Collaborating
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Web Appendix

I Proofs for Section IV

Throughout, let $x := \ln \frac{1-p}{p}$, so that in particular $x_0 = \ln \frac{1-\overline{p}}{\overline{p}}$, and $\beta := \alpha^{-1} - 1$.

Proof of Lemma 2: (Preliminaries.) Consider the objective function under a deadline T. We again use the fact that $\dot{p}_t = -p_t(1-p_t)\sum_i u_{i,t}$, and hence $p_t \sum_i u_{i,t} = d\log(1-p_t)/dt$. We can then rewrite expression (3) as

$$r\int_0^T \left(-\frac{\dot{p}_t}{1-p_t} + \alpha\left(\frac{\dot{p}_t}{p_t(1-p_t)} + u_{-i,t}\right)\right)\frac{1-\overline{p}}{1-p_t}e^{-rt}dt,$$

where $u_{-i,t} := \sum_{j\neq i} u_{j,t}$. Applying integration by parts to the objective, and ignoring irrelevant terms, we obtain

$$\int_0^T \left(r\alpha \ln\frac{p_t}{1-p_t} + \frac{r(\alpha-1)+\alpha u_{-i,t}}{1-p_t}\right)e^{-rt}dt - e^{-rT}\alpha\left(\frac{1-\alpha}{\alpha}\frac{1}{1-p_T} + \ln\frac{1-p_T}{p_T}\right).$$

Making the further change of variable $x_t = \ln((1-p_t)/p_t)$, and defining $\beta := 1/\alpha - 1$, player i maximizes:

$$\int_0^T (-x_t + e^{-x_t}(u_{-i,t}/r - \beta))e^{-rt}dt - \frac{e^{-rT}}{r}\left(\beta\left(1+e^{-x_T}\right) + x_T\right),$$
$$\text{such that } \dot{x}_t = u_{i,t} + u_{-i,t},$$

over functions $u_{i,t}$ in $[0,1]$, given the function $u_{-i,t}$.

The Hamiltonian for this problem is

$$H\left(u_{i,t}, x_t, \gamma_{i,t}\right) := (-x_t + e^{-x_t}(u_{-i,t}/r - \beta))e^{-rt} + \hat{\gamma}_{i,t}\left(u_{i,t} + u_{-i,t}\right),$$

and the salvage value is given by $\phi(x,T) := e^{-rT}\left(\beta\left(1+e^x\right) + x\right)/r$. We now drop the subscript i and, as in the proof of Theorem 1, we assume that $\overline{p} > \alpha$, which is equivalent to $x_0 < \ln\beta$. We also maintain the assumption on the discount rate given in (4), namely $1 + e^{-x_0}(1/r - \beta) > 0$.

(Necessary Conditions.) Define $\gamma_{i,t} := \hat{\gamma}_{i,t} e^{rt}$. By Pontryagin's principle, there must exist a continuous function $\gamma_{i,t}$ for each i, such that,

1. $u_{i,t}$ maximizes $\gamma_{i,t}(u_{i,t} + u_{-i,t})$;
2. $\dot{\gamma}_{i,t} = r\gamma_t + 1 + e^{-x_t}(u_{-i,t}/r - \beta)$;
3. $\gamma_{i,T} = \phi_x(x_T, T) = (\beta e^{-x_T} - 1)/r$.

We again consider a phase diagram in R^{n+1}, with dimensions $\gamma_1, ..., \gamma_n$, and x.

(Candidate Equilibrium.) Our candidate equilibrium strategy $u^*_{i,t}$ generates a trajectory that starts at $(\gamma_1, ..., \gamma_n, x_0) = (0, ..., 0, x_0)$, and has $u_{i,t} = u^*_{i,t} := r(\beta - e^{x_t})/(n-1)$ for $0 \leq t \leq \tilde{T}$, and $u_{i,t} = u^*_{i,t} := 1$ for $\tilde{T} < t \leq T$. This implies $\gamma_{i,t} = 0$ for $t \leq \tilde{T}$ and $\gamma_{i,t} > 0$ for $t > \tilde{T}$. The switching time $\tilde{T}$ is given by the solution to

$$T - \tilde{T} - T(x_{\tilde{T}}) = 0. \tag{1}$$

The function $T(x)$ in equation (1) is defined as

$$T(x) := \frac{1}{n+r} \ln \frac{(n+r)(\beta - e^x)}{\beta + 1}.$$

The equilibrium beliefs x^*_t are given by the solution to $\dot{x}_t = n u^*_{i,t}$. Therefore, for all $t \leq \tilde{T}$, we have $x^*_t = \ln \beta - \ln(1 + (\beta e^{-x_0} - 1)e^{-(n/(n-1))r\beta t})$, and for all $t > \tilde{T}$ we have $x^*_t = x^*_{\tilde{T}} + n(t - \tilde{T})$. It is immediate to verify that $T(x) < T_n(x)$, which is the time it takes for beliefs to reach α when n agents exert maximal effort: stopping occurs before beliefs have gone down to the Marshallian threshold.

We first verify that our candidate strategy is an equilibrium. For all $t \leq \tilde{T}$, agents exert effort at the interior level $u^*_t > 0$. At time $\tilde{T}$, agents switch to maximal effort. When $u^*_t = 1$, necessary condition 2 implies $\dot{\gamma}_{\tilde{T}} > 0$, and hence $\gamma_t > 0$ for all $t \in (\tilde{T}, T]$. Finally, continuity of the function γ^*_t requires that $\gamma^*_{\tilde{T}} = 0$. We therefore need to verify that the solution to the following differential equation,

$$\dot{\gamma}_t = r\gamma_t + 1 + e^{-x^*_t}((n-1)/r - \beta),$$

with boundary condition $\gamma_T = (\beta e^{-x^*_T} - 1)/r$, is equal to zero at $t = \tilde{T}$. Notice that γ_T is positive because $x_T \leq \ln \beta$. Using the fact that $x^*_t = x^*_{\tilde{T}} + n(t - \tilde{T})$, the solution to the differential equation

is given by

$$\gamma_t^* = \frac{(n-1-r\beta)\, e^{n\tilde{T}-x^*_{\tilde{T}}}}{r\,(n+r)}\left(e^{-(n+r)T+rt}-e^{-nt}\right)-\left(\frac{1}{r}-\frac{\beta}{r}e^{-r(T-t)-n\left(T-\tilde{T}\right)-x^*_{\tilde{T}}}\right). \tag{2}$$

The continuity of γ_t^* is verified by evaluating (2) at $t=\tilde{T}$, and setting the right-hand side equal to zero. We then obtain exactly equation (1), which defines $\tilde{T}$.

If $T(x_0)\geq T$, then $\gamma_t>0$ for all t, and agents exert $u_t^*\equiv 1$. This implies γ_t^* is given by (2), where we replace $x^*_{\tilde{T}}$ with x_0. It then suffices to verify that $\gamma_0^*>0$. Indeed, this is the case, because the right-hand side of (2) is increasing in t, decreasing in T, and it would be equal to zero at $t=0$ with a deadline of $T(x_0)>T$.

(Uniqueness.) We now rule out all other symmetric paths. Suppose that, in equilibrium, agents choose effort $u_t=1$ at some time $t_1<\tilde{T}$. This would imply $\gamma_{t_1}\geq 0$. However, if $u_{t_1}=1$, then necessary condition 2 and assumption (4) on the discount rate imply $\dot{\gamma}_{t_1}>0$. Therefore, we might as well consider the case of $\gamma_{t_1}>0$. In this case, agents exert maximal effort, and γ_t increases from time t on. However, let $x_t=x_{t_1}+n(t-t_1)$, and consider the solution to the differential equation $\dot{\gamma}_t=r\gamma_t+1+e^{-x_t}((n-1)/r-\beta)$ with boundary condition $\gamma_T=(\beta e^{-x_T}-1)/r$. Again, the solution γ_t will be strictly increasing. Furthermore, we will have $\gamma_{\tilde{T}}=0$ and therefore $\gamma_{t_1}<0$, since $t_1<\tilde{T}$, contradicting the assumption that $\gamma_{t_1}\geq 0$.

Now suppose agents continue to exert effort at the interior levels of $u_t=r(\beta-e^{x_t})/(n-1)$ for $t>\tilde{T}$. Denote the switching time to maximal effort by $t_2>\tilde{T}$, with $\gamma_{t_2}=0$. The implied path of γ_t^*, given the transversality condition, would then imply $\gamma_{t_2}>0$, which contradicts the assumption of interior effort levels for $t\leq t_2$.

Finally, suppose that agents choose effort $u_t=0$ at any time t_3. This requires $\gamma_{t_3}\leq 0$. However, $\gamma_{t_3}\leq 0$ and $u_{t_3}=0$ also imply $\dot{\gamma}_{t_3}<0$. Therefore, we consider the case of $\gamma_t<0$. In this case, agents exert no effort, and γ_t decreases for all $t\geq t_3$. In particular, this implies $\gamma_T<0$, which violates the transversality condition. Indeed, since $x_t<\ln\beta$ (because $u_t=0$ from time t_3 on), the transversality condition requires $\gamma_T>0$.

(Sufficiency.) The sufficient conditions in the proof of Theorem 1 did not rely on the infinite horizon, so this path is an equilibrium. □

Proof of Theorem 3: Let $V_i(\overline{p}) := V_i(\overline{p}, T(\overline{p}))$. If the deadline is such that effort switches to 1 at time $\tilde{T}$, the payoff of agent i is then

$$V_i(\tilde{T}) := (1-\overline{p})\left(\int_0^{\tilde{T}} \frac{(np_t-\alpha)}{1-p_t} u_{i,t}^* e^{-rt} dt + e^{-r\tilde{T}} \frac{V(p_{\tilde{T}})}{1-p_{\tilde{T}}}\right),$$

where p_t solves $\dot{p}_t = -p_t(1-p_t)nu_{i,t}^*$, $p_0 = \overline{p}$. Taking derivatives with respect to $\tilde{T}$, and considering the derivative at $\tilde{T}=0$ gives

$$\left.\frac{dV_i(\tilde{T})}{d\tilde{T}}\right|_{\tilde{T}=0} = \frac{\alpha((n-1)\overline{p}+r)-\overline{p}r}{(n-\alpha)(n-1)\overline{p}^2}\left((n-\alpha)\overline{p} - \alpha(n-\overline{p})\left(\frac{(n-\alpha)\overline{p}}{\alpha(n-\overline{p})-r(\overline{p}-\alpha)}\right)^{\frac{n}{n+r}}\right).$$

The derivative with respect to r of the term in parenthesis has a derivative equal to (up to a positive multiplicative constant)

$$\ln\left(\frac{(n-\alpha)\overline{p}}{\alpha(n-\overline{p})-r(\overline{p}-\alpha)}\right) - \left(\frac{(n-\alpha)\overline{p}}{\alpha(n-\overline{p})-r(\overline{p}-\alpha)} - 1\right) \leq 0, \tag{3}$$

so $\left.\frac{dV_i(\tilde{T})}{d\tilde{T}}\right|_{\tilde{T}=0}$ is decreasing in r. Since $\left.\frac{dV_i(\tilde{T})}{d\tilde{T}}\right|_{\tilde{T}=0,r=0} = 0$, it follows that $\left.\frac{dV_i(\tilde{T})}{d\tilde{T}}\right|_{\tilde{T}=0} \leq 0$. Because it is optimal to have agents choose level of effort $u=1$ as long as possible, the optimal value is $\tilde{T}=0$: agents should be given a deadline for which it is optimal to exert at a maximal rate immediately.

Finally, differentiating $T(\bar{p})$ with respect to r, one can show that $-\partial T/\partial r$ is exactly equal to expression (3), and so the optimal deadline is increasing in the discount rate. □

II Additional Proofs

A Observable Efforts with Deadlines

We identify an equilibrium in three phases. For a given prior belief $\bar{p}$, agents initially exert interior effort levels (phase 1); they then stop working for an interval of time (phase 2); and finally, they exert maximal effort until the deadline (phase 3). Either phase 1 or both phases 1 and 2 could be empty. We solve for the optimal strategies proceeding backwards in time, and so we start from the deadline T.

(Phase 3.) Suppose the posterior belief p_t and the remaining time $T-t$ are such that

all agents are exerting maximal effort $u_i = 1$. Under the Markov assumption, all agents will continue choosing $u_i = 1$ even after a deviation. Therefore, the individual incentives to deviate from maximal effort are unchanged from the unobservable case, and the proof of Lemma 2 extends to the observable case. In particular, we obtain a critical time $\tilde{t}$ after which agents can sustain maximal effort until the end of the game, given the deadline T and the current posterior p. This time is analogous to $\tilde{T}$ in equation (1), and it is given by

$$\tilde{t}\,(p) = T - \frac{1}{n+r} \ln \frac{(n-\alpha)p}{\alpha(n-p) - r(p-\alpha)}.$$

We now verify the optimality of maximal effort over the time interval $\left[\tilde{t}\,(p)\,, T\right]$. Consider the optimality equation

$$0 = \max_{u_i} \left\{ (u_i + u_{-i})\, p - u_i \alpha - (r + (u_i + u_{-i})p)\, V - (u_i + u_{-i})\, p\,(1-p)\, V_p + V_t \right\}, \tag{4}$$

and let $W\,(p,t)$ indicate the continuation value, given by the returns to n players exerting effort $u_{i,t} = 1$ from time t until the deadline T.

$$W\,(p,t) = (1-p) \int_0^{T-t} \frac{np_s - \alpha}{1-p_s} e^{-rs} ds,$$

with

$$p_s = \frac{p}{p + (1-p)e^{ns}}.$$

We then have

$$W\,(p,t) = \frac{p\,(n-\alpha)}{n+r} \left(1 - e^{-(n+r)(T-t)}\right) - \frac{\alpha}{r}\,(1-p) \left(1 - e^{-r(T-t)}\right).$$

Substitute $W\,(p,t)$ into the optimality equation (4), and consider the incentives to exert effort:

$$\frac{\partial W}{\partial u_i} = p - \alpha - pW - p\,(1-p)\, W_p. \tag{5}$$

It is immediate to verify that the right-hand side of (5) is equal to zero when $t = \tilde{t}\,(p)$. Therefore, agents are indifferent between effort levels along the frontier $\left(p, \tilde{t}\,(p)\right)$. Furthermore, differenti-

ating the right-hand side of (5) with respect to time, we obtain

$$\frac{d}{dt}\left(\frac{\partial W}{\partial u_i}\right) = p\exp\left(-\left(T-t\right)\left(n+r\right)\right)\left(n-\alpha\right) > 0,$$

so agents have strict incentives to work throughout the third phase.

(Phase 2.) If the deadline is long enough, so that $\tilde{t}\left(\bar{p}\right) > 0$, there exists a phase in which players do not exert any effort. In this "shirking" region, the equilibrium value is given by

$$\Omega\left(p,t\right) := e^{-r\left(\tilde{t}\left(p\right)-t\right)}W\left(p,\tilde{t}\left(p\right)\right).$$

We now use $\Omega\left(p,t\right)$ to construct the frontier $\hat{t}\left(p\right)$ that separates the region in the (p,t) space with interior effort from the shirking region. If effort is interior, by the optimality equation, the value function must satisfy the ordinary differential equation

$$p\alpha - V\left(p,t\right) - p\left(1-p\right)V_p\left(p,t\right) = 0, \tag{6}$$

which is obtained by setting by the right-hand side of (5) equal to zero. The general solution of equation (6) is given by

$$V\left(p,t\right) = 1-\alpha+\left(k\left(t\right)-\alpha\ln\frac{p}{1-p}\right)\left(1-p\right). \tag{7}$$

We impose the smooth pasting and value matching conditions of the functions $V\left(p,t\right)$ and $\Omega\left(p,t\right)$:

$$\begin{aligned} V\left(p,\hat{t}\left(p\right)\right) &= \Omega\left(p,\hat{t}\left(p\right)\right) \\ V_p\left(p,\hat{t}\left(p\right)\right) &= \Omega_p\left(p,\hat{t}\left(p\right)\right). \end{aligned}$$

We can then solve for the second switching frontier $\hat{t}\left(p\right)$, and obtain

$$\begin{aligned} \hat{t}\left(p\right) &= T - \frac{1}{n+r}\ln\frac{\left(n-\alpha\right)p}{\alpha\left(n+r\right)-p(\alpha+r)} - \frac{1}{r}\ln\left(1+\frac{\alpha^2\left(1-p\right)\left(n-1\right)}{\left(\alpha\left(n+r\right)-p(\alpha+r)\right)\left(p-\alpha\right)}\right) \\ &= \tilde{t}\left(p\right) - \frac{1}{r}\ln\left(1+\frac{\alpha^2\left(1-p\right)\left(n-1\right)}{\left(\alpha\left(n+r\right)-p(\alpha+r)\right)\left(p-\alpha\right)}\right), \end{aligned}$$

where the log term is always positive. We also obtain the equilibrium value of the constant of

integration $k(t)$ in equation (7):

$$k(t) = \frac{\Omega(\hat{p}(t), t) - (1-\alpha)}{1-\hat{p}(t)} + \alpha \ln \frac{\hat{p}(t)}{1-\hat{p}(t)},$$

where $\hat{p}(t)$ is the inverse function of $\hat{t}(p)$. Finally, we verify the optimality of zero effort in this phase. By construction (value matching), agents are indifferent between levels of effort on the frontier $\left(p, \hat{t}(p)\right)$. Now fix a p and evaluate how the expression is changing with t. We have

$$\frac{d[\text{LHS}(6)]}{dt} = -r(p-\alpha) < 0,$$

so agents have no incentives to exert effort at any time past the frontier $\hat{t}(p)$.

(Phase 1.) In the first phase, interior effort implies $u(p,t)$ must satisfy the optimality equation

$$V(p,t) = 1 - \alpha + (1-p)\left(\frac{\Omega(\hat{p}(t), t) - (1-\alpha)}{1-\hat{p}(t)} + \alpha \ln \frac{\hat{p}(t)}{1-\hat{p}(t)} \frac{1-p}{p}\right),$$

so it must be that

$$u(p,t) = \frac{rV(p,t) - V_t(p,t)}{\alpha(n-1)}.$$

Optimality of interior effort then follows by construction. Furthermore, the equilibrium evolution of beliefs is given by the solution to

$$\dot{p} = -p(1-p)nu(p,t).$$

The next figure illustrates the evolution of beliefs and the loci (p,t) separating the three phases.

B Learning-by-Doing

Following Doraszelski (2003), we model the accumulated knowledge of player i as

$$\dot{z}_i = u_i - \delta z_i,$$

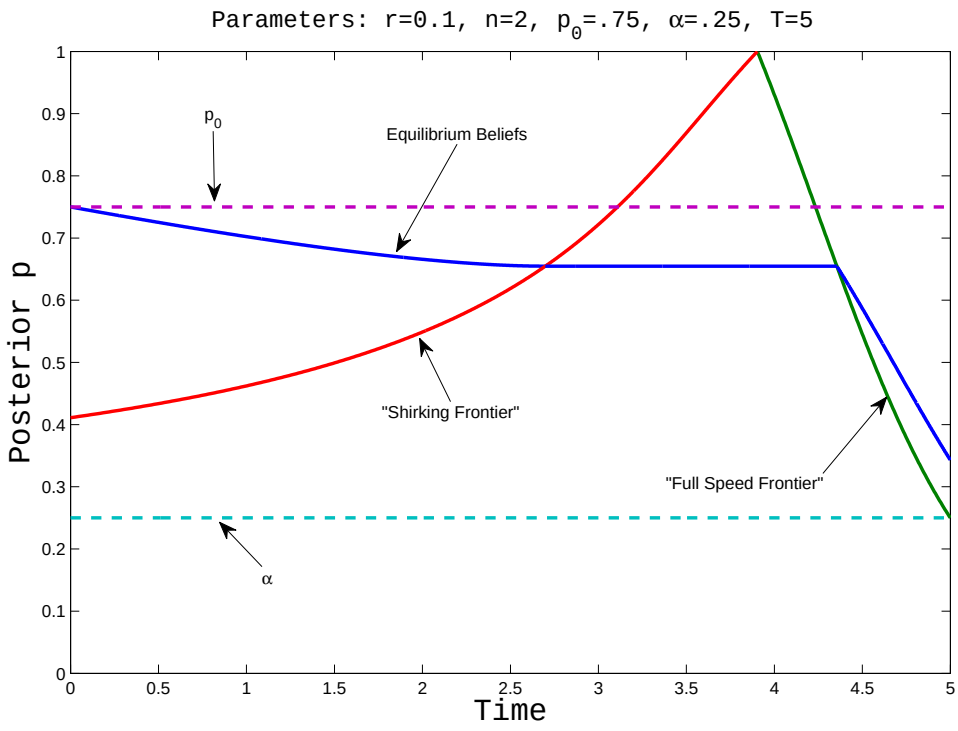


Figure 1: The three phases of the equilibrium

with $z_0 = 0$ (though this boundary condition is not necessary, as we will see). The arrival rate of a breakthrough, or human capital, for player i is given by

$$h_i = \lambda u_i + \rho z_i^{\phi}.$$

It follows that beliefs evolve according to

$$\dot{p}_t = -p_t \left(1 - p_t\right) \sum_i h_{i,t},$$

and therefore player i seeks to maximize:

$$\begin{aligned} V &= \int_0^{\infty} (p_t \sum_i h_{i,t} - \alpha u_{i,t}) e^{-\int_0^t (p_s \sum_j h_{j,s} + r) ds} dt \\ &= \int_0^{\infty} \left(p_t \sum_j h_{j,t} - \alpha u_{i,t} \right) \frac{1 - \bar{p}}{1 - p_t} e^{-rt} dt, \end{aligned}$$

by the usual integration by parts. Defining, as usual $x = \ln((1-p)/p)$, so that $\dot{x} = \sum_i h_{i,t}$, and ignoring the constant $(1-\bar{p})$, we have

$$V = \int_0^\infty (\dot{x}e^{-x} - (1+e^{-x})\alpha u_i)e^{-rt}dt.$$

Again, we integrate the first term by parts, and ignore the values at the endpoint (fixed, or zero), so that maximizing V is equivalent to maximizing

$$\begin{aligned} J &= -\int_0^\infty (re^{-x} + \alpha u_i(1+e^{-x}))e^{-rt}dt \\ &= r\int_0^\infty \left(-\frac{1}{\lambda}x + \frac{\sum_{j\neq i}\left(\lambda u_j + \rho z_j^\phi\right) + \rho z_i^\phi}{\lambda r}(1+e^{-x}) - e^{-x}\left(\frac{1}{\alpha} - \frac{1}{\lambda}\right)\right) e^{-rt}dt, \end{aligned}$$

subject to $\dot{x}_t = \sum_j h_{j,t}$, and $\dot{z}_i = u_i - \delta z_i$. When $\rho = 0$ and $\lambda = 1$ we recover the expression from the proof of Theorem 1,

$$\int_0^\infty (-x_t + e^{-x_t}(u_{-i,t}/r - \beta))e^{-rt}dt.$$

The Hamiltonian is

$$\begin{aligned} H &= \left(-\frac{1}{\lambda}x + \left(\sum_{j\neq i}\left(\lambda u_j + \rho z_j^\phi\right) + \rho z_i^\phi\right)\frac{1+e^{-x}}{\lambda r} - e^{-x}\left(\frac{1}{\alpha} - \frac{1}{\lambda}\right)\right) e^{-rt} \\ &\quad +\gamma\left((n-1)\left(\lambda u_j + \rho z_j^\phi\right) + \lambda u_i + \rho z_i^\phi\right) + \mu\left(u_i - \delta z_i\right). \end{aligned}$$

We now assume $\lambda = 1$ and we seek an interior solution. We then must have

$$\gamma + \mu = 0. \tag{8}$$

The co-state variables obey

$$\begin{aligned} \dot{\gamma} &= -\frac{\partial H}{\partial x} = \left(\frac{1}{\lambda} + e^{-x}\frac{(n-1)\lambda u + n\rho z^\phi}{\lambda r} - e^{-x}\left(\frac{1}{\alpha} - \frac{1}{\lambda}\right)\right) e^{-rt}, \\ \dot{\mu} &= -\frac{\partial H}{\partial z_i} = -\frac{\rho\phi z^{\phi-1}}{\lambda r}\left(1+e^{-x}\right)e^{-rt} - \gamma\rho\phi z^{\phi-1}e^{-rt} + \delta\mu. \end{aligned}$$

We let $g := \gamma e^{rt}$ and $m := \mu e^{rt}$, and then obtain

$$\begin{aligned}\dot{g} &= rg + 1 + e^{-x}\frac{(n-1)\,u + n\rho z^{\phi}}{r} - e^{-x}\beta,\\ \dot{m} &= -\frac{\rho\phi z^{\phi-1}}{r}\left(1+e^{-x}\right) - g\rho\phi z^{\phi-1} + (\delta + r)\,m.\end{aligned}$$

By differentiating condition (8), we can write

$$rg + 1 + e^{-x}\frac{(n-1)\,u + n\rho z^{\phi}}{r} - e^{-x}\beta - \frac{\rho\phi z^{\phi-1}}{r}\left(1+e^{-x}\right) - g\rho\phi z^{\phi-1} + (\delta + r)\,m = 0, \qquad (9)$$

and obtain an expression for g,

$$g = \frac{1}{\rho\phi z^{\phi-1} + \delta}\left(1 - e^{-x}\beta + e^{-x}\frac{(n-1)\,u + n\rho z^{\phi}}{r} - \frac{\rho\phi z^{\phi-1}}{r}\left(1+e^{-x}\right)\right). \qquad (10)$$

By differentiating condition (9), we obtain

$$\begin{aligned}& r\dot{g} - \dot{x}e^{-x}\frac{(n-1)\,u + n\rho z^{\phi}}{r} + e^{-x}\frac{(n-1)\,\dot{u} + n\rho\phi z^{\phi-1}\dot{z}}{r} + \dot{x}e^{-x}\beta - \frac{\rho\phi\,(\phi-1)\,z^{\phi-2}\dot{z}}{r}\left(1+e^{-x}\right)\\ +\quad & \dot{x}e^{-x}\frac{\rho\phi z^{\phi-1}}{r} - \dot{g}\rho\phi z^{\phi-1} - g\rho\phi\,(\phi-1)\,z^{\phi-2}\dot{z} - (r+\delta)\,\dot{g} = 0.\end{aligned}$$

We can then solve for $\dot{u}$, and write

$$\begin{aligned}\dot{u}\,(n-1) &= -\dot{x}\left(\beta r - (n-1)\,u - n\rho z^{\phi} + \rho\phi z^{\phi-1}\right)\\ &\quad - \left(n\rho\phi z - \rho\phi\,(\phi-1)\,(1+e^{x}) - re^{x}g\rho\phi\,(\phi-1)\right)z^{\phi-2}\dot{z}\\ &\quad + \left(\delta + \rho\phi z^{\phi-1}\right)\left(r^{2}e^{x}g + re^{x} + (n-1)\,u + n\rho z^{\phi} - r\beta\right), \qquad (11)\end{aligned}$$

where g is given by (10). We now have three autonomous ordinary differential equations, for $u\,(t)$, $x\,(t)$ and $z\,(t)$. We therefore define

$$\begin{aligned}\zeta\,(x) &= z\,(t)\\ \upsilon\,(x) &= u\,(t)\,,\end{aligned}$$

so that $\zeta'(x)\dot{x} = \dot{z}$ and $\upsilon'(x)\dot{x} = \dot{u}$. We then have

$$\zeta'(x) = \frac{\upsilon - \delta\zeta}{\dot{x}} \tag{12}$$

$$\upsilon'(x) = \frac{[\text{RHS}(11)]}{\dot{x}}, \tag{13}$$

with

$$\dot{x} = n(\lambda\upsilon + \rho\zeta^{\phi}).$$

In order to determine the terminal conditions, we construct a frontier (p, z) with the property that $u(p, z(p)) = 0$ represents a stopping point (remember that $p = (1 + e^{x})^{-1}$).

We can write the agents' equilibrium value as

$$\begin{aligned} 0 &= p(u_i + u_{-i} + h) - \alpha u_i + (1 - r - p(u_i + u_{-i} + h))V(p, h) \\ &\quad + V_h(p, h)\sum_i \phi\rho z_i^{\phi-1}(u_i - \delta z_i) - p(1-p)(u_i + u_{-i} + h)V_p(p, h), \end{aligned}$$

with

$$h := \rho\left(z_i^{\phi} + (n-1)z^{\phi}\right), \text{ and } \dot{h} = \sum_i \phi\rho z_i^{\phi-1}\dot{z}_i = \sum_i \phi\rho z_i^{\phi-1}(u_i - \delta z_i).$$

It follows that the optimality condition for effort provision is given by

$$0 = p - \alpha - pV + V_h\phi\rho z_i^{\phi-1} - p(1-p)V_p.$$

The continuation payoff, given beliefs p and total accumulated knowledge z, can be written as

$$\begin{aligned} V(p, h) &= 1 - r\int_0^{\infty}\left(1 - p + p\exp\left(-\frac{h}{s}\left(1 - e^{-st}\right)\right)\right)e^{-rt}dt \\ &= rp\int_0^{\infty}\left(1 - \exp\left(-\frac{h}{s}\left(1 - e^{-st}\right)\right)\right)e^{-rt}dt, \end{aligned}$$

where

$$s := \delta\phi, \text{ and } h = \rho\left(z_i^{\phi} + (n-1)z^{\phi}\right).$$

Therefore, the stopping frontier (p, z) must solve

$$p(1 - V) + V_{z_i} - p(1-p)V_p - \alpha = 0.$$

However, notice that

$$V_p = \frac{V}{p},$$

and we can therefore express the stopping frontier as the function $z^*(p)$ solving

$$p - V + V_z - \alpha = 0, \tag{14}$$

with

$$V_{z_i} = \phi\rho z_i^{\phi-1} V_h = \phi\rho z_i^{\phi-1} r p_0 \int_0^\infty \frac{1 - e^{-st}}{s} \exp\left(-\frac{h}{s}\left(1 - e^{-st}\right)\right) e^{-rt} dt.$$

Finally, a straightforward argument implies that $u = 0$ at the stopping point. This means we can solve (12) and (13) for $x \in [x_0, \bar{x}]$, with terminal conditions

$$\upsilon(\bar{x}) = 0, \zeta(\bar{x}) = z^*\left(\left(1 + e^{\bar{x}}\right)^{-1}\right).$$

With this procedure we can obtain the pre-images of points along the stopping frontier $z^*(p)$, and trace the paths back to an initial point $(p, z) = (p_0, 0)$.

We now provide illustrations of how parameters affect the frontier, and a generic path in (x, z) path. We start with the impact of the decay rate, δ (Figure 10).

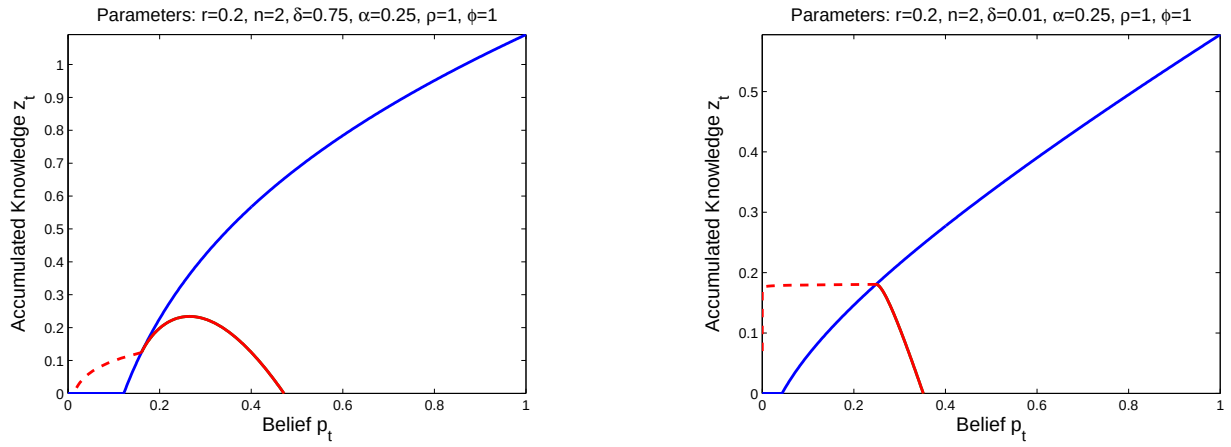


Figure 2: Public belief and accumulated knowledge, as a function of δ

Finally, we consider the impact of the relative importance of accumulated knowledge in the arrival rate of success, ρ (Figure 11).

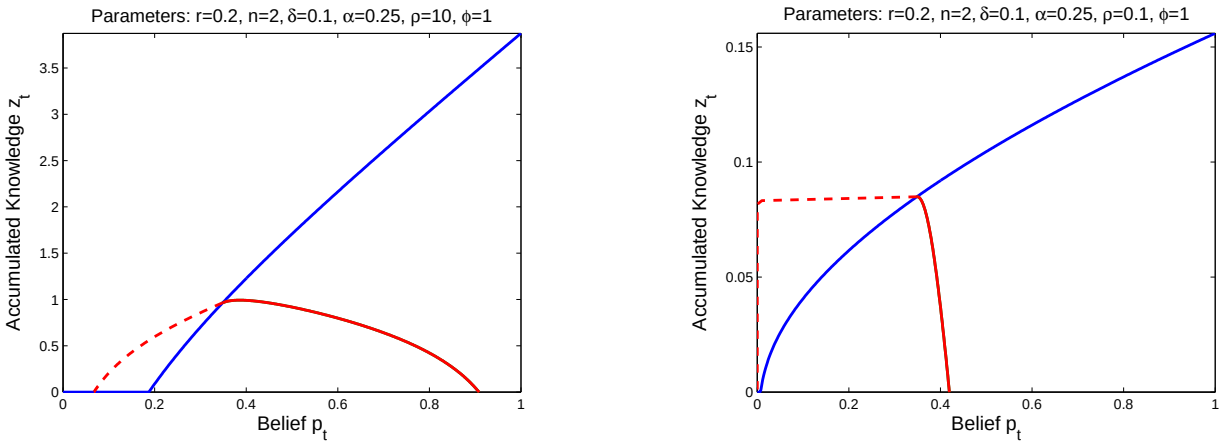


Figure 3: Public belief and accumulated knowledge, as a function of ρ