Dynamic Free Riding with Irreversible Investments: On-line Appendix

Abstract
In this appendix we present the proofs omitted in “Dynamic Free Riding with Irreversible Investments” by Marco Battaglini, Salvatore Nunnari and Thomas Palfrey.
1 Proof of Proposition 1

**Proposition 1.** For any $d, \delta, n$ and $y^o \in ([u']^{-1} (1 - \delta(1 - d)), [u']^{-1} (1 - \delta(1 - \frac{u'}{d})))$, there is an equilibrium with steady state $y^o$ in an irreversible economy. In all these equilibria convergence is monotonic and gradual.

Define $y^*(\delta, d, n) = [u']^{-1} (1 - \delta(1 - d)/n)$ and $y^{**}(\delta, d, n) = [u']^{-1} (1 - \delta(1 - d/n))$: these are the points at which

$$
y'(g) = \frac{1 - d - \frac{n(1-u'(g))}{\delta}}{1 - n}
$$

is, respectively, zero and one. Define $\bar{\gamma}(d, \delta) = [u']^{-1} (1 - \delta(1 - d))$: this is the point at which (1) is equal to $1 - d$. Note that $y^*(\delta, d, n) < \bar{\gamma}(d, \delta)$ and $\bar{\gamma}(d, \delta) < y^{**}(\delta, d, n)$. Moreover, since we are assuming that the planner interior solution is feasible ($y^*(\delta, d, n) < W/d$), we have $y^{**}(\delta, d, n) < W/d$. To construct an equilibrium with steady state $y^o \in [\bar{\gamma}(\delta, d), y^{**}(\delta, d, n)]$ we proceed in 3 steps.

**Step 1.** We first construct the strategies associated to a generic $y^o$. For a generic $y^o \in [\bar{\gamma}(d, \delta), y^{**}(\delta, d, n)]$, let $\bar{y}(g | y^o)$ be the solution of the differential equation (1) when we require the initial condition: $\bar{y}(y^o | y^o) = y^o$. Given $y^o$, moreover, let us define the two thresholds $g^3(y^o) = y^o/(1 - d)$ and $g^2(y^o) = \max \{ \min_{y \geq 0} \{ g | \bar{y}(g | y^o) \leq W + (1 - d)g \}, y^*(\delta, d, n) \}$. In words, the second threshold is the largest point between the point at which $\bar{y}(g | y^o)$ crosses from below $W + (1 - d)g$, and $y^*(\delta, d, n)$ (see Figure 1 in the paper for an example). It is easy to verify that, by construction, $g^3(y^o) \geq \bar{\gamma}(d, \delta)$; moreover, $\bar{y}(g | y^o) \in ((1 - d)g, W + (1 - d)g)$ with $\bar{y}'(g | y^o) \in [0, 1]$ and $\bar{y}''(g | y^o) \geq 0$ in $[g^2(y^o), y^o]$. For any $y^o \in [\bar{\gamma}(d, \delta), y^{**}(\delta, d, n)]$, we now define the investment function as follows:

$$y(g | y^o) = \begin{cases} 
\min \{ W + (1 - d)g, \bar{y}(g^2(y^o) | y^o) \} & g \leq g^2(y^o) \\
\bar{y}(g | y^o) & g^2(y^o) < g \leq y^o \\
y^o & y^o < g \leq g^3(y^o) \\
(1 - d)g & g > g^3(y^o)
\end{cases}$$

(2)

Note that when depreciation is zero, then $g^3(y^o) = y^o$ and $y'(g | y^o) = 1$ at $g = y^o$: so (2) coincides exactly with the investment function illustrated in Figure 1 in the paper. For future reference, define $g^1(y^o) = \max \{ 0, (\bar{y}(g^2(y) | y^o) - W) / (1 - d) \}$. This is the point at which $W + (1 - d)g = \bar{y}(g^2(y^o) | y^o)$, if positive. Since $\bar{y}(g^2(y) | y^o) < W + (1 - d)g^2(y^o)$, $g^1(y^o) \in [0, g^2(y^o)]$. We have:

**Lemma A.1.** $y(g | y^o) \in [g^2(y^o), y^o]$ for $g \in [g^2(y^o), y^o]$. 

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Proof. Because \( y(\cdot | y^o) \) is monotonic non-decreasing in \( g \in [g^2(y^o), y^o] \), for any \( g \in [g^2(y^o), y^o] \) we have \( y(\cdot | y^o) \in [y(g^2(y^o)) | y^o], y^o]). \) Since \( y(\cdot | y^o) \) has slope lower than one in \( [g^2(y^o), y^o] \) and \( y(y^o | y^o) = y^o \) for \( y^o \geq g^2(y^o) \), we must have \( y(g^2(y^o) | y^o) \geq g^2(y^o) \), so \( y(\cdot | y^o) \geq g^2(y^o) \) for \( g \in [g^2(y^o), y^o] \). Similarly, \( y(y^o | y^o) = y^o \) implies \( y(\cdot | y^o) \leq y^o \) for \( g \in [g^2(y^o), y^o] \). □

Step 2. We now construct the value functions corresponding to each steady state \( y^o \). For \( g \in [g^2(y^o), y^o] \) define the value function recursively as

\[
v(g | y^o) = \frac{W + (1-d)g - y(g | y^o)}{n} + u(y(g | y^o)) + \delta v(y(g | y^o)).
\] (3)

By Theorem 3.3 in Stokey, Lucas, and Prescott (1989), the right hand side of (3) is a contraction: it defines a unique, continuous and differentiable value function \( v(g | y^o) \) for this interval of \( g \). (Differentiability follows from the differentiability of \( y(\cdot | y^o) \).) Note that \( y(g | y^o) = \tilde{y}(g | y^o) \) for any \( g \in [g^2(y^o), y^o] \) and, by Lemma A.1, \( \tilde{y}(g | y^o) \in [g^2(y^o), y^o] \) for \( g \in [g^2(y^o), y^o] \). From the definition of \( \tilde{y}(g | y^o) \) and the discussion in Section 4 in the paper, it follows that \( u'(g) + \delta u'(\tilde{y}(g | y^o)) = 1 \) for any \( g \in [g^2(y^o), y^o] \). In the rest of the state space we define the value function recursively. In \( [g^1(y^o), g^2(y^o)] \), if \( g^1(y^o) < g^2(y^o) \), the value function is defined as:

\[
v(g | y^o) = \frac{W + (1-d)g - y(g^2(y^o) | y^o)}{n} + u(y(g^2(y^o) | y^o)) + \delta v(y(g^2(y^o) | y^o)).
\] (4)

where \( v(y(g^2(y^o) | y^o)) \) is well defined since \( y(g^2(y^o) | y^o) \in [g^2(y^o), y^o] \).

Lemma A.2. For \( g \in [g^1(y^o), y^o] \), \( u(g) + \delta u(g | y^o) \) is concave with slope larger or equal than 1.

Proof. If \( g^1(y^o) = g^2(y^o) \), the result is immediate. Assume therefore, \( g^1(y^o) < g^2(y^o) \). In this case \( g^2(y^o) = y^o(\delta, d, n) \). For any \( g \in [g^1(y^o), g^2(y^o)] \), \( y(g; y^o) = y^o(\delta, d, n) | y^o) \).

So we have \( v'(g | y^o) = (1-d)/n \) implying: \( u'(g) + \delta u'(y^o | y^o) = u'(g) + \delta(1-d)/n > 1 \) since \( g \leq g^2(y^o) = y^o(\delta, d, n) \). □

Consider \( g < g^1(y^o) \). In \( [g_{-1}, g^1(y^o)] \) the value function is defined as:

\[
v(g | y^o) = u(W + (1-d)g) + \delta v(W + (1-d)g | y^o)
\] (5)

where \( g_{-1} = \max \{0, [g^1(y^o) - W] / (1-d) \} \). Assume that we have defined the value function in \( g \in [g_{-(t+1)}, g_{-(t-1)}] \) as \( v_{-t} \), for all \( t \) such that \( g_{-(t-1)} > 0 \). Then we can define \( v_{-(t+1)} \) as (5) in \( [g_{-(t+1)}, g_{-t}] \) with \( g_{-(t+1)} = [g_{-t} - W] / (1-d) \).

Lemma A.3. For \( g \in [0, y^o] \), \( u(g) + \delta u(g | y^o) \) is concave with slope greater than or equal than 1.

Proof. We prove this by induction on \( t \). Consider now the interval \( [[g^1(y^o) - W] / (1-d), g^1(y^o)] \).

In this range we have \( v'(g | y^o) = [u'(W + (1-d)g) + \delta u'(W + (1-d)g | y^o)](1-d) \geq 1 - d, \)
For the same argument as above, moreover, $W > g$. We conclude that $u'(g) + \delta v'_{-1}(g|y^o)$ is concave, it has derivative larger than 1. Assume that we have shown that for $g \in [g_{-t}, g(y^o)]$, $u(g) + \delta v_{-t}(g|y^o)$ is concave and $u'(g) + \delta v_{-t}(g|y^o) > 1$. Consider in $g \in [g_{-(t+1)}, g_{-t}]$. We have:

$$v'(g|y^o) = [u'(W + (1-d)g) + \delta v'(W + (1-d)g|y^o)] (1-d) \geq 1 -d$$

since $W + (1-d)g \geq [g_{-t}, g^3(y^o)]$. So $u'(g) + \delta v'(g|y^o) \geq u'(g) + \delta (1-d) \geq 1$. By the same argument as above, moreover, $v$ is concave at $g_{-t}$. We conclude that for any $g \leq g^1$, $u(g) + \delta v(g|y^o)$ is concave and it has slope larger than 1. \[\blacksquare\]

For $g \in (y^o, g^3(y^o)]$ we define the value function as: $v(g|y^o) = \frac{W + (1-d)g - y^o}{n} + u(y^o) + \delta v(y^o|y^o)$.

**Lemma A.4.** For $g \leq g^3(y^o)$, $u(g) + \delta v(g|y^o)$ is concave with slope less than or equal to 1.

**Proof.** For $g \in (y^o, g^3(y^o)]$, $v'(g|y^o) = (1-d)/n$. Since $g \geq y^o \geq y^*(\delta, d, n)$, we have $u'(g) + \delta v'(g|y^o) = u'(g) + \delta (1-d)/n < 1$. Previous lemmas imply $u(g) + \delta v(g|y^o)$ is concave and has slope greater than or equal to 1 for $g \leq g^3(y^o)$. \[\blacksquare\]

Finally consider $g > g^3(y^o)$.

**Lemma A.5.** For any $g \geq g^3(y^o)$, $u(g) + \delta v(g|y^o)$ has slope less than or equal to 1.

**Proof.** In $g > g^3(y^o)$, we must have $(1-d)g \in [y^o, g^3(y^o)]$. From the proof of Lemma A.5 we know that $u'(g) + \delta v'(g) < 1$ for $g \in [y^o, g^3(y^o)]$, so we have:

$$v'(g) = (1-d) [u'(1-d)g] + \delta v'(1-d)g] < 1 -d$$

for $g > g^3(y^o)$. This fact implies that $u'(g) + \delta v'(g) < u'(g) + \delta (1-d)$ for any $g > g^3(y^o)$. Since $g^3(y^o) > y(\delta, d)$ we have $u'(g) + \delta (1-d) < u'(y(\delta, d)) + \delta (1-d) = 1$ for $g > g^3(y^o)$. It follows that $v^*(g)$ is has slope lower than 1 in $g > g^3(y^o)$. \[\blacksquare\]

From Lemmata A1-A5 we conclude that $u(g) + \delta v(g|y^o)$ has a global maximum at any $g \in [g^3(y^o), y^o]$.

**Step 3.** Define $x(g|y^o) = [W + (1-d)g - y(g|y^o)]/n$ and $i(g|y^o) = [y(g|y^o) - (1-d)g]/n$ as the levels of per capita private consumption and investment, respectively. Note that by construction, $x(g|y^o) \in [0, W/n]$. We now establish that $y(g|y^o)$, $x(g|y^o)$ and the associated value function $v(g|y^o)$ defined in the previous steps constitute an equilibrium. The fact that $v(g|y^o)$ describes the expected continuation value to an agent follows by construction. To see that $y(g|y^o)$ is an optimal reaction function given $v(g|y^o)$, note that an agent solves the following
We prove the result by contradiction. Suppose to the contrary there is a sequence of steady states of the steady state of the problem:

\[
\max_y \begin{cases} 
  u(y) - y + \delta v(y) \\
  y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y(g), \; y \geq \frac{(1-d)g}{n} + \frac{n-1}{n} y(g)
\end{cases}
\]

where \( y(g) = y(g|y^o) \). The investment function \( y(g|y^o) \) satisfies the constraints of this problem if \( y(g|y^o) \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y(g|y^o) \), so if \( y(g|y^o) \leq W + (1-d)g \); and if \( y(g|y^o) \geq \frac{(1-d)g}{n} + \frac{n-1}{n} y(g|y^o) \), so if \( y(g|y^o) \geq (1-d)g \). Both conditions are automatically satisfied by construction.

If \( g < g^1(y^o) \), we have \( u'(y) + \delta v'(y) \geq 1 \) for all \( y \in [(1-d)g, W + (1-d)g] \), so \( y(g|y^o) = W + (1-d)g \) is optimal. If \( g \geq g^2(y^o) \), \( u'(y) + \delta v'(y) < 1 \) for all \( y \in [(1-d)g, W + (1-d)g] \), so \( y(g|y^o) = (1-d)g \). In \( g \in (g^1(y^o), g^3(y^o)] \) a point maximizing \( u(y) + \delta v(y) \) is feasible and chosen, so again \( y(g|y^o) \) is an optimal choice.

**2 Proof of Proposition 2**

**Proposition 2.** For any \( \delta \) and \( n \), we have that \( \overline{y}_{IR}(\delta, d, n) - \underline{y}_{IR}(\delta, d, n) \to 0 \) as \( d \to 0 \). Moreover, there is \( \overline{d} > 0 \) such that for \( d < \overline{d} \), all equilibrium paths are gradual.

Consider a sequence \( d^m \to 0 \). For each \( d^m \) there is at least an associated equilibrium \( y_m(g) \), \( v_m(g) \) with steady state \( y^*_m \). To prove the result we proceed in two steps. In Section 2.1 we prove that for any \( \xi > 0 \), there is a \( \overline{m} \) such that for \( m > \overline{m} \), \( y_{IR}(\delta, d^m, n) \geq [u']^{-1} (1 - \delta) - \xi \). In Section 2.2 we prove that the steady state of any equilibrium can not be larger than \( [u']^{-1} (1 - \delta (1 - d/n)) \).

Since, as shown in Proposition 1, \( [u']^{-1} (1 - \delta (1 - d/n)) \) is an equilibrium steady state for any \( d \geq 0 \) and it converges to \( [u']^{-1} (1 - \delta) \), we must have \( \overline{y}_{IR}(\delta, d, n) - \underline{y}_{IR}(\delta, d, n) \to 0 \) as \( d \to 0 \). In Lemmata A.6 and A.7 presented in Section 2.2 we show that \( y'(g) \in (0, 1) \) in a local neighborhood of the steady state \( y^o \) if \( y^o > [u']^{-1} (1 - \delta (1 - d)/n) \). Since all equilibrium steady states converge to \( [u']^{-1} (1 - \delta) > [u']^{-1} (1 - \delta/n) \), this implies that that convergence of \( g \) to the steady state is gradual in all equilibria if \( d \) is sufficiently small.

**2.1 The lower bound**

We prove the result by contradiction. Suppose to the contrary there is a sequence of steady states \( y^o_m \), with associated equilibrium investment and value functions \( y_m(g) \), \( v_m(g) \), and an \( \xi > 0 \) such that \( y^o_m < \gamma(0) - \xi \) for any arbitrarily large \( m \), where \( \gamma(d) = [u']^{-1} (1 - \delta (1 - d)) \). Define \( y^0_m(g) = y_m(g) \), and \( y^j_m(g) = y_m(y^j_{m-1}(g)) \) and consider a marginal deviation from the steady state.
from $y_m^0$ to $y_m^0 + \Delta$. By the irreversibility constraint we have $y_m(g) \geq (1 - d^m)g$. Using this property and the fact that $y_m^0$ is a steady state, so $y_m^j(y_m^0) = y_m^0$, we have:

$$y_m(y_m^0 + \Delta) - y_m(y_m^0) \geq (1 - d^m)(y_m^0 + \Delta) - y_m^0 = (1 - d^m)\Delta - d^m y_m^0$$

This implies that, as $m \to \infty$, for any given $\Delta$: $[y_m(y_m^0 + \Delta) - y_m^0] / \Delta \geq 1 + o_1(d^m)$ where $o_1(d^m) \to 0$ as $m \to 0$. We now show with an inductive argument that a similar property holds for all iterations $y_m^j(y_m^0)$. Assume we have shown that: $[y_m^{j-1}(y_m^0 + \Delta) - y_m^0] / \Delta \geq 1 + o_{j-1}(d^m)$ where $o_{j-1}(d^m) \to 0$ as $m \to 0$. We must have: $y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0 \geq (1 - d^m)y_m^{j-1}(y_m^0 + \Delta) - y_m^0$. We therefore have: $y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0 \geq y_m^{j-1}(y_m^0 + \Delta) - y_m^0 - d^m y_m^{j-1}(y_m^0 + \Delta)$, so we have:

$$\frac{y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0}{\Delta} \geq \frac{y_m^{j-1}(y_m^0 + \Delta) - y_m^0 - d^m y_m^{j-1}(y_m^0 + \Delta)}{\Delta} \geq 1 + o_j(d^m) \quad (7)$$

where $o_j(d^m) = o_{j-1}(d^m) - d^m y_m^{j-1}(y_m^0 + \Delta)$, so $o_j(d^m) \to 0$ as $m \to 0$.

We can write the value function after the deviation to $y_m^0 + \Delta$ as:

$$V(y_m^0 + \Delta) = \sum_{j=0}^{\infty} \delta^{j-1} \left[ W + (1 - d^m)y_m^{j-1}(y_m^0 + \Delta) - y_m^0 \right] + u(y_m^j(y_m^0 + \Delta))$$

For any given function $f(x)$, define $\Delta f(x) = f(x + \Delta) - f(x)$. We can write:

$$\Delta V(y_m^0)/\Delta = \sum_{j=0}^{\infty} \delta^{j-1} \left[ \frac{(1-d^m)\Delta y_m^{j-1}(y_m^0 + \Delta) - \Delta y_m^j(y_m^0)}{\Delta} + \frac{u'(y_m^0) + o(\Delta)}{\Delta} \right] y_m^j(y_m^0)/\Delta$$

$$\geq \sum_{j=0}^{\infty} \delta^{j-1} \left[ \frac{(1-d^m)(1+o_{j-1}(d^m)) - (1+o_j(d^m))}{\Delta} + \frac{u'(y_m^0) + o(\Delta)}{\Delta} \right] (1 + o_j(d^m))$$

where $o(\Delta) \to 0$ as $\Delta \to 0$. In the first equality we use the fact that if we choose $\Delta$ small, since $y_m(g)$ is continuous, $\Delta y_m^j(y_m^0)$ is small as well. This implies that

$$\frac{u(y_m^j(y_m^0 + \Delta)) - u(y_m^j(y_m^0))}{\Delta} \geq \frac{u'(y_m^0) + o(\Delta)}{1 - \delta}$$

converges to $u'(y_m^0)$ as $\Delta \to 0$. The inequality in 8 follows from (7). Given $\Delta$, as $m \to \infty$, we therefore have $\lim_{m \to \infty} \Delta V(y_m^0)/\Delta \geq \frac{u'(y_m^0)}{1 - \delta}$. We conclude that for any $\varepsilon > 0$, there must be a $m_\varepsilon$ such that for any $\Delta \in (0, m_\varepsilon)$ there is a $m_\Delta$ guaranteeing that $\Delta V(y_m^0)/\Delta \geq \frac{u'(y_m^0)}{1 - \delta} - \varepsilon$ for $m > m_\Delta$. After a marginal deviation to $y_m^0 + \Delta$, therefore, the change in agent’s objective function is:

$$u'(y_m^0) + \delta \Delta V(y_m^0)/\Delta - 1 \geq \frac{u'(y_m^0)}{1 - \delta} - \delta \varepsilon - 1$$
for $m$ sufficiently large. A necessary condition for the un-profitability of a deviation from $y^0_m$ to $y^0_m + \Delta$ is therefore: $y^0_m \geq [u']^{-1}(1 - \delta + \delta \varepsilon (1 - \delta))$. Since $\varepsilon$ can be taken to be arbitrarily small, for an arbitrarily large $m$, this condition implies $y^0_m \geq \bar{y}(0) - \xi/2$, which contradicts $y^0_m < \bar{y}(0) - \xi$.

We conclude that $y_{IR}(\delta, d, n) \rightarrow \bar{y}(0)$ as $d \rightarrow 0$.

2.2 The upper bound

Suppose to the contrary that there is stable steady state at $y^o > [u']^{-1}(1 - \delta (1 - d/n))$. We must have $y^o \in \left([u']^{-1}(1 - \delta (1 - d/n)), W/d\right]$, since it is not feasible for a steady state to be larger than $W/d$. Consider a left neighborhood of $y^o$, $N_\varepsilon(y^o) = (y^o - \varepsilon, y^o)$. The value function can be written in $g \in N_\varepsilon(y^o)$ as:

$$v(g) = u(y(g)) + \delta v(y(g)) - y(g) + \frac{W + (1-d)g}{n} + (1 - 1/n) y(g)$$

(9)

where $y(g)$ is the equilibrium strategy associated to $y^o$. In $N_\varepsilon(y^o)$ the constraint $y \geq \frac{1-d}{n} g + \frac{n-1}{n} y(g)$ cannot be binding (else we would have $y(g) = (1-d)g$, but this is not possible in a neighborhood of $y^o > 0$). We consider two cases.

**Case 1.** Suppose first that $y^o < W/d$. We must therefore have that $y(g) < W + (1-d)g$ in $N_\varepsilon(y^o)$, so the constraint $y \leq \frac{W + (1-d)g}{n} + \frac{n-1}{n} y$ is not binding. The solution is in the interior of the constraint set of (6), and the objective function $u(y(g)) + \delta v(y(g)) - y(g)$ is constant for $g \in N_\varepsilon(y^o)$.

**Lemma A.6.** If $y^o > [u']^{-1}(1 - \delta (1 - d)/n)$, then there is a left neighborhood $N_\varepsilon(y^o)$ in which $y(g)$ is not constant.

**Proof.** Suppose to the contrary that, for any $N_\varepsilon(y^o)$, there is an interval in $N_\varepsilon(y^o)$ in which $y(g)$ is constant. Using the expression for $v(g)$ as presented above, we must have $v'(g) = (1-d)/n$ for any $g$ in this interval. Since $N_\varepsilon(y^o)$ is arbitrary, then we must have a sequence $g^m \rightarrow y^o$ such that $v'(g^m) = (1-d)/n \forall m$. We can therefore write:

$$\lim_{\Delta \rightarrow 0} \frac{v(y^o) - v(y^o - \Delta)}{\Delta} = \lim_{\Delta \rightarrow 0} \lim_{m \rightarrow \infty} \frac{v(g^m) - v(g^m - \Delta)}{\Delta}$$

$$= \lim_{m \rightarrow \infty} \lim_{\Delta \rightarrow 0} \frac{v(g^m) - v(g^m - \Delta)}{\Delta} = \frac{1-d}{n}$$

where the second equality follows from the continuity of $v(g)$. This implies that $v^-(y^o)$, left derivative of $v(g)$ at $y^o$, is well defined and equal to $\frac{1-d}{n}$. Consider now a marginal reduction of $g$ at $y^o$. The change in utility is (as $\Delta \rightarrow 0$):

$$\Delta U(y^o) = u(y^o - \Delta) - u(y^o) + \delta \left[v(y^o - \Delta) - v(y^o)\right] + \Delta$$

$$= \left[1 - \left(u'(y^o) + \delta \frac{1-d}{n}\right)\right] \Delta$$
In order to have \( \Delta U(y^o) \leq 0 \), we must have \( u'(y^o) + \delta (1 - d)/n \geq 1 \). This implies \( y^o \leq [u']^{-1} (1 - \delta (1 - d)/n) \), a contradiction. Therefore, if there is stable steady state at \( y^o > [u']^{-1} (1 - \delta (1 - d)/n) \), then \( y(g) \) is not constant in \( N_{\varepsilon}(y^o) \). \hfill \Box

Lemma A.6 implies that there is a left neighborhood \( N_{\varepsilon}(y^o) \) in which \( u(g) + \delta v(g) - g \) is constant if \( y^o > [u']^{-1} (1 - \delta (1 - d)/n) \) (since otherwise \( y(g) \) would be constant). Moreover, since \( y^o \) is a stable steady state and \( y(g) \) is strictly increasing, \( g \in N_{\varepsilon}(y^o) \) implies \( y(g) \in N_{\varepsilon}(y^o) \) for any open left neighborhood \( N_{\varepsilon}(y^o) = (y^o - \varepsilon', y^o) \subset N_{\varepsilon}(y^o) \). These observations imply:

**Lemma A.7.** If \( y^o > [u']^{-1} (1 - \delta (1 - d)/n) \), then there is a left neighborhood \( N_{\varepsilon}(y^o) \) in which

\[
y'(g) = \frac{n}{n-1} \left( \frac{1 - u'(g)}{\delta} - \frac{1 - d}{n} \right)
\]

**Proof.** There is a \( N_{\varepsilon}(y^o) \) and a constant \( K \) such that \( \delta v(g) = K + g - u(g) \) for \( g \in N_{\varepsilon}(y^o) \). Hence \( v(g) \) is differentiable in \( N_{\varepsilon}(y^o) \). Moreover, \( y(g) \in N_{\varepsilon}(y^o) \) for all \( g \in N_{\varepsilon}(y^o) \). Hence \( u(y(g)) + \delta v(y(g)) - y(g) \) is constant in \( g \in N_{\varepsilon}(y^o) \) as well. These observations and the definition of \( v(g) \) imply that \( v'(g) = \frac{1-d}{n} + (1 - \frac{1}{n}) y'(g) \) in \( N_{\varepsilon}(y^o) \). Given that \( u'(g) + \delta v'(g) = 1 \) in \( g \in N_{\varepsilon}(y^o) \), we must have: \( u'(g) + \delta v'(g) = u'(g) + \delta \left[ \frac{1-d}{n} + (1 - \frac{1}{n}) y'(g) \right] = 1 \) which implies (10) for any \( g \in N_{\varepsilon}(y^o) \). \hfill \Box

Let \( g^m \) be a sequence in \( N_{\varepsilon}(y^o) \) such that \( g^m \to y^o \). We must have

\[
y^-(y^o) = \lim_{\Delta \to 0} \frac{y(y^o) - y(y^o - \Delta)}{\Delta} = \lim_{\Delta \to 0} \lim_{m \to \infty} \frac{y^m(y^o) - y^m(y^o - \Delta)}{\Delta} = \lim_{m \to \infty} \lim_{\Delta \to 0} \frac{y^m(y^o) - y^m(y^o - \Delta)}{\Delta} = \frac{n}{n-1} \left( \frac{1 - u'(y^o)}{\delta} - \frac{1 - d}{n} \right)
\]

(11)

where \( y^-(y^o) \) is the left derivative of \( y(g) \) at \( y^o \), the second equality follows from continuity and the last equality follows from Lemma A.7 and the fact that under the starting assumption we have \( y^o > [u']^{-1} (1 - \delta (1 - d)/n) > [u']^{-1} (1 - \delta (1 - d)/n) \). Consider a state \((y^o - \Delta)\). For \( y^o \) to be stable we need that for any small \( \Delta \):

\[
y(y^o - \Delta) \geq y^o - \Delta = y(y^o) + (y^o - \Delta) - y^o
\]

where the equality follows from the fact that \( y(y^o) = y^o \). As \( \Delta \to 0 \), this implies \( y^-(y^o) \leq 1 \) in \( N_{\varepsilon}(y^o) \). By (11), we must therefore have: \( \frac{n}{n-1} \left( \frac{1 - u'(y^o)}{\delta} - \frac{1 - d}{n} \right) \leq 1 \). This implies: \( y^o \leq [u']^{-1} (1 - \delta (1 - d)/n) \), a contradiction.

**Case 2.** Assume now that \( y^o = W/d \) and consider first the case in which it is a strict local maximum of the objective function \( u(y) + \delta v(y) - y \). In this case in a left neighborhood \( N_{\varepsilon}(y^o) \), we have that the upper bound \( y \leq \frac{W + (1-d)g}{n} + \frac{n-1}{n} y(g) \) is binding: implying \( y(g) = W + (1-d)g \)
in \( N_\varepsilon(y^o) \). We must therefore have a sequence of points \( g^m \to y^o \) such that \( g^m = y(g^{m-1}) \) and \( y(g^m) = W + (1-d)g^m \forall m \). Given this, we can write:

\[
v(g^m) = u(g^{m+1}) + \delta v(g^{m+1}) = u(g^{m+1}) + \delta [u(g^{m+2}) + \delta v(g^{m+2})] = \sum_{j=0}^{\infty} \delta^j u(W + (1-d)g^{m+j})
\]

We therefore must have that \( v(g^m) \) is differentiable and \( \delta v'(g^m) = \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W+(1-d)g^{m+j}). \) Since \( u'(g^m) + \delta v'(g^m) \geq 1 \), we have \( u'(g^m) + \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W+(1-d)g^{m+j}) \geq 1 \) for all \( m \). Consider the limit as \( m \to \infty \). Since \( u'(g) \) is continuous and \( g^m \to y^o \), we have:

\[
1 \leq \lim_{m \to \infty} \left[ u'(g^m) + \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W+(1-d)g^{m+j}) \right] = u'(y^o) + \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(y^o) = \frac{u'(y^o)}{1-\delta(1-d)}
\]

This implies \( y^o \leq [u']^{-1} (1-\delta(1-d)) < [u']^{-1} (1-\delta(1-d/n)) \), a contradiction. Assume now that \( y^o = W/d \), but it is not a strict maximum of \( u(y) + \delta v(y) - y \) in any left neighborhood. It must be that \( u(y) + \delta v(y) - y \) is constant in some left neighborhood \( N_\varepsilon(y^o) \). If this were not the case, then in any left neighborhood we would have an interval in which \( y^o \) is constant, but this is impossible by Lemma A.6. But then if \( u(y) + \delta v(y) - y \) is constant in some \( N_\varepsilon(y^o) \), the same argument as in Step 1 implies a contradiction. \( \blacksquare \)

### 3 Proof of Proposition 4

**Proposition 4.** For any \( d > 0 \) and \( n \), there is a \( \overline{\delta} < 1 \) such that the most efficient SPE path in a RIE and the most efficient SPE path in a IIE coincide with the Pareto efficient investment path for any \( \delta > \overline{\delta} \). Hence, neither the most efficient SPE path in a RIE nor the most efficient SPE path in a IIE are characterized by gradualism for any \( \delta > \overline{\delta} \).

We first show that there is a \( \delta_1 < 1 \), such that for \( \delta > \delta_1 \) the efficient path is a SPE path in an irreversible investment economy. To this goal, we first define the equilibrium strategies and establish some key properties. Let \( y^M(g;\delta) \), \( v^M(g;\delta) \) be, respectively, the investment function and the value function of the Markov equilibrium with the lowest steady state characterized in Proposition 2 when the discount factor is \( \delta \) and the rate of depreciation is \( d \). Let \( g^M(d;\delta) = [u']^{-1} (1-\delta(1-d)/n) \) be the associated steady state. It is easy to see that, for any \( d \) and \( n \), \( g^M(d;\delta) < y^*_\delta(d;\delta, n) \) for all \( \delta \in [0,1] \). Define \( y^M_j(g;\delta;\delta) \) recursively with \( y^0_M(g;\delta) = g \) and \( y^M_j(g;\delta;\delta) = y^M(y^M_{j-1}(g;\delta;\delta);\delta, \delta) \). For any \( g \), \( y^M_j(g;\delta;\delta) \to g^M(d;\delta) \) as \( j \to \infty \). It follows that \( \lim_{\delta \to 1} [(1-\delta) v^M(g;\delta;\delta)] = (W - d g^M(d,1)) / n + u(g^M(d,1)) \). Let \( y^\delta(g;\delta;\delta) \) be the
efficient investment function characterized in Section 3 with steady state \( g^P(d, \delta) = y^P_\delta(d, d, n) \), and let \( v^P(g; d, \delta) \) be the associated expected utility for a player. Similarly, it is easy to see that

\[
\lim_{\delta \to 1} [(1 - \delta) v^P(g; d, \delta)] = (W - dg^P(d, 1)) / n + u(g^P(d, 1)),
\]

where \( y^P(g; d, \delta) \) be the efficient investment function characterized in Section 3 with steady state \( g^P(d, \delta) = y^P_\delta(d, d, n) \). It follows that

\[
\lim_{\delta \to 1} [(1 - \delta) v^P(g; d, \delta)] > \lim_{\delta \to 1} [(1 - \delta) v^M(g; d, \delta)].
\]

Associated to an aggregate investment function \( y^l(g; d, \delta), l = \{M, P\} \), we have the individual contribution function: \( i^l(g; d, \delta) = [y^l(g; d, \delta) - (1 - d)g] / n \). To construct the equilibrium, consider the following trigger strategies. If \( g_\tau = y^P_t(g_\tau; g, \delta) \) for all \( \tau \leq t \), then \( i^l(g_\tau; d, \delta) = i^P(g; d, \delta) \), where \( i^l_t(g_\tau) \) is the investment at time \( t \) of an agent. If \( \exists \tau \leq t \) such that \( g_\tau \neq y^P(g; d, \delta) \), then \( i^l(g_\tau) = i^M(g; d, \delta) \). Note that, by construction, deviations are not profitable after a \( \tau \) such that \( g_\tau \neq y^P_t(g_\tau; g, \delta) \). For the remaining histories note that the average utility of a deviating agent must converge to \( (1 - \delta) v^M(g; d, \delta) \) such that for \( \delta > \delta_1 \) no deviation is profitable.

The result that we also have a \( \delta_2 < 1 \), such that for \( \delta > \delta_2 \) the efficient path is a SPE path in a reversible investment economy can be proven analogously. From Battaglini et al. [2012], we know that there is a Markov equilibrium \( \vec{y}^M(g; d, \delta), \vec{v}^M(g; d, \delta) \) with steady state \( \vec{y}^M(d, \delta) \leq (u')^{-1}(1 - \delta(1 - d)/n) \), and so strictly lower than the steady state \( g^P(d, 1) \) of the planner’s solution for all \( \delta \in [0, 1] \). Proceeding exactly as above we can see that

\[
\lim_{\delta \to 1} [(1 - \delta) v^P(g; d, \delta)] > \lim_{\delta \to 1} [(1 - \delta) v^M(g; d, \delta)].
\]

Associated to an aggregate investment function \( \vec{y}^M(g; d, \delta) \) we define as above the individual contribution function:

\[
i^M(g; d, \delta) = [\vec{y}^M(g; d, \delta) - (1 - d)g] / n.
\]

To construct the equilibrium, consider the following trigger strategies. If \( g_\tau = y^P_t(g_\tau; g, \delta) \) for all \( \tau \leq t \), then \( i^M(g_\tau; d, \delta) = i^P(g; d, \delta) \), where \( i^M(g_\tau) \) is the investment at time \( t \) of an agent. If \( \exists \tau \leq t \) such that \( g_\tau \neq y^P_t(g_\tau; g, \delta) \), then \( i^M(g_\tau) = i^M(g; d, \delta) \). Note that, by construction, deviations are not profitable after a \( \tau \) such that \( g_\tau \neq y^P_t(g_\tau; g, \delta) \). For the remaining histories note that the average utility of a deviating agent must converge to \( (1 - \delta) v^M(g; d, \delta) \) such that for \( \delta > \delta_2 \) no deviation is profitable. Given this, the statement of the proposition follows immediately by defining \( \vec{\delta} = \max(\delta_1, \delta_2) \).