Online Appendix: “Optimal taxation and debt with uninsurable risks to human capital accumulation”

Piero Gottardi, Atsushi Kajii, and Tomoyuki Nakajima
March 6, 2015

1 Proofs

1.1 Proof of Lemma 1

The proof of this lemma uses an argument similar to Epstein and Zin (1991) and Angeletos (2007). Since the idiosyncratic shocks, \( \theta_{i,t} \), are i.i.d. across individuals and across periods, the utility maximization problem of each individual can be expressed as:

\[
V_t(x) = \max_{c, \eta_h} \left\{ (1 - \beta)c^{1 - \frac{1}{\psi}} + \beta \left( E_t[V_{t+1}(x')^{1-\gamma}] \right)^{\frac{1 - \frac{1}{\psi}}{1 - \gamma}} \right\}
\]

s.t.

\[
x' = (x - c) \left[ R_{k,t+1}(1 - \eta_h) + R_{h,t+1}\theta' \eta_h \right] \geq 0,
\]

\[
c \in [0, x], \quad \eta_h \in [0, 1].
\]

Here, \( V_t(x) \) is the value function for the utility maximization problem of an individual whose total wealth is \( x \) at the beginning of period \( t \). We conjecture that there exists a (deterministic) sequence \( \{v_t\}_{t=0}^\infty \), with \( v_t \in \mathbb{R}_+ \) for all \( t \), such that

\[
V_t(x) = v_t x
\]

Using this conjecture and the budget constraint, we obtain

\[
\left( E_t[V_{t+1}(x')^{1-\gamma}] \right)^{\frac{1}{1-\gamma}} = v_{t+1}(x - c) \left\{ E_t \left[ (R_{k,t+1}(1 - \eta_h^t) + R_{h,t+1}\theta' \eta_h^t)^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}}
\]

It follows that in the above maximization problem the individual chooses the portfolio \( \eta_h \) so as to solve the following maximization problem:

\[
\eta_h = \arg \max_{\eta_h^t \in [0, 1]} \left\{ E_t \left[ (R_{k,t+1}(1 - \eta_h^t) + R_{h,t+1}\theta' \eta_h^t)^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}}
\]

Let \( \rho_{t+1} \) denote the maximized value in this problem. Note that neither \( \eta_h \) nor \( \rho_{t+1} \) depends on the initial state \( x \). That is, under the conjectured value function, all individuals would choose the same portfolio and the same certainty-equivalent rate of return.

Given the certainty-equivalent rate of return, \( \rho_{t+1} \), the level of consumption is chosen so as to solve

\[
\max_{c \in [0, x]} \left\{ (1 - \beta)c^{1 - \frac{1}{\psi}} + \beta [v_{t+1}\rho_{t+1}(x - c)]^{1 - \frac{1}{\psi}} \right\}^{\frac{1}{1 - \frac{1}{\psi}}}
\]
The first-order condition for this problem is

$$(1 - \beta)c^{-\frac{1}{\psi}} = \beta v_{t+1}^{1-\frac{1}{\psi}} \rho_{t+1}^{1-\frac{1}{\psi}} (x - c)^{-\frac{1}{\psi}}$$

which leads to

$$\eta_c = \left\{ 1 + \left( \frac{\beta}{1 - \beta} \right)^{\psi} (v_{t+1} \rho_{t+1})^{\psi-1} \right\}^{-1}$$

where $\eta_c = \frac{c}{x}$.

On the other hand, the Bellman equation implies

$$v_{t-\frac{1}{\psi}} = (1 - \beta)\eta_c^{1-\frac{1}{\psi}} + \beta (v_{t+1} \rho_{t+1})^{1-\frac{1}{\psi}} (1 - \eta_c)^{1-\frac{1}{\psi}}$$

This equation and the above first-order condition for $c$ imply that

$$v_{t}^{\psi-1} = (1 - \beta)^{\psi} + \beta v_{t+1}^{\psi-1} \rho_{t+1}^{\psi-1}$$

The bounded solution to this difference equation is

$$v_t = (1 - \beta)^{\frac{\psi}{\psi-1}} \left\{ 1 + \sum_{s=0}^{\infty} \prod_{j=0}^{s} \left( \beta^{\psi} \rho_{t+1+j}^{\psi-1} \right) \right\}^{\frac{1}{\psi-1}}$$

Also, the consumption rate $\eta_c$ is

$$\eta_{c,t} = (1 - \beta)^{\psi} v_t^{1-\psi}$$

It is straightforward to verify that, constructed in this way, $\{V_t(x), \eta_c, \eta_h\}$ indeed characterizes the solution to the utility maximization problem. The rest of the lemma follows immediately.

1.2 Proof of Proposition 3

Totally differentiating constraint (36) of problem (35), we obtain

$$(\ddot{r} - F_k + F_h - \ddot{w}) \, d\eta_h - (1 - \eta_h) \, d\ddot{r} - \eta_h \, d\ddot{w} = 0.$$ 

Evaluating this expression at the benchmark equilibrium, where $G_t = B_t = 0$, $\ddot{r}_t = \ddot{F}_k$ and $\ddot{w}_t = \ddot{F}_h$, for all $t$, yields

$$(1 - \ddot{\eta}_h) \, d\ddot{r} + \ddot{\eta}_h \, d\ddot{w} = 0.$$ 

Thus, to satisfy the balanced budget, $\ddot{r}$ and $\ddot{w}$ must satisfy the following relationship around $(\ddot{r}, \ddot{w}) = (\ddot{F}_k, \ddot{F}_h)$:

$$\frac{d\ddot{w}}{d\ddot{r}} = -\frac{1 - \ddot{\eta}_h}{\ddot{\eta}_h}.$$
Hence the effect of a marginal change in $\hat{r}$, taking into account the induced change in $\tilde{w}$ via the government budget constraint, is given by $\frac{\partial}{\partial \hat{r}} - \frac{1-\tilde{w}}{\eta_h} \frac{\partial}{\partial \tilde{w}}$ and will be denoted by $\frac{d}{d\hat{r}}$. Since the lifetime utility is increasing in $\rho$, for each $t$, it suffices to show that $\frac{d\rho}{d\hat{r}} > 0$.

The envelope theorem implies that $\frac{\partial\rho}{\partial \eta_h} = 0$ at the benchmark equilibrium. It follows that

$$\frac{d\rho}{d\hat{r}} = \hat{\rho}\gamma E \left[ \hat{R}_x(\theta)^{-\gamma} \left\{ (1 - \hat{\eta}_h) + \theta \hat{\eta}_h \frac{d\tilde{w}}{d\hat{r}} \right\} \right],$$

where $\hat{R}_x(\theta) \equiv (1 - \delta_k + \hat{F}_k)(1 - \hat{\eta}_h) + (1 - \delta_h + \hat{F}_h)\theta \hat{\eta}_h$. Since $E(\theta) = 1$, we have

$$E \left[ \hat{R}_x(\theta)^{-\gamma}(1 - \theta) \right] = \text{Cov}(\hat{R}_x(\theta)^{-\gamma}, 1 - \theta) > 0,$$

where the inequality follows from the fact that both $\hat{R}_x(\theta)^{-\gamma}$ and $1 - \theta$ are decreasing functions of $\theta$. Given that $\hat{\eta}_h < 1$, this proves that $\frac{d\rho}{d\hat{r}} > 0$.

It remains to show that the after-tax rental rate of capital, $\hat{r}$, and the tax rate on capital income, $\tau_k$, move in the opposite directions around the benchmark equilibrium. Since $\tau_k = 1 - \hat{\tau}/\hat{F}_k$, we have

$$\frac{d\tau_k}{d\hat{r}} = -\frac{\hat{F}_k + (-\hat{F}_{kk} + \hat{F}_{kh})\frac{d\rho}{d\hat{r}}}{\hat{F}_k^2}. \tag{43}$$

Differentiating the individual first order conditions (15) yields

$$\left\{ \Phi_{\hat{r}} - \frac{1 - \hat{\eta}_h}{\hat{\eta}_h} \Phi_{\tilde{w}} \right\} d\hat{r} + \Phi_{\eta_h} d\eta_h = 0,$$

so that

$$\frac{d\eta_h}{d\hat{r}} = \frac{\frac{1 - \hat{\eta}_h}{\hat{\eta}_h} \Phi_{\tilde{w}} - \Phi_{\hat{r}}}{\Phi_{\eta_h}}. \tag{44}$$

Thus we obtain

$$\frac{d\tau_k}{d\hat{r}} = \frac{1}{\hat{F}_k^2} \frac{-\hat{F}_k \Phi_{\eta_h} + (-\hat{F}_{kk} + \hat{F}_{kh}) \left( \frac{1 - \hat{\eta}_h}{\hat{\eta}_h} \Phi_{\tilde{w}} - \Phi_{\hat{r}} \right)}{\Phi_{\eta_h}} < 0,$$

since by Assumption 1 we have $\Phi_{\tilde{w}} > 0$, $\Phi_{\hat{r}} < 0$, while $\Phi_{\eta_h} < 0$ follows from the strict concavity of $\rho(\hat{r}, \tilde{w}, \eta_h)$ and $F_{kh} = (1 - \alpha)\alpha k^{\alpha - 1}h^{-\alpha} > 0$. This completes the proof.

### 1.3 Proof of Proposition 4

We are interested in the welfare effect of a marginal variation of $\overline{\theta}_{T+1}$ evaluated at $\overline{\theta}_{T+1} = 0$, that is the sign of $dv_0/d\overline{\theta}_{T+1}|_{\overline{\theta}_{T+1}=0}$. Denote the variables solving the Ramsey problem under (37) as $v_t(\overline{\theta}_{T+1})$, $\rho_t(\overline{\theta}_{T+1})$, etc.. It is immediate to see that its solution is the same as under (34) for all periods except two,

$$\rho_t(\overline{\theta}_{T+1}) = \rho^0, \quad \forall t \neq T + 1, T + 2 \tag{45}$$
We have \( \rho_{T+2}(\bar{b}_{T+1}) = \rho^R(\bar{b}_{T+1}, 0, \eta_{c,T+1}(\bar{b}_{T+1})) \). Recalling again (12), we obtain

\[
v_{T+1}(\bar{b}_{T+1}) = \left\{ (1 - \beta)\psi + \beta\psi \rho_{T+2}(\bar{b}_{T+1})^{\psi^{-1}} v_{T+2}(\bar{b}_{T+1})^{\psi^{-1}} \right\}^{\psi^{-1}}. \tag{46}\]

Here, note that (45) implies \( \partial v_{T+2}/\partial \bar{b}_{T+1} = 0 \). In addition, \( \partial \rho^R(0, 0, \eta_c)/\partial \eta_c = 0 \). Differentiating then \( v_{T+1}(\bar{b}_{T+1}) \) with respect to \( \bar{b}_{T+1} \) and evaluating it at \( \bar{b}_{T+1} = 0 \) yields

\[
\left. \frac{dv_{T+1}}{db_{T+1}} \right|_{\bar{b}_{T+1}=0} = \beta\psi(\rho^R)^{\psi-2}\rho_1^\circ v^\circ, \tag{47}\]

where \( \rho_1^\circ \equiv \partial \rho^R(b, b', \eta_c^\circ)/\partial b \), evaluated at \( b = b' = 0 \).

Next, consider the expression analogous to (46) for date \( T \):

\[
v_T(\bar{b}_{T+1}) = \left\{ (1 - \beta)\psi + \beta\psi (\rho_{T+1}(\bar{b}_{T+1}))^{\psi^{-1}} v_{T+1}(\bar{b}_{T+1})^{\psi^{-1}} \right\}^{\psi^{-1}}. \tag{48}\]

Its derivative with respect to \( \bar{b}_{T+1} \), evaluated at \( \bar{b}_{T+1} = 0 \), using (47) and again the fact that \( \partial \rho^R/\partial \eta_c |_{\bar{b}_{T+1}=b_T=0} = 0 \), equals

\[
\left. \frac{dv_T}{db_{T+1}} \right|_{\bar{b}_{T+1}=0} = \beta\psi(\rho^\circ)^{\psi-2}v^\circ \left[ \rho_2^R + \beta\psi(\rho^\circ)^{\psi^{-1}} \rho_1^R \right],
\]

where \( \rho_2^R \equiv \partial \rho^R(b, b', \eta_c^\circ)/\partial b' \) evaluated at \( b = b' = 0 \).

Let us denote then by \( \lambda(b, b', \eta_c) \) the Lagrange multiplier on the flow budget constraint for the government in problem (32) and by \( \eta_h(b, b', \eta_c), \tilde{\tau}(b, b', \eta_c), \tilde{w}(b, b', \eta_c), \) and \( R_s(b, b', \eta_c) \) its solution. Using the envelope property and the fact that \( b, b' \) only appear in constraint (31) of the problem,

\[\text{Hence from (12) we get } v_t(\bar{b}_{T+1}) = v^\circ, \quad \forall t \geq T + 2, \text{ and } dv_0/dv_T > 0, \text{ so that} \]

\[
\frac{dv_0}{db_{T+1}} \bigg|_{\bar{b}_{T+1}=0} \gtrless 0 \quad \iff \quad \frac{dv_T}{db_{T+1}} \bigg|_{\bar{b}_{T+1}=0} \gtrless 0.
\]

\[27\text{Here and in what follows we omit the dependence of } \rho^R \text{ on } g \text{ whenever } g \text{ is constant across periods.} \]

\[28\text{To see this, recall from the definition of } \rho^R(b, b', \eta_c) \text{ in (32) that } \eta_c \text{ affects } \rho^R \text{ only through the government budget constraint (31). Consider the associated function:} \]

\[
f(b, b', \eta_c, \eta_h, \tilde{\tau}, \tilde{w}, R_s) = g + (1 - \delta_h + \tilde{\tau})b - (1 - \eta_h)R_s b' - F [(1 - \eta_c)(1 - \eta_h) - b, (1 - \eta_h)\eta_h]
+ \tilde{\tau}[(1 - \eta_c)(1 - \eta_h) - b] + \tilde{w}(1 - \eta_h)\eta_h
\]

We have \( \left. \frac{\partial f}{\partial \eta_c} \right|_{b = b' = 0} = 0 \) and so, by the envelope theorem we get the claimed property.

\[29\text{The superscript } ^\circ \text{ indicates, as in the main text, variables evaluated at a solution of the Ramsey problem under the constraint } b_t = g_t = 0 \text{ for all } t. \]
we obtain, when $b_t = g_t = 0$ for all $t$:\footnote{To better understand the form of these expressions, notice that, as we see from (31), a marginal increase of $b_{T+1}$ relaxes this constraint at $T + 1$ yielding a gain of $\lambda^o (1 - \eta^c) R^o_z$, while tightening this constraint at $T + 2$ with a loss of $\lambda^o \beta^\psi (\rho^o)^{\psi-1} (1 - \delta_k + F^o_k)$ (recall that $\rho^1_R$ is multiplied by $\beta^\psi (\rho^o)^{\psi-1}$ in the expression of $dv_T/db_{T+1}$). Since $(1 - \eta^c) = \beta^\psi (\rho^o)^{\psi-1}$, the comparison of these two reduce to the comparison between $R^o_z$ and $(1 - \delta_k + F^o_k)$.}

\[
\begin{align*}
\rho^1_R &= -\lambda^o (1 - \delta_k + F^o_k), \\
\rho^2_R &= \lambda^o \beta^\psi (\rho^o)^{\psi-1} R^o_z,
\end{align*}
\]

since

\[
\eta^c = 1 - \beta^\psi (\rho^o)^{\psi-1}.
\]

Therefore,

\[
\frac{dv_T}{db_{T+1}} = \xi [R^o_z - (1 - \delta_k + F^o_k)],
\]

where

\[
\xi \equiv \beta^2 (\rho^o)^{2\psi-3} \lambda^o \nu^o
\]

and $\xi > 0$ since $\lambda^o > 0$, as we show next. As argued in Section 3.1, when $b_t = g_t = 0$ for all $t$, problem (32) reduces to (35).

Let us write the solution to (10) as $\eta_h(\tilde{r}, \tilde{w})$. Then the first order conditions for $\tilde{r}$ and $\tilde{w}$ in problem (35) are given by

\[
\begin{align*}
0 &= \frac{\partial \rho}{\partial \tilde{r}} - (1 - \eta^c) \lambda^o + \left[ \frac{\partial \rho}{\partial \eta_h} + \lambda^o (F^o_k - F^o_h + \tilde{r}^o - \tilde{w}^o) \right] \frac{\partial \eta_h}{\partial \tilde{r}}, \\
0 &= \frac{\partial \rho}{\partial \tilde{w}} - \eta^c \lambda^o + \left[ \frac{\partial \rho}{\partial \eta_h} + \lambda^o (F^o_k - F^o_h + \tilde{r}^o - \tilde{w}^o) \right] \frac{\partial \eta_h}{\partial \tilde{w}}.
\end{align*}
\]

From the second equation, recalling that under Assumption 1 we have $\frac{\partial \eta_h}{\partial \tilde{w}} > 0$ and $\frac{\partial \eta_h}{\partial \tilde{r}} < 0$, we obtain

\[
\lambda^o (F^o_k - F^o_h + \tilde{r}^o - \tilde{w}^o) = -\frac{\partial \rho}{\partial \tilde{w}} + \eta^c \lambda^o.
\]

Substituting then this equation into the first equation above, and solving for $\lambda^o$, we get

\[
\lambda^o = \left( 1 - \eta^c - \frac{\eta^c}{\partial \eta_h} \frac{\partial \eta_h}{\partial \tilde{w}} \right)^{-1} \left( \frac{\partial \rho}{\partial \tilde{r}} - \frac{\partial \rho}{\partial \tilde{w}} \frac{\partial \eta_h}{\partial \tilde{r}} \right) > 0,
\]

where the sign of the inequality follows from the fact that $\eta^c_h \in (0,1)$, $\frac{\partial \rho}{\partial \tilde{r}} > 0$ and $\frac{\partial \rho}{\partial \tilde{w}} > 0$. 

1.4 Proof of Proposition 5

The Lagrangean for problem (33), using (12) and (14) to substitute for \( \rho_{t+1} \) and \( \eta_{c,t} \), is

\[
v_0 + \sum_{t=0}^{\infty} \lambda_t^v \left\{ (1 - \beta)^\psi + \beta^\psi \rho^R(b_t, b_{t+1}, (1 - \beta)^\psi v_t^{1-\psi} v_{t+1}^{\psi-1} - v_t^{\psi-1} \right\}.
\]

The first-order condition with respect to \( b_{t+1} \) is then

\[
\lambda_t^v \beta^\psi \rho_{t+1} \rho_2 \rho_{t+1} v_{t+1}^{\psi-1} + \lambda_{t+1}^v \beta^\psi \rho_{t+2} \rho_{t+2} v_{t+2}^{\psi-1} = 0,
\]

where \( \rho_{t+1} \equiv \rho^R(b_t, b_{t+1}, \eta_{c,t}) \), \( \rho_{2,t+1} \equiv \partial \rho^R(b_t, b_{t+1}, \eta_{c,t})/\partial b_{t+1} \), and \( \rho_{1,t+2} \equiv \partial \rho^R(b_{t+1}, b_{t+2}, \eta_{c,t+1})/\partial b_{t+1} \). The first-order condition for \( v_{t+1} \) is

\[
\lambda_t^v \beta^\psi \rho_{t+1} v_{t+1}^{\psi-2} + \lambda_{t+1}^v \beta^\psi \rho_{t+2} (1 - \beta)^\psi (1 - \psi) v_{t+1}^{\psi-2} \lambda_{t+1}^v v_{t+1}^{\psi-1} = 0,
\]

where \( \rho_{t+2} \equiv \partial \rho^R(b_{t+1}, b_{t+2}, \eta_{c,t+1})/\partial \eta_{c,t+1} \).

In a steady-state equilibrium, equation (50) reduces to

\[
\rho_{2}^R + \frac{\lambda_{t+1}^v}{\lambda_t^v} \rho_{1}^R = 0
\]

and equation (51) to

\[
\frac{\lambda_{t+1}^v}{\lambda_t^v} = \beta^\psi \rho_{t+1}^{\psi-1} \left( 1 - \beta^\psi \rho_{t+1}^{\psi-1} (1 - \beta)^\psi (1 - \psi) \frac{\rho_{t+1}^{\psi-1}}{\rho} \right)^{-1},
\]

where the term in parenthesis captures the effect on \( \rho \) of the change in the savings rate, given by the second term in (51), which only arises (as we saw in footnote 30) when debt is nonzero.

By a similar argument to the one in the proof of Proposition 4 above, at a steady state equilibrium the derivative of \( \rho^R \) with respect to \( b \) and \( b' \) satisfies

\[
\frac{\rho_{1}^R}{\rho_{2}^R} = \frac{1 - \delta_k + F_k}{(1 - \eta_c)R_x} = \frac{1 - \delta_k + F_k}{\beta^\psi \rho_{1}^{\psi-1} R_x}.
\]

where, for the second equality, we used again (14), \( \eta_c = (1 - \beta)^\psi v_{t+1}^{1-\psi} \), and constraint (12), \( v_{t+1}^{\psi-1} = (1 - \beta)^\psi + \beta^\psi \rho_{1}^{\psi-1} v_{t+1}^{\psi-1} \), of problem (33).

Combining (52)-(54) and using again (14), yields the claimed result:

\[ R_x = (1 - \delta_k + F_k) \left[ 1 - (1 - \psi)\beta^\psi \rho_{1}^{\psi-2} \rho_{1}^{\psi-1} \right]^{-1}. \]
2 Sufficient conditions for Assumption 1

Let us rewrite problem (9) more compactly as

$$\max_{\eta_h \geq 0} E [u (r (1 - \eta_h) + \theta w \eta_h)] ,$$

where, with a slight abuse of notation, \(r\) denotes \(1 - \delta + \bar{r}\), \(w\) denotes \(1 - \delta_h + \bar{w}\), and the function \(u(.)\) is increasing, concave and with a constant coefficient of relative risk aversion \(\gamma\). Letting \(\eta_h^*\) be an interior solution of (9), the properties stated in Assumption 1 are equivalent to \(\frac{\partial \eta_h^*}{\partial r} < 0\) and \(\frac{\partial \eta_h^*}{\partial w} > 0\), as already noticed in the main text. Setting \(R \equiv \theta w - \alpha\), problem (9) may also be written as

$$\max_{\eta_h \geq 0} E [u (r + R \eta_h)] ,$$

(55)

when \(\alpha = r\). Problem (55) is often referred to as the standard portfolio choice problem. Hereafter, we shall use some results on such problem reported in Gollier (2004).\(^{31}\)

From Proposition 9 in Gollier (2004) it follows that, when the coefficient of relative risk aversion \(\gamma\) is not larger than one, any first order stochastic improvement in \(R\) increases the optimal value of \(\eta_h\). Since an increase in \(w\) induces such an improvement, we conclude that \(\frac{\partial \eta_h^*}{\partial w} > 0\) if \(\gamma \leq 1\).

Note that an increase in \(r\), keeping \(R\) (that is, \(\alpha\)) constant, constitutes an increase in wealth and so from Proposition 8 in Gollier (2004) it follows that this change induces a decrease in \(\eta_h^*\) if \(u\) exhibits decreasing absolute risk aversion. With constant relative risk aversion, \(u\) indeed exhibits decreasing absolute risk aversion. There is then a second effect of the increase in \(r\), given by the change in \(R\) : an increase in \(\alpha\) induces a first order worsening on \(R\) and so reduces \(\eta_h^*\) if \(\gamma \leq 1\). Hence we conclude that \(\frac{\partial \eta_h^*}{\partial r} < 0\) if \(\gamma \leq 1\).

Having established that the stated properties always hold when \(\gamma \leq 1\), we show next that, when \(\gamma > 1\), they hold for some family of distributions of \(\theta\). Assuming that \(\theta\) is a continuous random variable with density function \(g(t)\) differentiable almost everywhere, we shall show below that the stated comparative statics properties hold if both \(t \frac{g'(t)}{t} \) and \(\frac{g'(t)}{t}\) are non-increasing in \(t\). The condition hold for example when \(\theta\) is a uniform distribution over some interval, or a Pareto distribution (i.e., the density function is a power function).

To establish the result we build on Proposition 17 in Gollier (2004), stating that, if \(u(.)\) is strictly increasing, then any improvement in \(R\) in monotone likelihood ratio (MLR) increases the optimal value \(\eta_h^*\) of problem (55). That is, if \(R\) and \(R'\) are distinct continuous random variables with density \(f_R\) and \(f_{R'}\) respectively, the optimal value \(\eta_h^*\) under \(R'\) is larger than that under \(R\) if \(f_{R'}(t) / f_R(t)\) is non decreasing in \(t\).

Since \(R = \theta w - \alpha\), \(\Pr [R \leq z] = \Pr [\theta \leq (z + r) / w]\) and so the density function \(f(z)\) of \(R\) is

given by

\[ f(z) = \frac{d}{dz} \int_0^{(z+r)/w} g(t) \, dt = \frac{1}{w} g \left( \frac{z + r}{w} \right). \tag{56} \]

So in order to use the above proposition to establish the property \( \frac{\partial \eta^*_h}{\partial w} > 0 \), it suffices to show that for any \( \hat{w} > w \) \( \frac{1}{w} g \left( \frac{z + \hat{r}}{w} \right) / \frac{1}{w} g \left( \frac{z + r}{w} \right) \) is non decreasing in \( z \). Taking a monotone (logarithmic) transformation and differentiating with respect to \( z \), this condition obtains when

\[ \frac{1}{w} g' \left( \frac{z + \hat{r}}{w} \right) - \frac{1}{w} g' \left( \frac{z + r}{w} \right) \geq 0, \]

that is, when

\[ \frac{1}{w} g' \left( \frac{z + \hat{r}}{w} \right) \text{ is non-decreasing in } w, \]

at any \( w > 0 \), for given \( z \) and \( r \). Since the map \( w \mapsto (z + r)/w \) is monotonic and decreasing, setting \( t = (r + z)/w \), the condition above can be equivalently stated as

\[ t g'(t) g(t) \text{ is non-increasing in } t. \]

Next, we use the same proposition to derive a condition guaranteeing that \( \frac{\partial \eta_h}{\partial r} < 0 \). Recalling the argument above regarding the effect of increasing \( r \) keeping \( R \) constant, when \( u(\cdot) \) exhibits decreasing absolute risk aversion, it suffices to show that the optimal value of \( \eta_h^* \) decreases as \( \alpha \) in \( R = w\theta - \alpha \) increases, keeping \( r \) fixed. Hence we derive next a condition on \( g(t) \) such that a decrease in \( \alpha \) induces a MLR improvement: that is, for any \( \hat{\alpha} < \alpha \) \( \frac{1}{w} g \left( \frac{z + \hat{\alpha}}{w} \right) / \frac{1}{w} g \left( \frac{z + \alpha}{w} \right) \) is non decreasing in \( z \). Arguing analogously as in the previous case, we can show that this property holds if \( g' \left( \frac{z + \hat{\alpha}}{w} \right) / g \left( \frac{z + \alpha}{w} \right) \) is non increasing in \( \alpha \) at any \( \alpha > 0 \), where \( z \) and \( w \) are fixed. So changing variables we conclude that \( \frac{\partial \eta_h}{\partial r} < 0 \) holds if

\[ \frac{g'(t)}{g(t)} \text{ is non-increasing in } t. \]

3 Exogenous government purchases

Here we extend our analysis to the case where the public expenditure policy is specified in terms of an exogenous sequence of absolute levels of expenditure \( \{G_t\}_{t=0}^\infty \) (rather than per unit of total wealth). We will obtain conditions characterizing the Ramsey steady state which are analogous to those obtained in Proposition 5 and Corollary 6. Hence, also in the case of exogenous \( G_t \), the capital income tax rate must be positive in the long run, as long as the effect on the saving rate is small enough.

When the sequence \( \{G_t\}_{t=0}^\infty \) is exogenously given, we can no longer use the recursive approach followed in the paper to solve the Ramsey problem in the case where \( \{g_t\}_{t=0}^\infty \) is exogenously given.
We solve instead the problem in a more direct way. Given $X_0$ and $b_0$, the Ramsey problem consists in the maximization of $v_0$ with respect to $\{b_{t+1}, X_{t+1}, v_{t+1}, \bar{r}_{t+1}, \bar{w}_{t+1}\}_{t=0}^{\infty}$ subject to

$$v_t^{\psi-1} = (1 - \beta)^{\psi} + \beta^\psi p_{t+1}^{\psi-1} v_{t+1}$$

$$G_{t+1} \over X_t + (1 - \delta_k + \bar{r}_{t+1}) b_t = (1 - \eta_{c,t}) R_{x,t+1} b_{t+1} + F(k_t, h_t) - \bar{r}_{t+1} k_t - \bar{w}_{t+1} h_t$$

$$X_{t+1} = (1 - \eta_{c,t}) R_{x,t+1} ,$$

where $\eta_{h,t}, \eta_{c,t}, \rho_{t+1}, R_{x,t+1}, k_t,$ and $h_t$ are the following functions of $\bar{r}_{t+1}, \bar{w}_{t+1}, b_t,$ and $v_t$:

$$\eta_{h,t} = \eta_h(\bar{r}_{t+1}, \bar{w}_{t+1}) \equiv \arg \max_{\eta_h} \rho(\bar{r}_{t+1}, \bar{w}_{t+1}, \eta_h),$$

$$\rho_{t+1} = \rho(\bar{r}_{t+1}, \bar{w}_{t+1}) \equiv \max_{\eta_h} \rho(\bar{r}_{t+1}, \bar{w}_{t+1}, \eta_h),$$

$$R_{x,t+1} = R_x(\bar{r}_{t+1}, \bar{w}_{t+1}) \equiv (1 - \delta_k + \bar{r}_{t+1})(1 - \eta_h(\bar{r}_{t+1}, \bar{w}_{t+1})) + (1 - \delta_h + \bar{w}_{t+1}) \eta_h(\bar{r}_{t+1}, \bar{w}_{t+1}),$$

$$\eta_{c,t} = \eta_c(v_t) \equiv (1 - \beta)^{\psi}(v_t)^{1-\psi} ,$$

$$k_t = k(\bar{r}_{t+1}, \bar{w}_{t+1}, b_t, v_t) \equiv (1 - \eta_c(v_t))(1 - \eta_h(\bar{r}_{t+1}, \bar{w}_{t+1})) - b_t ,$$

$$h_t = h(\bar{r}_{t+1}, \bar{w}_{t+1}, v_t) \equiv (1 - \eta_c(v_t)) \eta_h(\bar{r}_{t+1}, \bar{w}_{t+1}) ,$$

The Lagrangean for this problem is then:

$$v_0 + \sum_{t=0}^{\infty} \left[ \lambda_{v,t} \left\{ (1 - \beta)^{\psi} + \beta^\psi \rho(\bar{r}_{t+1}, \bar{w}_{t+1})^{\psi-1} v_{t+1}^{\psi-1} - v_t^{\psi-1} \right\} ight.$$ 

$$+ \lambda_{b,t} \left\{ [1 - \eta_c(v_t)] R_x(\bar{r}_{t+1}, \bar{w}_{t+1}) b_{t+1} + F[k(\bar{r}_{t+1}, \bar{w}_{t+1}, b_t, v_t), h(\bar{r}_{t+1}, \bar{w}_{t+1}, v_t)] 

- \bar{r}_{t+1} k(\bar{r}_{t+1}, \bar{w}_{t+1}, b_t, v_t) - \bar{w}_{t+1} h(\bar{r}_{t+1}, \bar{w}_{t+1}, v_t) - G_{t+1} \over X_t - (1 - \delta_k + \bar{r}_{t+1}) b_t \right\} 

+ \lambda_{x,t} \left\{ [1 - \eta_c(v_t)] R_x(\bar{r}_{t+1}, \bar{w}_{t+1}) - X_{t+1} \over X_t \right\} \right].$$

The first order conditions for $v_t, b_t,$ and $\bar{r}_{t+1}$ are so, respectively,\(^{32}\)

$$0 = -\lambda_{v,t} v_t^{\psi-2} \over \psi - 1 + \lambda_{v,t-1} \beta^\psi v_{t+1}^{\psi-1} + \lambda_{b,t} \eta_{c,t}^{\psi}(v_t) \left\{ -R_{x,t+1} b_{t+1} - F_{k,t}(1 - \eta_h,t) - F_{h,t} \eta_h,t + \bar{r}_{t+1}(1 - \eta_h,t) + \bar{w}_{t+1} \eta_h,t \right\}$$

$$- \lambda_{x,t} \eta_{c,t}^{\psi}(v_t) R_{x,t+1},$$

$$0 = \lambda_{b,t-1} (1 - \eta_{c,t}) R_{x,t} - \lambda_{b,t} (1 - \delta_k + F_{k,t}) ,$$

$$0 = (\psi - 1) \lambda_{v,t} \beta^\psi R_{v,t}^{\psi-2} p_{t+1}^{\psi-1} + \lambda_{b,t} \left\{ (1 - \eta_{c,t}) R_{x,t+1} b_{t+1} + F_{k,t} k_{r,t} + F_{h,t} h_{r,t} - k_t - \bar{r}_{t+1} k_{r,t} - \bar{w}_{t+1} h_{r,t} - b_t \right\}$$

$$+ \lambda_{x,t} (1 - \eta_{c,t}) R_{x,t+1} .$$

\(^{32}\)To derive the steady state condition determining the tax rate on capital we do not have to use the first-order conditions with respect to $\bar{w}_{t+1}$ or $X_{t+1}$. But, of course, we would need those conditions to derive all the steady state equilibrium variables.
where $\eta'_c(v_t) \equiv d\eta_c(v_t)/dv_t$, $F_{k,t} \equiv \partial F(k_t, h_t)/\partial k_t$, $F_{h,t} \equiv \partial F(k_t, h_t)/\partial h_t$, $\rho_{r,t+1} \equiv \partial \rho(\tilde{r}_{t+1}, \tilde{w}_{t+1})/\partial \tilde{r}_{t+1}$, $R_{x,t,t+1} \equiv \partial R_x(\tilde{r}_{t+1}, \tilde{w}_{t+1})/\partial \tilde{r}_{t+1}$, $k_{r,t} \equiv \partial k(\tilde{r}_{t+1}, \tilde{w}_{t+1}, b_t, v_t)/\partial \tilde{r}_{t+1}$, and $h_{r,t} \equiv \partial h(\tilde{r}_{t+1}, \tilde{w}_{t+1}, v_t)/\partial \tilde{r}_{t+1}$.

Assuming that $G_t$ grows at an exogenous, constant rate $\gamma$, we focus again our attention on a steady state (balanced growth path) where all the variables in equations (57)-(59) remain constant, except for the Lagrange multipliers, $\lambda_{v,t}$, $\lambda_{b,t}$, and $\lambda_{x,t}$ that grow at the same rate:

$$\frac{\lambda_{v,t}}{\lambda_{v,t-1}} = \frac{\lambda_{b,t}}{\lambda_{b,t-1}} = \frac{\lambda_{x,t}}{\lambda_{x,t-1}} \equiv \gamma.$$ 

Since $\rho$ is constant we have $v = (1 - \beta)^\psi / (1 - \beta^\psi \rho^\psi - 1)$. Also, $\eta_c = (1 - \beta)^\psi v^{1-\psi}$, and so

$$\beta^\psi \rho^{\psi - 1} = 1 - \eta_c.$$

It then follows from equation (57) that, along a balanced growth path,

$$\frac{\lambda_{v,t}}{\lambda_{v,t-1}} = (1 - \eta_c) + \Lambda \eta'_c(v),$$

where $\Lambda$ is the term

$$\Lambda \equiv \frac{\psi - 1}{\psi \psi^2} \left[ \frac{\lambda_{b,t}}{\lambda_{v,t-1}} \left\{ -R_x b - F_k(1 - \eta_h) - F_h \eta_h + \tilde{r}(1 - \eta_h) + \tilde{w} \eta_h \right\} - \frac{\lambda_{x,t}}{\lambda_{v,t-1}} R_x \right],$$

a constant given the fact that all Lagrange multipliers grow at the same rate.

We can then use equation (58) to derive the following steady-state condition which is the counterpart of the one in Proposition 5:

$$R_x = (1 - \delta_k + F_k) \left[ 1 + \frac{\Lambda \eta'_c(v)}{1 - \eta_c} \right].$$

(60)

Just as in the case of a constant, exogenously given level of $g$, this condition implies that at a Ramsey steady state the average rate of return on consumers’ portfolios, $R_x$, is equal to the before tax return on physical capital (or equivalently the cost of government debt), $1 - \delta_k + F_k$, augmented with the effect of public debt on the saving rate, $\Lambda \eta'_c/(1 - \eta_c)$. As long as the latter effect is small, we get again $R_x \approx 1 - \delta_k + F_k$, which implies that the optimal capital tax rate is positive in the long run: $\tau_k > 0$.

When $\psi = 1$, again the effect on the saving rate vanishes, so that condition (60) reduces to

$$R_x = 1 - \delta_k + F_k,$$

which is identical to the condition derived in Corollary 6.

4 Algorithm to solve the model numerically

The Ramsey equilibrium for our model can be computed in a straightforward way. The function $\rho^R(b, b', \eta_c)$ is computed as the solution to the maximization problem defined in (32). Then the steady state value of $b$ is obtained by solving equation (39).
The transitional dynamics is computed for the calibrated economy where $\psi = 1$. In this case $\eta_c$ is constant, so the function above can be written simply as $\rho^R(b, b')$ and (30) simplifies to

$$\ln(v_0) = \sum_{t=0}^{\infty} \beta^{t+1} \ln(\rho_{t+1}).$$

In the dynamic programming formulation, the Ramsey problem (33) can be written as

$$\ln v(b) = \max_{b'} \beta \ln \rho^R(b, b') + \beta \ln v(b').$$

This problem is solved by discretizing the state space and by the value function iteration.

5 Transient dynamics

The Ramsey equilibrium converges to the steady state only in one period. Figure 1 in this appendix illustrates the transitional dynamics of the Ramsey equilibrium, starting from the “baseline equilibrium” in Table 2 in the main text.
Figure 1: Transitional dynamics of the Ramsey equilibrium starting from the baseline equilibrium.