right)\left(2+\tfrac{\partial R^*}{\partial p}\right)+\left(m^o(1-\lambda)\tfrac{\partial^2 z^*}{\partial p^2}+a\tfrac{\partial^2 \omega^*}{\partial p^2}\right)(p^*-c+R^*)\right] \\ & -\left(\tfrac{\partial \omega^*}{\partial p}(p^*-c+R^*)+\omega^*(1+\tfrac{\partial R^*}{\partial p})\right)^2\end{aligned}$$

is the determinant of the Hessian of V^*. Note that $\Delta > 0$, given the fact that the Hessian of V^* is negative definite at the global maximizer (p^*, a^*). Inside the square brackets in (31), the first term is negative because $\Phi'' > 0$ (Assumption 1), $\frac{\partial z^*}{\partial p} < 0$ and $\frac{\partial \omega^*}{\partial p} < 0$ (Proposition 1). To sign the second term inside the square brackets in (31), note that (5) and (11) together yield

$$\omega^* + \frac{\partial \omega^*}{\partial p}(p^* - c + R^*) = \frac{(1-\theta^*)\frac{\partial \alpha^*}{\partial p} z^*(p^* - c + R^*)}{1 + [\theta^* + (1-\theta^*)\alpha^*]\frac{a^*}{m^o(1-\lambda)}} < 0, \tag{32}$$

because $\frac{\partial \alpha^*}{\partial p} < 0$ and all other terms in the ratio on the r.h.s. of (32) are positive. Consequently, the expression in square brackets in (31) is negative, implying $\frac{\partial p^*}{\partial c} > 0$.

To sign $\frac{\partial p^*}{\partial m^o}$, suppose, to obtain a contradiction, that for two firm types with marginal cost c and sizes $m_2^o > m_1^o$, it holds that $p^*(m_2^o, c) \leq p^*(m_1^o, c)$. Given the continuity of the pricing policy, one can find a firm type (m_3^o, c) where $m_3^o \in [m_1^o, m_2^o]$ such that $p^*(m_3^o, c) = p^*(m_2^o, c)$ and $p^*(m^o, c)$ is non-increasing over the interval $(m_3^o - \varepsilon, m_3^o)$ for some $\varepsilon > 0$. Then, by Proposition 1(iv), $W^*(p^*(m_3^o, c), m_3^o) > W^*(p^*(m^o, c), m^o)$ for all $m^o \in [m_3^o - \varepsilon, m_3^o)$. Therefore, firm (m_3^o, c) can raise its price slightly without a reduction in its probability of sale and increase its profit per consumer, a contradiction with $p^*(m_3^o, c)$ being a value maximizer for firm (m_3^o, c).

Part (*ii*). If the exit probability $X^*(p, m^o)$ does not depend on m^o directly, $W^*(p, m^o)$ also does not depend on m^o directly, and it reduces to a function, $W^*(p)$, of price only. At this point, we allow for the possibility that in equilibrium $W^*(p^*)$ depends on firm size indirectly through the optimal price p^*. By definitions (4) and (5), the demand functions $z^*(p, m^o)$ and $\omega^*(p, m^o)$ are then also functions of price only: $z^*(p)$ and $\omega^*(p)$. Because a larger size now confers no additional surplus to any consumer, the demand functions for previous-period customers and the new customers acquired through ads must also be identical, i.e. $z^*(p) \equiv \omega^*(p)$. Consequently, (11) reduces to

$$(m^o(1-\lambda) + a)\, z^* + (m^o(1-\lambda) + a)\frac{dz^*}{dp}\,(p - c + R^*) = 0.$$

Dividing through by $m^o(1-\lambda) + a > 0$ and rearranging terms, a firm's optimal price can be written as

$$p^* = \frac{\varepsilon_{z^*}(p^*)}{\varepsilon_{z^*}(p^*) - 1}(c - R^*).$$

The price depends on the elasticity $\varepsilon_{z^*}(p^*)$ of the individual demand function $z^*(p^*)$, which is independent of firm size. Furthermore, R^*, which is the change in the continuation value of the firm due to a change in its size m^* at the beginning of the next period, is also independent of m^o. Consequently, price is not a function of m^o. ■

Proof of Proposition 3. The proof of Proposition 2(i) and the strict convexity of Φ (Assumption 1(i)) imply that a unique value-maximizing $a^*(m^o, c)$ exists. Taking the total derivative of (11) and (13) with respect to c yields

$$\frac{\partial a^*}{\partial c} = -\frac{1}{\Delta}\left[\left(m^o(1-\lambda)\frac{\partial z^*}{\partial p} + a^*\frac{\partial \omega^*}{\partial p}\right)\left(\omega^* + \frac{\partial \omega^*}{\partial p}(p^* - c + R^*)\right) - \omega^* Y^*\right], \qquad (33)$$

where

$$Y^* = 2\left(m^o(1-\lambda)\frac{\partial z^*}{\partial p} + a^*\frac{\partial \omega^*}{\partial p}\right) + \left(m^o(1-\lambda)\frac{\partial^2 z}{\partial p^2} + a^*\frac{\partial^2 \omega^*}{\partial p^2}\right)(p^* - c + R^*).$$

To sign (33), note that $\Delta > 0$, as shown earlier. Second, the first term inside the brackets in (33) is positive, following from (32) and that $\frac{\partial z^*}{\partial p} < 0$ and $\frac{\partial \omega^*}{\partial p} < 0$ (Proposition 1). Next, $\Delta > 0$ and the properties of the Hessian of V^* at (p^*, a^*) imply

$$Y^* < \frac{\left(\frac{\partial \omega^*}{\partial p}(p^* - c + R^*) + \omega^*(1 + \frac{\partial R^*}{\partial p})\right)^2}{\omega^* \frac{\partial R^*}{\partial a} - \Phi''} - \frac{1}{\Phi''}\omega^{*2}\left(\frac{\partial R^*}{\partial p}\right)^2 < 0,$$

where the sign follows because $\omega^* \frac{\partial R^*}{\partial a} - \Phi'' < 0$ (because $\Delta > 0$) and $\Phi'' > 0$ (Assumption 1). As a result, the second term inside the brackets in (33) is negative. Consequently, the entire term inside the brackets in (33) is positive. Thus, (33) is negative.

To see the monotonicity of a^* in m^o, consider two firms (m_1^o, c) and (m_2^o, c) such that $m_1^o < m_2^o$. Firm (m_2^o, c) has a higher expected return in the current period from a marginal ad

$$\omega^*(p_1^*, m_1^o)(p_1^* - c) < \omega^*(p_2^*, m_2^o)(p_2^* - c).$$

To see the last inequality, note that $p_2^* > p_1^*$ (Proposition 2). Furthermore, it must be that $\omega^*(p_1^*, m_1^o) < \omega^*(p_2^*, m_2^o)$. Suppose not. Firm (m_2^o, c) could then charge price p_1^* and have a probability of sale per ad $\omega^*(p_1^*, m_2^o) > \omega^*(p_1^*, m_1^o)$ by Proposition 1. Firm (m_2^o, c) could then afford to raise its price and still have a higher probability of sale and higher profit per ad. Thus, the marginal ad brings higher marginal profit in the current period to firm (m_2^o, c). Because firm (m_2^o, c) also has a lower likelihood of exit for all future periods, the consumer acquired through the marginal ad also brings a larger expected marginal benefit to firm

(m_2^o, c) from the next period onwards. The value effect of the marginal ad must therefore be larger for firm (m_2^o, c). Then, by (13) and the fact that Φ' is strictly increasing, firm (m_2^o, c) must send more ads. ■

Proof of Proposition 4. The properties of m^* follow directly from its definition and the properties of a^* and z^*. Since both m^* and p^* are strictly increasing in m^o, so is r^*. To see the monotonicity of r^* in c, note that

$$\frac{\partial r^*}{\partial c} = \frac{\partial a^*}{\partial c}\omega^* p^* + \left[\left((1-\lambda)m^o\frac{\partial z^*}{\partial p} + a^*\frac{\partial \omega^*}{\partial p}\right)p^* + ((1-\lambda)m^o z^* + a^*\omega^*)\right]\frac{\partial p^*}{\partial c} \tag{34}$$

Rearranging (11), and then substituting it in (34), we obtain

$$\begin{aligned}\frac{\partial r^*}{\partial c} &= \frac{\partial a^*}{\partial c}\omega^* p^* + \left((1-\lambda)m^o\frac{\partial z^*}{\partial p} + a^*\frac{\partial \omega^*}{\partial p}\right)(c - R^*)\frac{\partial p^*}{\partial c} \\ &< \frac{\partial a^*}{\partial c}\omega^* p^* + \left((1-\lambda)m^o\frac{\partial z^*}{\partial p} + a^*\frac{\partial \omega^*}{\partial p}\right)p^*\frac{\partial p^*}{\partial c} < 0,\end{aligned}$$

where the first inequality follows because $p^* - c + R^* > 0$, and the second inequality from the fact that $\frac{\partial a^*}{\partial c} < 0$, $\frac{\partial z^*}{\partial p} < 0$, $\frac{\partial \omega^*}{\partial p} < 0$, and $\frac{\partial p^*}{\partial c} > 0$. ■

Theorem 1. Given the values of the parameters of the model except κ, there is some $\kappa' > 0$ such that there exists a unique stationary equilibrium with a positive advertising and positive entry and exit, i.e. $M^* > 0$ and $x^*(c) > 0$ for some $c \in [\underline{c}, \infty)$, as long as $\kappa < \kappa'$.

Proof of Theorem 1. Given any equilibrium measure of firms μ^*, the exit threshold $c^*(m^o)$ is uniquely defined for any $m^o \geq 0$ by Proposition 1. Since $V^*(m^o, c) = 0$ for $c \geq c^*(m^o)$, we can write for any m^o

$$\int_{c^*(m^o)}^{\infty} V^*(m^o, c)h(c)dc = 0. \tag{35}$$

For positive entry in equilibrium, free entry condition (9) must hold with equality

$$\int_{\underline{c}}^{\infty} V^*(0, c)h(c)dc = \kappa. \tag{36}$$

Let $S = \{(m', c') : m' = m^*(m^o, c),\ c' \leq c^*(m')\}$ be the set of firm of types reachable from firm type (m^o, c), conditional on staying in the industry. Define the operator L_{c^*} as

$$L_{c^*}(m^o, c; S) = \begin{cases} \int_S h(c')dc', \text{if } c \leq c^*(m^o) \\ 0, \text{ otherwise,} \end{cases}$$

with norm $\|L_{c^*}\| < 1$. Invariance of μ^* in stationary equilibrium requires

$$\mu^* = L_{c^*}\mu^* + M^* H, \tag{37}$$

where $L_{c^*}\mu^*(S) = \int L_{c^*}(m^o, c; S)d\mu^*$. A stationary equilibrium with positive entry and exit is then given by a triplet $(c^*(\cdot), M^*, \mu^*)$ that satisfies equations (35)-(37) simultaneously.

Next, consider the advertising equilibrium corresponding to a given measure μ, an entry mass M, and an exit rule $c(\cdot)$. Let $E = \{\mu, M, c(\cdot)\}$. Let A denote the space of continuous, bounded functions defined over firm types under E, i.e. $a(E) \equiv a(\cdot, \cdot; E) : [\underline{m}, \overline{m}] \times [\underline{c}, \infty) \to \mathbb{R}^+$, where $\underline{m} > 0$ and $\overline{m} < \infty$ are the minimum and maximum firm sizes under E. Endow A with the sup-norm. Define $T : A \to A$ as the operator that matches any $a \in A$ to some $T(a) \in A$ that results from the firms' optimal choices of advertising given the distribution of consumer surplus $G(W)$ generated by the function a under E. We will first show that, corresponding to any E, there exists a unique advertising equilibrium such that the advertising policy $a(E)$ adopted by firms under E generates a distribution of consumer surplus across ads that renders the same advertising policy $a(E)$ optimal for E. We will then show the existence and uniqueness of a triplet $E^* = \{\mu^*, M^*, c^*(\cdot)\}$ that satisfies (35)-(37). The following lemmas accomplish these tasks.

Lemma 2. Given any E, T has a unique fixed point $a(E)$, i.e. $T(a(E)) = a(E)$.

Lemma 3. For any configuration of the model's parameters (except κ) that satisfies the model's assumptions, there exists some $\kappa' > 0$ such that, given any $\kappa < \kappa'$, there is a unique $E^* = \{\mu^*, M^*, c^*(\cdot)\}$ which satisfies (35)-(37).

Proof of Lemma 2. Given E and any $a \in A$, consider the cumulative advertising $A(m^o, c; E)$ made by firms that offer at most as much surplus as firm (m^o, c). In other words,

$$A(m^o, c; E) = \int_{\underline{m}}^{\overline{m}} \int_{c(x)}^{\infty} \Gamma(x, y, A(x, y; E)) d\mu(x, y), \tag{38}$$

where $c(x)$ is the marginal cost level such firm $(x, c(x))$ offers as much surplus as firm (m^o, c) does. Note that $A(m^o, c; E)$ is continuous and monotonic in its arguments m^o and c. Using

(13), the integrand Γ is defined as

$$\begin{aligned}\Gamma(x, y, A(x, y; E)) &= a(x, y; E) \\ &= \Phi'^{-1}\left(\omega(p(x, y), x; A)\left[(p(x, y; A) - y) + \beta \int \tfrac{\partial V(m, c'; A)}{\partial m} h(c')dc'\right]\right) \\ &= \Phi'^{-1}(\Lambda(x, y, A(x, y; E)).\end{aligned}$$

The dependence of the functions ω, p, and V on $A(x, y; E)$ is made explicit. Equation (38) is a Volterra-Fredholm type integral equation. We will show that (38) has a unique fixed point, $A^*(m^o, c; E)$, in the space of continuous, bounded, and non-decreasing functions defined over (m^o, c) pairs. For any $A_1(x, y; E)$ and $A_2(x, y; E)$ in that space, an application of Mean Value Theorem implies

$$\|\Gamma(x, y, A_1(x, y; E)) - \Gamma(x, y, A_2(x, y; E))\| \leq \|D(x, y)\| \, \|A_1(x, y; E) - A_2(x, y; E)\|,$$

where $\|\cdot\|$ denotes the sup-norm, and

$$D(x, y) = \frac{1}{\Phi''(\Lambda(x, y, A(x, y))} \frac{\partial \Lambda(x, y, A(x, y))}{\partial A(x, y)}.$$

Γ is bounded and continuous in its arguments, because Φ'^{-1} and Λ are both continuous. $\|D(x, y; A)\|$ is also bounded, continuous, and integrable. Moreover the function $A(x, y; E)$ is bounded because $a(x, y; E) < \infty$ for all (x, y). Therefore, the conditions for the existence and uniqueness of a solution to the integral equation (38) are satisfied (See, e.g., Proposition 2 in Hacia (1997)). As a result, there exists a unique, continuous, non-decreasing $A(E) \equiv A(m^o, c; E)$ that satisfies (38). Consequently, the advertising function $a(E) \equiv a(m^o, c; E)$ associated with $A(E)$ also exists and it is unique. In other words, the operator $T(a)$ has a unique fixed point $a(E)$.

Proof of Lemma 3. Given the existence and uniqueness of an advertising equilibrium corresponding to a given E, we now focus on the existence and uniqueness of a triplet E satisfying equations (35)-(37). We build on the basic arguments in Hopenhayn (1992). Assumptions A3, A4 and A5 in Hopenhayn (1992) apply with little modification: A3(a) is satisfied because H is continuous in current period's cost shock, and H is invariant to previous period's shock, implying continuity. A3(b) is replaced with an invariant *i.i.d.* process here, but the fact that current cost shock affects end-of-period firm size implies that firm's future value depends on the current cost shock, allowing the exit threshold to be defined as $c^*(m^o)$, as described in the text. A4 is also satisfied because in any period the probability, $1-H(c)$, of observing a cost shock greater than c is positive for any c. Finally, A5 is satisfied because the distribution of entrants' cost shocks, H, is continuous.

Existence: Let D be the set of continuous, bounded, non-decreasing functions defined as $c(\cdot) : \mathbb{R}^+ \rightarrow [\underline{c}, \infty)$. First, we show that, for any $c(\cdot) \in D$ and $M > 0$, an invariant measure $\mu(c(\cdot), M)$ exists, i.e.

$$\mu(c(\cdot), M) = L_c\mu(c(\cdot), M) + MH.$$

Equivalently, μ satisfies

$$\mu(c(\cdot), M) = M(\mathcal{I} - L_c)^{-1}H,$$

where I is the identity operator and $(I - L_c)^{-1}$ is the inverse operator for $I - L_c$. Following steps in Lemma 4 of Hopenhayn (1992), let L_c^n be the composition of L_c n times with itself and $L_c^0 = I$. The assumption that $H(c) < 0$ for all $c \in [\underline{c}, \infty)$ implies that the norm $\|L_c^n\| < 1$, so $(I - L_c^n)^{-1} = \sum_{t=0}^{\infty} L_c^{nt}$ following from Kolmogorov and Fomin (1970, Chapter 6, Section 23, Theorem 4, p. 231). The existence of $(I - L_c)^{-1}$ then follows because $\|L_c^n\|$ is non-increasing in n. Thus, $\mu(c(\cdot), M)$ satisfies invariance.

Next, we will say that the invariant measure $\mu(c(\cdot), M)$ is continuous on D if for all $c(\cdot) \in D$ and for every sequence $c_n(\cdot) \rightarrow c(\cdot)$, we have $\mu(c_n(\cdot), M) \rightarrow \mu(c(\cdot), M)$. Continuity with respect to M is defined similarly, but simply on $\mathbb{R}^+$. Using arguments similar to Lemma 5 in Hopenhayn (1992), it can be shown that the invariant measure $\mu(c(\cdot), M)$ is jointly continuous, strictly increasing in M, and non-increasing in $c(\cdot)$, i.e. for two exit schedules $c_1(\cdot)$, $c_2(\cdot) \in D$ such that $c_2(m^o) \succ c_1(m^o)$, we have $\mu(c_2(\cdot), M) \leq \mu(c_1(\cdot), M)$ in the sense that $\int_c u(m^o, c)d\mu_2(m^o, \cdot) \leq \int_c u(m^o, c)d\mu_1(m^o, \cdot)$ for any non-decreasing function u and for any given m^o.

For any exit rule $c(\cdot)$, define the entry mass $M^e(c(\cdot))$ implicitly as

$$V^e(\mu(c(\cdot), M^e)) = \int_{\underline{c}}^{\infty} V^*\left(0, c; \mu(c(\cdot), M^e)\right) h(c)dc = \kappa.$$

In other words, for the invariant measure μ, $M^e(c(\cdot))$ is the mass of entrants that are needed for the expected discounted profit for entrants, V^e, to be equal to the cost of entry under the exit rule $c(\cdot)$. Also, define $M^x(c(\cdot))$ as the mass of entrants such that for the invariant measure μ, the exit rule $c(\cdot)$ is optimal, i.e. for all $m^o \geq 0$

$$\int_{c(m^o)}^{\infty} V^*(m^o, c; \mu(c(\cdot), M^x))h(c)dc = 0.$$

A stationary equilibrium with positive entry and exit exists if and only if there is a function $c^*(\cdot)$ such that $M^e(c^*(\cdot)) = M^x(c^*(\cdot))$. This amounts to showing that the functions M^e and M^x intersect at least once.

It can be shown, analogous to Lemma 6 in Hopenhayn (1992), that the function $M^x : D \to \mathbb{R}^+$ is well-defined, continuous and strictly increasing, i.e. for any two functions $c_1(\cdot)$, $c_2(\cdot) \in D$ such that $c_2(\cdot) \succ c_1(\cdot)$, we have $M^x(c_2(\cdot)) > M^x(c_1(\cdot))$. In particular, let $Z(m^o, c; c(\cdot), M) = \int\limits_{c(m^o)}^{\infty} V(m^o, c; \mu(c(\cdot), M))h(c)dc$. It can be shown that Z is strictly increasing in m^o, strictly decreasing in c, non-decreasing in $c(\cdot)$ and strictly decreasing in M. Note that as $M \to \infty$, $Z(m^o, c; c(\cdot), M) \to -\frac{1}{1-\beta}f$, because $\Pi(m^o, c) \to -f$. Furthermore, as $M \to 0$, $\mu(c(\cdot), M) \to 0$. Then, by Assumption 1(iii) for small $M > 0$, $\Pi(0, c(0)) > 0$ for all $c(\cdot)$. Therefore, $Z(m^o, c; c(\cdot), M) > 0$ for small M. Thus, for all (m^o, c), there exists a unique M such that $Z(m^o, c; c(\cdot), M) = 0$. Therefore, M^x is the unique value such that $Z(m^o, c(m^o); \mu(c(\cdot), M^x)) = 0$ for all $m^o \geq 0$. Continuity of M^x follows from the continuity of Z, and because $Z(m^o, c(m^o); \mu(c(\cdot), M))$ is strictly decreasing in M, M^x is strictly increasing in $c(\cdot)$.

Similarly, following Lemma 7 in Hopenhayn (1992), it can be shown that M^e is continuous and non-increasing on D as long as $V^e(0) > \kappa$. As $M \to \infty$, $V^e(\mu(c(\cdot), M^e)) \to 0$ and as $M \to 0$, $V^e(0) > \kappa$. Because $V^e(\mu(c(\cdot), M))$ is continuous and strictly decreasing in M, there exists M^e such that $V^e(\mu(c(\cdot), M^e)) = \kappa$. M^e is continuous because it is the minimizer of the continuous function $|V^e(\mu(c(\cdot), M)) - \kappa|$. M^e is also non-increasing because $\mu(c(\cdot), M)$ is non-increasing in $c(\cdot)$.

Now note that for any $m^o > 0$, $V(m^o, c; \mu(c(\cdot), M)) > V(0, c; \mu(c(\cdot), M))$ for $c > c^*(m^o)$ because V is strictly increasing in m^o. The fact that $V(m^o, c; \mu(c(\cdot), M))$ has a maximum at $\underline{c}$ then implies that $M^x(c(\cdot)) > M^e(c(\cdot))$ for $c(\cdot) \equiv \underline{c}$. This implies that either there exists some function $c^*(\cdot)$ such that $M^x(c^*(\cdot)) < M^e(c^*(\cdot))$, or $M^x(c(\cdot)) > M^e(c(\cdot))$ for all $c(\cdot) \in D$. In the former case, there exists an equilibrium where $M^x(c^*(\cdot)) = M^e(c^*(\cdot)) > 0$ for some

$c^*(\cdot) \in D$, and in the latter there exists a stationary equilibrium with no entry and exit, i.e. for some $\bar{c} < \infty$ and the exit rule $c^*(\cdot) \equiv \bar{c}$, we have $M^x(c^*(\cdot)) = M^e(c^*(\cdot)) = 0$. Overall, then, there exists a stationary equilibrium.

For positive entry and exit in equilibrium, entry must be relatively easy. If the entry cost satisfies $\kappa < \kappa^* = V^e(\mu^*)$ for some $\kappa^* > 0$, then there exists some $c^*(\cdot) \in D$ such that $M^x(c^*(\cdot)) < M^e(c^*(\cdot))$. Unless the advertising cost and fixed cost is very high, potential entrants have positive expected profit, i.e. $V^e(\mu^*) > 0$, so that $\kappa^* > 0$. Therefore, an equilibrium with positive entry and exit exists.

Uniqueness: Suppose that there are two exit schedules $c_1^*(\cdot)$ and $c_2^*(\cdot)$ such that the corresponding measures μ_1^* and μ_2^* constitute stationary equilibria with positive entry and exit. Assume, without loss of generality, that $c_1^*(0) > c_2^*(0)$, that is, to survive the marginal entrant needs to be more efficient in economy 2 than in economy 1. Then, we must have $V_2^*(0, c_1^*(0)) < V_2^*(0, c_2^*(0)) = 0$ and $V_1^*(0, c_1^*(0)) = 0$. Thus, there must exist some firm type $(0, c)$ such that $V_2^*(0, c) < V_1^*(0, c)$. But free entry requires $V^e(\mu_1^*) = V^e(\mu_2^*) = \kappa$. Therefore, firm value cannot be lower for all entrant types under μ_2^* compared to μ_1^*, and must increase for some. In other words, if the values of all entrant types move in the same direction in response to a change in the measure of firms, the free entry condition is violated under μ_2^* and the two equilibria cannot coexist. We will show that this is the case. Assume, without loss of generality, that when the equilibrium measure of firms μ_1^* changes to μ_2^*, the profit of the most efficient entrant decreases, i.e. $V_2^*(0, \underline{c}) < V_1^*(0, \underline{c})$. If the profits of all entrants with marginal cost higher than $\underline{c}$ also decrease, then the equilibrium is unique. Take any cost level $c' \in (\underline{c}, \min\{c_1^*(0), c_2^*(0)\}]$. Consider the set of firm types that provide a consumer at least as much surplus as firm $(0, c)$ under measure μ_i, $i = 1, 2$

$$B_i(0, c) = \{(x, y) : W^*(x, y) \geq W^*(0, c)\}.$$

Since $V_2^*(0, \underline{c}) < V_1^*(0, \underline{c})$, we must have $\mu_2(B_2(0, \underline{c})) > \mu_1(B_1(0, \underline{c}))$ because profit is strictly decreasing in the measure of firms that offer more consumer surplus than firm type $(0, \underline{c})$. This implies $M_2^* > M_1^*$, i.e. entry mass must be higher for μ_2^*. But since μ^* is strictly increasing in M, we must then also have $\mu_2(B_2(0, c')) > \mu_1(B_1(0, c'))$, and therefore $V_2^*(0, c') < V_1^*(0, c')$ for all $c' \in (\underline{c}, \min\{c_1^*(0), c_2^*(0)\}]$. Thus, we have $V^e(\mu_2^*) < V^e(\mu_1^*) = \kappa$ and the free entry condition is violated under μ_2^*, implying uniqueness. This completes the proof of Lemma 3 and Theorem 1. ■

Proof of Proposition 5. Part (i) Let F_T^* and f_T^* denote, respectively, the cumulative distribution function and the density of firm size for firms of age $T \geq 1$. We will show that $F_T^*(m) < F_{T-1}^*(m)$ for all $T \geq 1$. Let $c^*(m, m^o) \in [\underline{c}, \infty)$ be the largest cost shock that results in a firm size of at least m at the end of a period, starting from a previous period size of m^o. The function $c^*(m, m^o)$ is decreasing in its first argument and increasing in its second argument. To see this claim, note that $c^*(m, m^o)$ is implicitly defined by the law of motion (7) as

$$m^o(1-\lambda)\widetilde{z}^*(m^o, c^*(m, m^o)) + a^*(m^o, c^*(m, m^o))\widetilde{\omega}^*(m^o, c^*(m, m^o)) = m.$$

Total differentiation with respect to m^o and m gives

$$\begin{aligned}
\frac{\partial c^*(m, m^o)}{\partial m^o} &= -\frac{(1-\lambda) + \frac{\partial a^*}{\partial m^o} + m^o(1-\lambda)\frac{\partial \widetilde{z}^*}{\partial m^o} + a^*\frac{\partial \widetilde{\omega}^*}{\partial m^o}}{\frac{\partial a^*}{\partial c}\widetilde{z}^* + m^o(1-\lambda)\frac{\partial \widetilde{z}^*}{\partial c} + a^*\frac{\partial \widetilde{\omega}^*}{\partial c}} > 0, \\
\frac{\partial c^*(m, m^o)}{\partial m} &= \left(\frac{\partial a^*}{\partial c}\widetilde{z}^* + m^o(1-\lambda)\frac{\partial \widetilde{z}^*}{\partial c} + a^*\frac{\partial \widetilde{\omega}^*}{\partial c}\right)^{-1} < 0,
\end{aligned}$$

where the inequalities follow from the signs of the individual terms established in earlier propositions. The conditional probability that the current period size is at most m can then be written as

$$P^*(m|m^o) = \begin{cases} 0 & \text{if } c^*(m, m^o) < \underline{c}, \\ 1 - H(c^*(m, m^o)) & \text{if } c^*(m, m^o) \in [\underline{c}, c^*(m^o)). \end{cases}$$

Therefore, for age $T = 1$

$$F_1^*(m) = 1 - H(c^*(m, 0)),$$

because new firms start with no customers. For age $T = 2$,

$$F_2^*(m) = \int_{\underline{m}_1^*}^{\overline{m}_1^*} \left[1 - H(c^*(m, m^o))\right] f_1^*(m^o)dm^o = 1 - \int_{\underline{m}_1^*}^{\overline{m}_1^*} H(c^*(m, m^o))f_1^*(m^o)dm^o,$$

where $\underline{m}_1^* > 0$ and $\overline{m}_1^*$ are the minimum and maximum firm sizes at age $T = 1$. But note that

$$\int_{\underline{m}_1^*}^{\overline{m}_1^*} H(c^*(m, m^o))f_1^*(m^o)dm^o > H(c^*(m, 0)),$$

because $H(c^*(m,m^o)) > H(c^*(m,0))$ for all $m^o > 0$, as $c^*(m,m^o)$ is strictly increasing in m^o. Thus, $F_2^*(m) < F_1^*(m)$ for all $m \in [0, \overline{m}_2^*]$. For any $T \geq 3$, one can write

$$F_{T-1}^*(m) = 1 - \int_{\underline{m}_{T-2}^*}^{\overline{m}_{T-2}^*} H(c^*(m,m^o)) f_{T-2}^*(m^o) dm^o, \tag{39}$$

$$F_T^*(m) = 1 - \int_{\underline{m}_{T-1}^*}^{\overline{m}_{T-1}^*} H(c^*(m,m^o)) f_{T-1}^*(m^o) dm^o. \tag{40}$$

Note that $\overline{m}_T^* > \overline{m}_{T-1}^*$ and $\underline{m}_T^* = \underline{m}_1^*$ for $T \geq 1$. Now, suppose that $F_{T-1}^*(m) < F_{T-2}^*(m)$. Because $H^*(c^*(m,m^o))$ is strictly increasing in m^o, first order stochastic dominance, together with (39) and (40), implies $F_T^*(m) < F_{T-1}^*(m)$. By induction, it must then hold that $F_T^*(m) < F_{T-1}^*(m)$ for any arbitrary T.

Part (ii) Follows from part (i).

Part (iii) Follows from part (i) and Proposition 1. ■

B Simulation algorithm

We outline the simulation algorithm used for comparative statics. The baseline advertising technology, $\Phi(a) = \chi a^{\varphi}$, generates positive advertising for any $\varphi > 1$. To see this claim, note that Assumption 1(iii) is satisfied because $\lim_{a \to 0} \Phi'(a) = \Phi'(0) = \chi \varphi 0^{\varphi - 1} = 0$ for $\varphi > 1$. Therefore, the marginal cost of advertising at $a = 0$, represented by the second term in the l.h.s. of (13), is zero. What remains to be shown is that the marginal return to advertising represented by the first term in the l.h.s. of (13) is positive, and thus exceeds the marginal cost at $a = 0$. A feasible, but not necessarily optimal, action available to any firm is to charge monopoly price and sell, at least, to its captive consumers. This action yields positive revenue per ad for a firm, because, as long as advertising is not free, there is always a positive mass of consumers whose only ad is the ad they received from the firm. Formally, the revenue per ad for a type (m^o, c) firm when it charges its monopoly price $p^m(m^o, c) \geq s$ is

$$\omega^*(p^m, m^o)(p^m - c + R^*) \geq \Omega^*(0)(p^m - c + R^*) > \Omega^*(0)(p^m - c) > \Omega^*(0)(s - c),$$

where the first inequality follows from the fact that $\omega^*(p^m, m^o) \geq \Omega^*(0)$, the second from $R^* \geq 0$, and the third from $p^m \geq s$. Because $\Omega^*(0) > 0$, the last term, $\Omega^*(0)(s-c)$, is positive as long as $s > c$, which is satisfied for all $c < 4$ because $s = 4$ and $c \sim U[1, 5]$. Consequently, there is positive advertising as long as the fixed cost is not too high and the industry is not empty, i.e. the entry cost is not too high. For the industry to be non-empty, in a given period there must be some firm types which do not exit, and the expected value of entry must not be lower than the entry cost. The conditions for a non-empty industry are checked as part of the simulation algorithm below.

The algorithm is based on Theorem 1 and involves nested value function iterations. The simulation setup requires a grid of firm types. We partition the support of the marginal cost distribution $U[1, 5]$ into a discrete set $\{c_1, ..., c_n\}$, where $c_1 = 1$ and $c_n = 5$. We also use a grid of firm size levels $\{m_1, ..., m_k\}$, where $m_1 = 0$ and $m_k = \overline{m}$, where $\overline{m}$ is taken sufficiently large to contain the upper bound on firm size in stationary equilibrium. The algorithm has the following steps:

1. Start with initial guesses for the maximum price charged, $p_m \geq s$, the value function, $V(m^o, c)$, the surplus function, $W(p, m^o)$, the measure of firms, $\mu(m^o, c)$, and the cumulative distribution of ads, $Q(p, m^o)$ (or equivalently advertising policy function $a(m^o, c)$). We look for fixed points by simultaneously iterating on these variables and functions.[1] As initial values, we choose $p_m = 4.5$, $V(m^o, c) \equiv \alpha_1 m^o + \alpha_2 c$ for some $\alpha_1 \geq 0$, and $\alpha_2 \leq 0$, $W(p, m^o) \equiv 0$, $W^{\varnothing} = 0$, and $\Delta\mu(m^o, c) = b \geq 0$ such that $\sum_{m^o} \sum_c \Delta\mu(m^o, c) = \sum_{m^o} \sum_c b = 1$, implying that $b = \frac{1}{nk}$. $Q(p, m^o)$ is also set to be the $c.d.f.$ of a two-dimensional discrete uniform distribution over $\{p_1, ..., p_l\} \times \{m_1, ..., m_k\}$. Given $W(p, m^o)$ and $Q(p, m^o)$, compute the individual demand functions as

$$
\begin{aligned}
z(p, m^o) &= \sum_{i=0}^{T} \frac{e^{-A} A^i}{i!} Q^i(p, m^o), \\
\omega(p, m^o) &= Q_s(p, m^o) \sum_{i=0}^{T} \frac{e^{-A} A^i}{i!} Q^i(p, m^o)
\end{aligned}
\tag{41}
$$

[1] There are two alternative methods of iteration on these functions. One method is to construct separate loops of iteration for each function. The second method is simultaneous iteration of all or a subset of the functions under one single loop. The first method is more robust, but computationally slower. The second method is faster with potential problems of robustness. To deal with the robustness issues, small step sizes for updating functions may be used. We tried both methods, and given the dimensionality of the setup, we found that using the second method with small step sizes is easier.

where $Q_s(p, m^o)$ is the probability that among all the sales that take place, the firm type that makes a randomly selected sale is not preferred to firm type (p, m^o). Since we do not have $Q_s(p, m^o)$ available initially, we start by assuming that $Q_s(p, m^o) = Q(p, m^o)$, and update it in further iterations as a computed $Q_s(p, m^o)$ becomes available. The upper limit T in the summations (41) is set to a large enough number ($T = 20$) and the term $\frac{e^{-A}A^i}{i!}$ follows from a Poisson approximation to $\Psi(i)$.

2a. Compute $\frac{\Delta z(p_i, m^o)}{\Delta p_i} = \frac{z(p_i, m^o) - z(p_{i-1}, m^o)}{p_i - p_{i-1}}$, $\frac{\Delta \omega(p_i, m^o)}{\Delta p_i} = \frac{\omega(p_i, m^o) - \omega(p_{i-1}, m^o)}{p_i - p_{i-1}}$, and $\frac{\Delta V(m_j^o, c)}{\Delta m_j^o} = \frac{V(m_j^o, c) - V(m_{j-1}^o, c)}{m_j^o - m_{j-1}^o}$ over a grid for p_{kn} and m_n^o. Solve the discrete-space version of the corresponding first order conditions (11) and (13), and find the optimal price and advertising levels $a_{i,j}$ and $p_{i,j}$ for each firm type using

$$p_{ij} = c_j - \frac{m_i^o(1-\lambda)z(p_{ij}, m_i^o) + a_{ij}\omega(p_{ij}, m_i^o)}{m_i^o(1-\lambda)\frac{\Delta z(p_{ij}, m_i^o)}{\Delta p_{ij}} + a_{ij}\frac{\Delta \omega(p_{ij}, m_i^o)}{\Delta p_{ij}}} - \beta \sum_k \frac{\Delta V(m^*, c_k')}{\Delta m^*} h(c_k'). \tag{42}$$

where $m^* = m_i^o(1-\lambda)z(p_{ij}, m_i^o) + a_{ij}\omega(p_{ij}, m_i^o)$, and

$$a_{ij} = \left[\frac{\omega(p_{ij}, m_i^o)\left[(p_{ij} - c_j) + \beta \sum_k \frac{\Delta V^o(m^*, c_k')}{\Delta m^*} h(c_k')\right]}{\varphi \chi} \right]^{\frac{1}{\varphi - 1}}. \tag{43}$$

2b. Compute period sales and profits for all firm types

$$m_{ij}^* = m_i^o(1-\lambda)z(p_{ij}, m_i^o) + a_{ij}\omega(p_{ij}, m_i^o),$$

$$\Pi(m_i^o, c_j) = m_{ij}^*(p_{ij} - c_j) - \Phi(a_{ij}) - f.$$

Compute total number of ads

$$A = \sum_i \sum_j a_{ij} \Delta \mu(m_i^o, c_j).$$

Obtain the updated value function

$$V'(m_i^o, c_j) = \max\{0, \Pi(m_i^o, c_j) + \beta \sum_{k=1}^{n} V(m_{ij}^*, c_k) h(c_k)\}.$$

Compute the exit indicator

$$X(p_{ij}, c_j) = \begin{cases} 0, & \text{if } \Pi(m_i^o, c_j) + \beta \sum_{k=1}^{n} V(m_{ij}^*, c_k) h(c_k) \geq 0, \\ 1, & \text{otherwise.} \end{cases}$$

2c. Obtain the updated surplus functions for consumers,

$$W^{\varnothing\prime} = \frac{\beta(1-\lambda)}{(1-\beta(1-\lambda)e^{-A})} \sum_{i=1}^{T} \frac{e^{-A}A^i}{i!} \sum_k W_k \Delta G^i(W_k)$$

$$\begin{aligned}
W'(p_{ij}, m_j^o) &= s - p_{ij} \\
&+\beta(1-\lambda)\Bigg\{ e^{-A}[(1-X(p_{ij}, m_j^o)) \sum_k W(p'_{kj}, m^*_{ij})]h(c'_k) + X(p_{ij}, m_j^o)W^{\varnothing} \\
&+\sum_{i=1}^{T} \frac{e^{-A}A^i}{i!} \Bigg[(1-X(p_{ij}, m_j^o)) \sum_k \max\{W_k, W(p'_{kj}, m^*_{ij})\}\Delta G^i(W_s) \\
&+ X(p_{ij}, m_j^o)) \sum_k W_k \Delta G^i(W_k)\Bigg]\Bigg\}.
\end{aligned} \tag{44}$$

2d. Obtain the implied distribution of ads across firm types $Q'(p_{ij}, m_i^o)$ using the optimal number of ads a_{ij} sent by each firm type and the measure of firms $\mu(m_i, c_j)$, i.e.

$$Q'(p_{ij}, m_i^o) = \frac{1}{A} \sum_{(q,r) \text{ s.t. } W(p_{qr}, m_q) \leq W(p_{ij}, m_i)} a_{qr} \Delta\mu(m_q^o, c_r).$$

Also, obtain the implied distribution of sales across firm types $Q'_s(p_{ij}, m_i^o)$

$$R = \sum_i \sum_j m^*_{ij} \Delta\mu(m_i^o, c_j),$$

$$Q'_s(p_{ij}, m_i^o) = \frac{1}{R} \sum_{(q,r) \text{ s.t. } W(p_{qr}, m_q^o) \leq W(p_{ij}, m_i^o)} m^*_{qr} \Delta\mu(m_q^o, c_r).$$

3. Calculate the distances

$$\begin{aligned}
d(Q, Q') &= \max_{i,j} |Q(p_{ij}, m_i^o) - Q'(p_{ij}, m_i^o)|, \\
d(V, V') &= \max_{i,j} |V(m_i^o, c_j) - V'(m_i^o, c_j)|, \\
d(W, W') &= \max_{i,j} |W(p_{ij}, m_i^o) - W'(p_{ij}, m_i^o)|.
\end{aligned}$$

If $d(Q, Q') < \varepsilon$, $d(V, V') < \eta$, $d(W, W') < \sigma$ for small $\varepsilon, \eta, \sigma > 0$, then go to step 4. Otherwise, set $V \equiv \alpha V + (1-\alpha)V'$, $W \equiv \alpha W + (1-\alpha)W'$, and $Q \equiv \alpha Q + (1-\alpha)Q'$ (with $\alpha = 0.1$) and go back to Step 2.

4. Obtain the entry mass M using the discrete version of the free entry condition,

$$M = \sup\{M : \sum_{i=1}^{n} V'(0, c_i)h(c_i) = \kappa\},$$

and compute the exit rule $c(m^o)$ point-wise as

$$c(m_i^o) = \max\{c_j : V'(m_i^o, c_j) > 0 \text{ and } V'(m_i^o, c_{j+1}) \leq 0\}.$$

5. Using M, and $c(m_i^o)$, and $\mu(m_i^o, c_j)$, obtain the updated measure

$$\mu'(m_i^o, c_j) = \sum_{m \leq m_i^o} \sum_{c' \leq \min\{c_j, c^o(m^o)\}} \mu(m, c') + M \sum_{c' \leq \min\{c_j, c^o(0)\}} h(c').$$

6. Calculate the distance

$$d(\mu, \mu') = \max_{(i,j)} |\mu(m_i^o, c_j) - \mu'(m_i^o, c_j)|.$$

If $d(\mu, \mu') < \xi$ for some small $\xi > 0$, stop. If $M > 0$ and $c(m^o) > c_1$, equilibrium entails positive entry and exit. Otherwise, set $\mu(m^o, c) = \alpha\mu(m^o, c) + (1 - \alpha)\mu'(m^o, c)$, $V \equiv \alpha V + (1 - \alpha)V'$, $W \equiv \alpha W + (1 - \alpha)W'$, $Q \equiv \alpha Q + (1 - \alpha)Q'$ (with $\alpha = 0.1$) and go back to Step 2.

C Calculation of Total Factor Productivity

Plant (establishment) level TFP is aggregated to the firm level by weighting each plant's TFP measure in a given industry by its share of total value of firm's shipments in that industry. Recognizing that many plants manufacture several products that fall within more than just one 4-digit SIC industry, we use the multifactor superlative index number for the revenue-based productivity measurement. This index measures a plant's productivity relative to other plants in its main 4-digit SIC industry – the industry in which the plant has the largest value of shipments. For details on this index, see the discussion of Malmquist productivity indices in Section 4 of Caves, Christensen and Diewert (1982). A similar approach is followed by Bernard, Redding and Schott (2010). An establishment's output is measured using deflated (real) value of shipments plus nominal inventory investments. The inputs are measured using deflated value of equipment, deflated value of plant, total labor hours, deflated value of materials, and deflated value of energy. Labor inputs are measured as an establishment's production-worker hours adjusted by multiplying the production-worker hours by the ratio of total payroll to payroll of production workers. Equipment and plant inputs are establishment's book values for their structure and equipment stocks deflated using sector-specific deflators from the Bureau of Economic Analysis. Materials and energy inputs are based on establishment's expenditures on materials and energy deflated using the input price indices in NBER Productivity Database. For the calculation of 4-digit industry-level cost shares, materials and energy expenditures and payments to labor are aggregated across establishments in the industry. Industry level cost of capital is obtained by first multiplying an establishment's real capital stock with the capital rental rates for the 2-digit industry an establishment belongs to, and then aggregating across establishments in the 4-digit industry. Each input cost is divided by the total cost at the industry level to obtain the cost share of the input. The quantity-based (physical) TFP is calculated in the same way as in Appendix A2 of Foster, Haltiwanger and Syverson (2008). Because the firms in the sample for price regressions are highly specialized in their primary products and these products are highly homogenous, we do not use a multi-product productivity measurement for the case of physical TFP.

D Industry Definitions

A constraint on the selection of the industries in Table 3 was the switch from SIC codes to NAICS codes in 1997, and a further revision in 2002. These changes coincided with a period during which Internet-based commerce diffused rapidly, rendering construction of consistent time series of industry aggregates difficult for other candidate industries. The 1987 SIC (which also applies to 1992), 1997 NAICS and 2002 NAICS codes are comparable for all industries except for Life Insurance. For Life Insurance, the discrepancy due to code revisions is small, as explained below. The following information about the industries analyzed in Table 3 is taken from US Census Bureau's Industry Statistics Sampler.

<u>New Car Dealers</u> (SIC 5511 / NAICS 441110) Establishments primarily engaged in retailing new automobiles and light trucks, such as sport utility vehicles, and passenger and

cargo vans, or retailing these new vehicles in combination with activities, such as repair services, retailing used cars, and selling replacement parts and accessories.

Book Stores (SIC 5942/NAICS 451211) Establishments primarily engaged in retailing new books.

Camera and Photographic Supplies Stores (SIC 5946/NAICS 443130) Establishments primarily engaged in either retailing new cameras, photographic equipment, and photographic supplies or retailing new cameras and photographic equipment in combination with activities, such as repair services and film developing.

Prerecorded Tape, Compact Disc, and Record Stores (SIC 5735/NAICS 451220) Establishments primarily engaged in retailing new prerecorded audio and video tapes, compact discs (CDs), digital video discs (DVDs), and phonograph records.

Travel Agencies (SIC 4724/NAICS 561510) Establishments primarily engaged in acting as agents in selling travel, tour, and accommodation services to the general public and commercial clients.

Life Insurance (SIC 6311/NAICS 524113 and 524130) Establishments primarily engaged in underwriting life insurance. These establishments are operated by enterprises that may be owned by stockholders, policyholders, or other carriers. The sales classified under SIC 3611 are made up of 95% of the sales classified under NAICS 524113 plus nearly all of the sales classified under NAICS 524130. However, NAICS 524130 contains only 2.2% of the total establishments in SIC 6311 and have a small contribution to total sales. Therefore, we exclude NAICS 524130 in calculating the industry statistics for 1997 and 2002.

E Additional Tables and Figures

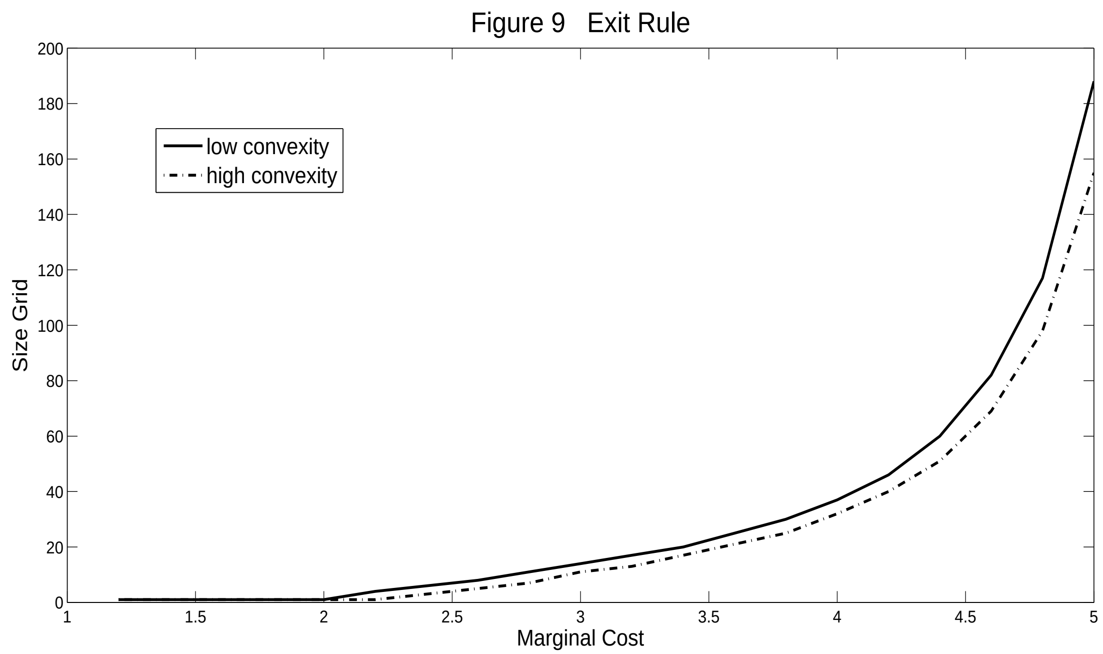

Figure 9: Convexity of information cost: Exit rules

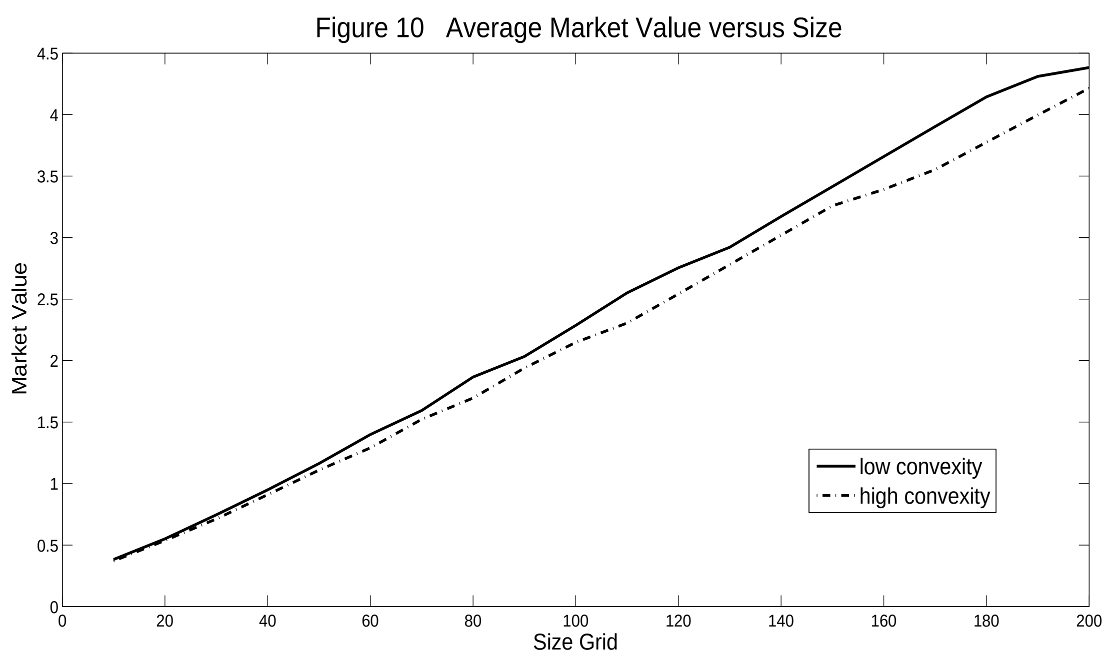

Figure 10: Convexity of information cost: Firm size-value relationship ("0"=smallest)

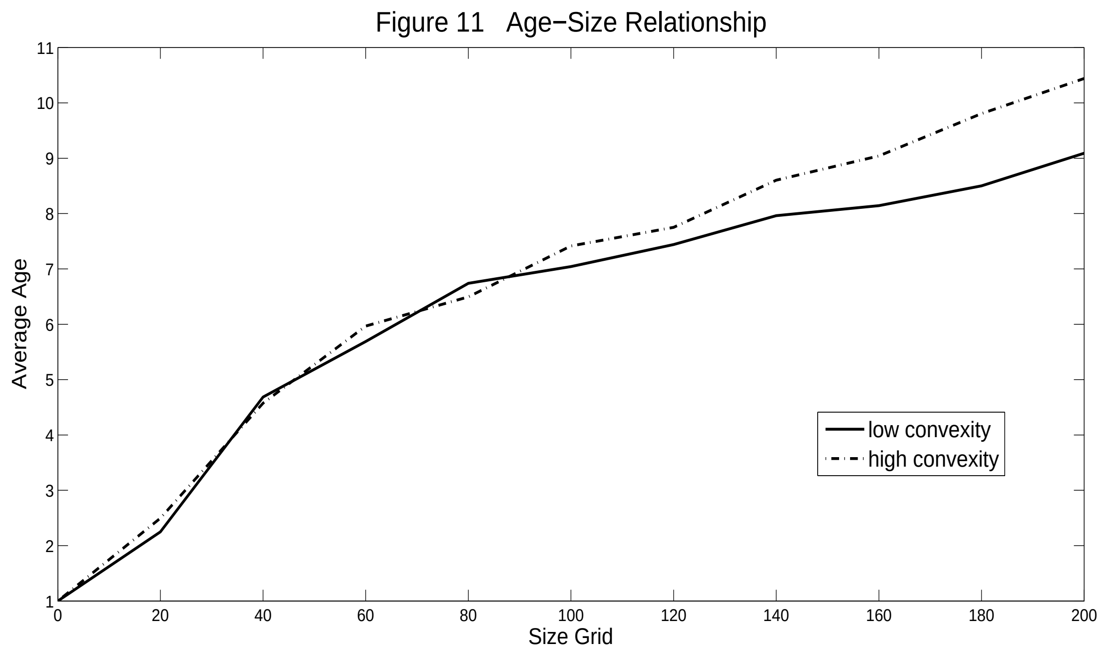

Figure 11: Convexity of information cost: Firm age-size relationship ("0"=smallest)

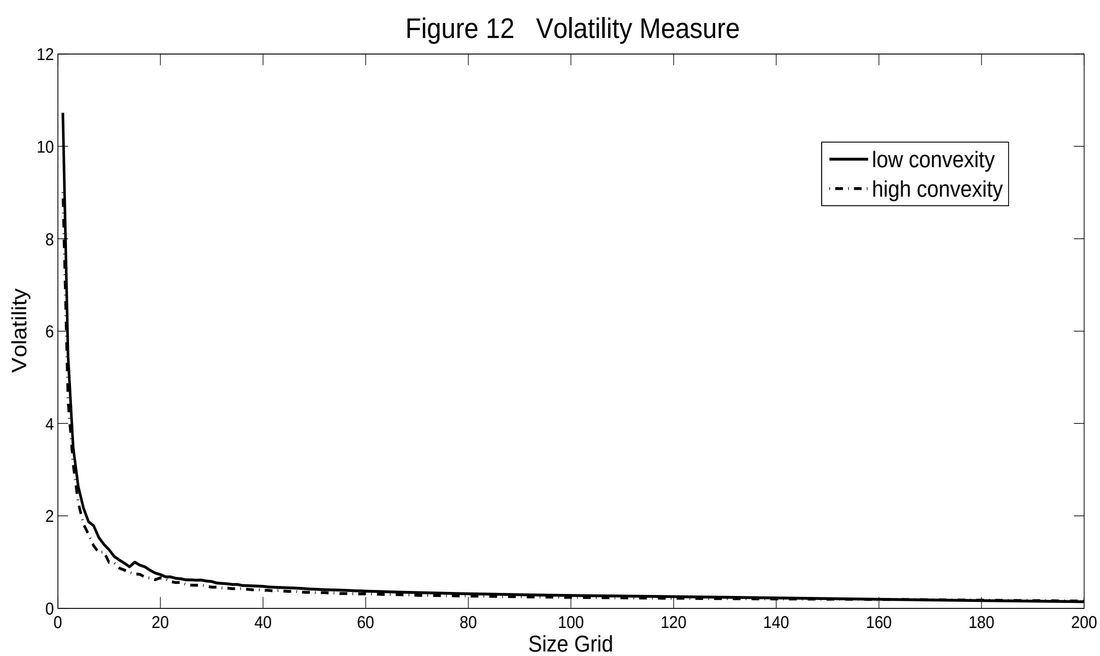

Figure 12: Convexity of information cost: Firm size-volatility relationship ($Vol(m^o)$) ("0"=smallest)

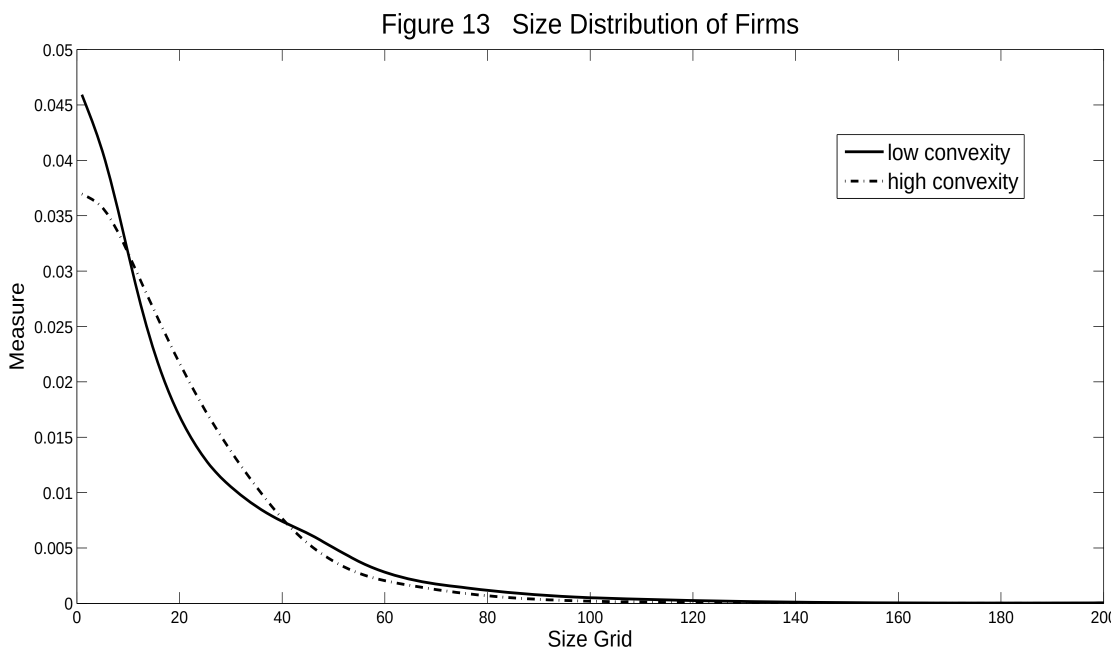

Figure 13: Convexity of information cost: Density of firm size

	Dependent Variable: 1997 Advertising Expenditures ($K)			
	IV		V	
Independent variables:	OLS	Tobit	OLS	Tobit
1996 Value of Shipments ($K)	0.012***	0.013***	-	-
	[0.003]	[0.001]		
1996 Number of Employees	-	-	4.543***	5.113***
			[1.007]	[0.084]
1997 TFP	0.194**	1.341**	0.155**	1.298**
	[0.045]	[0.402]	[0.041]	[0.403]
Multi-unit dummy	334.9	5,771.1***	175.3	5,234.1***
	[298.9]	[345.2]	[367.8]	[276.4]
Industry fixed effects	Y	Y	Y	Y
Inverse Mills Ratio (IMR)	0.149***	-	0.147***	-
	[0.041]		[0.042]	
R^2 / *Log Likelihood*	0.36	-201,878	0.28	-202,345

Notes: All variables are in levels. OLS estimation is based on Heckman's two-step correction. IMR is the ratio of the standard normal density to the standard normal c.d.f. predicted from the selection equation.
Coefficients in OLS estimation (except for that of IMR) are marginal effects that take selection into account.
Corrected standard errors in parentheses. (*), (**), and (***) indicate significance at 10%, 5%, and 1% respectively.
Industry fixed effects are based on 1997 SIC codes (4-digit).

Table A1. The relationship between firm advertising expenditures, size, and productivity (ASM Sample)