Systemic Risk and Stability in Financial Networks (Online Appendix)

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This appendix contains the proofs of Propositions 9–12, Propositions A.1–A.4, and Corollary A.1 omitted from the main body of the paper.

Proof of Proposition 9

Proof of part (a). First consider the complete financial network. If $\epsilon < \epsilon_p^*$, at least one bank does not default. Given the symmetry, all $n-p$ banks that are not hit with a negative shock do not default either, implying that the complete network is the most stable and resilient financial network in the face of small shocks.

Now consider the ring financial network and assume that p consecutive banks, labeled $i+1$ through $j = i+p$, are hit with negative shocks. An immediate observation is that all banks in default also form a connected chain, say of length $\tau \geq p$, the last of which is labeled $s = i+\tau$. In view of Lemma B.6, bank i does not default, as it is the bank furthest away from the realized shocks. As a result, as long as $y > y_p^* = (n-p)(a-v)$, in the unique payment equilibrium of the financial network, all banks can meet their senior liabilities v in full. This can be established by verifying that bank j — which is the bank facing the most amount of potential distress — can pay its senior debts. In particular,

$$x_{j,j-1} = y + (p-1)(a-\epsilon-v),$$

guaranteeing that $x_{j,j-1} + a - \epsilon > v$. Given that all banks can meet their senior liabilities, we have

$$x_{s+1,s} = y + \tau(a-v) - p\epsilon$$

where $s = i+\tau$ is the index of the last bank on the chain that defaults. On the other hand, given that $s+1$ does not default, we have $y \leq a - v + x_{s+1,s}$. As a result,

$$\tau = \left\lceil \frac{p\epsilon}{a-v} \right\rceil - 1 \geq \frac{p\epsilon}{a-v} - 1.$$

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Hence, when shocks hit p consecutive banks on the credit chain, the number of bank failures reaches the upper bound established by Lemma B.5, implying that the ring network is the least resilient financial network. $\square$

Proof of part (b). The proof follows a logic similar to that of Proposition 6. We first prove that if $\epsilon > \epsilon_p^*$, then the complete network is the least stable and resilient financial network. In particular, we show that all banks default. By Lemma B.6, the p distressed banks default on their senior liabilities. The remaining $n-p$ banks do not default only if

$$(n-p-1)\frac{y}{n-1} + (a-v) \geq y.$$

The above inequality, however, can hold only if $y < \hat{y}_p = (n-1)(a-v)/p \leq y_p^*$. Hence, the complete network is the least resilient and the least stable financial network as all n banks default.

We next show that if $\epsilon > \epsilon_p^*$, then all n banks in the ring network fail as well. Suppose not, and that there exists a bank that can pay all its creditors in full. On the other hand, by Lemma B.6, there is also a bank that defaults on its senior liabilities v. Consider the path on the ring network connecting bank j to bank l, such that (i) j defaults on its senior debt; (ii) l pays all its creditors in full; and (iii) all banks on the path default but can pay back their senior debt. Denote the length of the path connecting j to l by τ (see Figure C.1), and suppose that there are h negative shocks realized on this path.

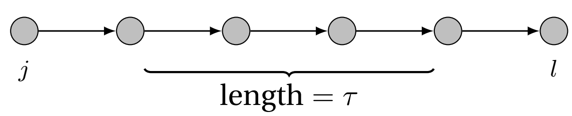

Figure C.1. There are τ banks connecting j to l, all of which default, but can meet their senior liabilities.

Given that j does not pay anything to its junior creditor (which is the first bank on the path connecting it to l) and that l does not default, we have $(\tau+1)(a-v) - h\epsilon \geq y$, implying that

$$\begin{aligned}\tau &> \frac{y_p^* + h\epsilon_p^*}{a-v} - 1 \\ &= n - p - 1 + \frac{hn}{p}.\end{aligned}$$

On the other hand, the remaining $p-h$ shocks hit banks that are not on the path connecting j to l. Thus, the total number of defaults is at least $\tau + p - h$, implying

$$\sharp\text{defaults} > n - 1 + h\left(\frac{n}{p} - 1\right) \geq n - 1.$$

This, however, contradicts the assumption that at least one bank does not default.

Finally, consider a δ-connected financial network with the corresponding partition (S, S^c) such that $|S^c| = p$. Note that by definition, $\max\{y_{ij}, y_{ji}\} \leq \delta y$ for all $i \in S$ and $j \in S^c$. Therefore, for any bank $i \in S$, it must be the case that $\sum_{j\in S} y_{ij} \geq y - p\delta y$. On the other hand, in the case that all p negative shocks hit the banks in S^c, any bank i can meet its liabilities in full as long

$$a - v + \sum_{j\in S} y_{ij} \geq y.$$

Thus, as long as $\delta < (a-v)/(py)$, then no bank in S defaults, establishing that the given financial network is strictly more stable than the complete financial network. □

Proof of part (c). An argument similar to the one invoked in the proof of part (b) shows that the when $\epsilon > \epsilon_p^*$ and $y > \hat{y}_p$, all banks in the complete network default. Therefore, the complete network is the least stable and resilient financial network.

To prove that the ring financial network is more stable than the complete network, we show that there exists a realization of the shocks for which at least one bank in the ring network does not default. In particular, consider the situation in which p consecutive banks, labeled 1 through p, are hit with negative shocks. By Lemma B.6, in the unique payment equilibrium, bank p defaults on its senior debt. Therefore, the length of the cascade of defaults following bank p, denoted by τ, satisfies

$$\tau(a-v) < y \leq (\tau+1)(a-v).$$

Thus, the number of defaults in the whole network is

$$\begin{aligned} \sharp\text{defaults} &= p + \tau \\ &< p + \frac{y_p^*}{a-v} \\ &= n, \end{aligned}$$

implying that at least one bank does not default. Hence, the ring network is strictly more stable than the complete network. □

Proof of Proposition 10

The proofs of parts (a) and (b) are similar to those of Propositions 4 and 6, respectively, and are thus omitted. To prove part (c), first consider the complete financial network and without loss of generality, assume that bank 1 is hit with the negative shock. It is easy to verify that, as long as $\epsilon_*(\zeta) < \epsilon < \epsilon^*(\zeta)$, the unique payment equilibrium is given by

$$\begin{aligned} (x_1, \ell_1) &= (a - v - \epsilon + \zeta A + y, A) \\ (x_i, \ell_i) &= \left(y, \frac{\epsilon - n(a-v) - \zeta A}{\zeta(n-1)}\right) \end{aligned}$$

Therefore, the total liquidation across the financial network satisfies

$$\sum_{i=1}^{n} \zeta \ell_i = \epsilon - n(a - v). \tag{0}$$

Next consider the ring financial network. Again, it is east to verify that if bank 1 is hit with the negative shock, then the total amount of liquidation across the financial network satisfies

$$\sum_{i=1}^{n} \zeta \ell_i = \tau \zeta A + [\epsilon - (\tau + 1)(a - v) - \tau \zeta A]^+, \tag{0}$$

where $\tau = \lceil \epsilon/(a - v + \zeta A) \rceil - 1$ is the number of defaults. Comparing (0) to (0) then immediately implies that the extent of liquidation in the complete financial network is strictly smaller than the ring financial network. Hence, the former is strictly more stable and resilient than the latter.

Finally, consider the financial network depicted in Figure 2 with $q = 1$. Suppose that bank 1 is hit with a negative shock, which immediately implies that banks 1 and 2 default and liquidate their projects entirely, whereas all other banks can meet their liabilities in full. Consequently,

$$\sum_{i=1}^{n} \zeta \ell_i = 2 \zeta A.$$

Comparing the above to (0) shows that as long as $\epsilon > \epsilon_*(\zeta) + \zeta A$, then the given 0-connected network is strictly more stable and resilient than the complete financial network. □

Proof of Proposition 11

The proof of this proposition is similar to those of Propositions 4 and 6, and hence is omitted.

Proof of Proposition 12

Proof of part (a). The proof closely follows the proof of the second part of Lemma B.6. In particular, suppose that $\epsilon > (a - v) \sum_{k=1}^{n} \theta_k / \theta_j$, but all banks can meet their liabilities to the senior creditors in full. Thus, by the definition of the payment equilibrium,

$$z_i + \sum_{k \neq i} x_{ik} \geq \theta_i v + \sum_{k \neq i} x_{ki},$$

for all banks i. Summing over all i implies

$$a \sum_{i=1}^{n} \theta_i - \theta_j \epsilon \geq v \sum_{i=1}^{n} \theta_i,$$

which is a contradiction. Thus, the distressed bank j defaults on its liabilities to the senior creditors.
□

Proof of part (b). In the presence of a large shock to bank j, all other banks default if and only if $x_i < \theta_i y$ for all i, where x_i's are the solutions to the following collection of equations:

$$x_i = \theta_i(a - v) + \sum_{k \neq j} q_{ik} x_k.$$

Comparing the above equation to (5), however, implies that $x_i = (a - v)\widehat{m}_{ij}$. Thus, all banks default if and only if $\widehat{m}_{ij} < \theta_i m^*$, completing the proof. $\square$

Proof of Proposition A.1

Let $\{\chi_t\}_{t \geq 0}$ denote the discrete-time, discrete-space Markov chain with the transition probability matrix Q; that is, $\mathbb{P}(\chi_{t+1} = j | \chi_t = i) = q_{ij}$. Also, let τ_{ij} denote the number of time steps that it takes to visit state j for the first time; that is, $\tau_j = \min\{t \geq 0 : \chi_t = j\}$. Therefore, the mean hitting time of state j conditional on starting from state i is given by

$$\begin{aligned}
\mathbb{E}_i[\tau_j] &= \sum_{t=1}^{\infty} t\mathbb{P}(\tau_j = t | \chi_0 = i) \\
&= \sum_{t=1}^{\infty} \sum_{k=1}^{n} t\mathbb{P}(\tau_j = t, \chi_1 = k | \chi_0 = i) \\
&= \sum_{k=1}^{n} q_{ik} \sum_{t=1}^{\infty} t\mathbb{P}(\tau_j = t | \chi_1 = k),
\end{aligned}$$

implying that the mean hitting times satisfy the following fixed point equation:

$$\mathbb{E}_i[\tau_j] = 1 + \sum_{k=1}^{n} q_{ik} \mathbb{E}_k[\tau_j].$$

This equation, however, is identical to equation (3). Furthermore, given the argument in the proof of Lemma 1, the equation has a unique solution. Therefore, $\mathbb{E}_i[\tau_j] = m_{ij}$. $\square$

Proof of Proposition A.2

Levin, Peres, and Wilmer (2009, Lemma 10.10) show that in any reversible Markov chain and for any three states i, j and k, we have $\mathbb{E}_i[\tau_j] + \mathbb{E}_j[\tau_k] + \mathbb{E}_k[\tau_i] = \mathbb{E}_j[\tau_i] + \mathbb{E}_k[\tau_j] + \mathbb{E}_i[\tau_k]$. On the other hand, by Proposition A.1, the harmonic distances in the financial network are equal to the mean hitting times in the corresponding Markov chain, establishing (A1). $\square$

Proof of Corollary A.1

Pick an arbitrary bank k, and create an ordering of the rest of the banks according to the value of $m_{ik} - m_{ki}$. More specifically, let bank i appear before bank j if $m_{ik} - m_{ki} \geq m_{jk} - m_{kj}$. Proposition A.2, on the other hand, requires (A1) to hold. Consequently, it must be the case that $m_{ij} \geq m_{ji}$. $\square$

Proof of Proposition A.3

Kirkland and Neumann (2012, Theorem 6.2.1) show that Markov chain mean hitting times satisfy the triangle inequality. Thus, by Proposition A.1, the harmonic distances satisfy the triangle inequality as well. □

Proof of Proposition A.4

Equation (B25) in the proof of Lemma 1 establishes that

$$\frac{1}{n}\sum_{j\neq i} m_{ij} = \sum_{k=2}^{n} \frac{1}{1-\lambda_k},$$

where $\{\lambda_2, \ldots, \lambda_n\}$ are the $n-1$ smallest eigenvalues of matrix Q (that is, excluding $\lambda_1 = 1$). Given that the right-hand side above is independent of i, it is immediate that the average harmonic distance from bank i to all other banks is an invariant property of the financial network. □

References

Kirkland, Stephen J. and Michael Neumann (2012), *Group Inverses of M-matrices and Their Applications*, volume 26. CRC Press.

Levin, David Asher, Yuval Peres, and Elizabeth Lee Wilmer (2009), *Markov Chains and Mixing Times*. American Mathematical Society, Providence, RI.