

Online Appendix for “Multiproduct Search and the Joint Search Effect”

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March 2014

This appendix consists of three parts: First, I discuss the issue of (symmetric) equilibrium existence. Second, I explore the possibility of asymmetric equilibrium. Third, I extend the analysis in the paper to the case when firms sell and consumers buy more than two products, and derive the price formula in the uniform distribution example.

1 About (Symmetric) Equilibrium Existence

This section discusses the existence of symmetric pure-strategy equilibria.¹ We have known that the system of first-order conditions has a unique solution. Hence, if the first-order conditions are sufficient for defining equilibrium prices, then a unique symmetric pure-strategy equilibrium exists. The sufficiency of the first-order condition is guaranteed if a firm’s profit function defined in (9) in the paper is single-peaked at $\varepsilon = \mathbf{0}$ given the other firm charges $\mathbf{p}$ which solves the first-order conditions. As explained below, a general investigation of this problem is hard. I can solve this issue only partially: First, section 1.1 below shows that in the case with an exponential distribution, the profit function is indeed single-peaked. Second, section 1.2 below reports numerical results for several often used regular distributions which support the single-peakedness of the profit function.

I focus on the case with two symmetric products (i.e., $F_1 = F_2$). Let p be the equilibrium price for each product. For convenience, I slightly change the notation and the way of

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¹In our model, each firm’s pricing strategy space is a subset of $\mathbb{R}^2$ and we can also make it compact without loss of generality. The profit function is continuous in prices as long as the density functions f_i , $i = 1, 2$, are continuous. Then Theorem 1 in Becker and Damianov (2006) ensures that our model has a symmetric equilibrium in mixed strategies. (See Becker and Damianov (2006): “On the Existence of Symmetric Mixed Strategy Equilibria”, *Economics Letters*, 90(1), 84-87.)

expressing the deviation profit function. Suppose firm II sticks to the equilibrium prices, while firm I unilaterally deviates to $(p - x, p - y)$. Denote by $Q(x, y)$ the demand for firm I's product 1. Then $Q(y, x)$ is the demand for firm I's product 2 due to symmetry, and firm I's deviation profit is

$$\pi(x, y) \equiv (p - x)Q(x, y) + (p - y)Q(y, x) .$$

For the consumers who visit firm I first, their stopping rule is characterized by the reservation frontier $\phi_{\Delta}(u)$ which satisfies

$$\zeta(u + x) + \zeta(\phi_{\Delta}(u) + y) = s ,$$

where $\zeta(z) = \int_z^{\bar{u}} [1 - F(u)] du$. (Charging a price lower than its rival by x is equivalent to supplying a product with match utility higher than its rival by x .) Then we have

$$\phi_{\Delta}(u) = \phi(u + x) - y .$$

We extend the support of $\phi_{\Delta}(\cdot)$ to the whole domain and stipulate that $\phi_{\Delta}(u) > \bar{u}$ for $u < a - x$ such that $F(\phi_{\Delta}(u)) = 1$ and $f(\phi_{\Delta}(u)) = 0$.² While for those consumers who visit firm II first, they adopt the equilibrium stopping rule since they hold the equilibrium belief about firm I's prices.

The demand for firm I's product 1 is then

$$Q(x, y) = \frac{1}{2} \int_{\underline{u}}^{\bar{u}} [1 - F(\phi_{\Delta}(u))(1 - F(u + x))] dF(u) + \frac{1}{2} \int_{\underline{u}}^{\bar{u}} F(\phi(u))(1 - F(u - x)) dF(u) .$$

(Note that $\phi(u) > \bar{u}$ for $u < a$ such that $F(\phi(u)) = 1$ and $f(\phi(u)) = 0$.) For a consumer who visits firm I first and values its product 1 at u_1 , she will continue to visit firm II if her valuation for firm I's product 2 is below $\phi_{\Delta}(u_1)$. In that case, she will eventually buy product 1 from firm II if $v_1 > u_1 + x$. So the integrand in the first term is the probability that this consumer will buy product 1 from firm I (i.e., the probability that she either stops searching immediately and so buys both products at firm I or returns to firm I and buy its product 1 after visiting firm II). For a consumer who visits firm II first and values its product 1 at v_1 , she will come to firm I if her valuation for firm II's product 2 is below $\phi(v_1)$. She will then buy firm I's product 1 if $u_1 + x > v_1$. The whole event occurs with probability $F(\phi(v_1))(1 - F(v_1 - x))$. This is the integrand in the second term. The fraction 1/2 in

²When the match utility has a bounded support, we should be careful about the boundary problem (which does not matter for the local deviation analysis in the paper but matters here for the global deviation analysis). For example, when $\bar{u} < \infty$ and $y > 0$, at $u = a - x$ the curve of $\phi_{\Delta}(\cdot)$ has a vertical segment from $\bar{u} - y$ to $\bar{u}$. Similarly, when $\bar{u} < \infty$ and $x > 0$, $\phi_{\Delta}(u) = a - y$ for $u \geq \bar{u} - x$ and so the curve of $\phi_{\Delta}(\cdot)$ has a horizontal segment.

front of each term is because half of the consumers visit each firm first. In the following, we ignore $1/2$ since it does not affect the analysis.

A general discussion. In general, the profit function $\pi(x, y)$ is *not* even quasi-concave. $\pi(x, y)$ is quasi-concave if the upper counter set $\{(x, y) : \pi(x, y) \geq t\}$ is convex for any t . But from the numerical results reported later, we can see that the upper counter set is not convex for relatively small t . However, in all the numerical examples $\pi(x, y)$ is still single-peaked at $x = y = 0$. This suggests the sufficiency of the first-order conditions for defining symmetric equilibrium prices.

In the one-dimensional case, quasi-concavity is essentially equivalent to single-peakedness. But in the two(or higher)-dimensional case, quasi-concavity is a stronger condition than single-peakedness. (Quasi-concavity requires all upper counter sets be convex, while single-peakedness only requires all upper counter sets have no holes (or be *simply connected*).) In our model, quasi-concavity is a too strong condition to impose. But I am not aware of any proper concepts between quasi-concavity and single-peakedness which can be easily checked. This is a main reason why it is hard to generally investigate the sufficiency of first-order conditions for equilibrium prices.

One possible way to make progress is to show that $\pi(x, y)$ is quasi-concave (or even concave) on an area around $(0, 0)$, and beyond that area it does not exceed $\pi(0, 0)$. This is the method I will use to deal with the exponential example below.³

Before moving to the exponential example, let me explain more about why I only work out that example. When we deal with specific examples, there are two main challenges: (i) The reservation frontier function $\phi(\cdot)$ should take a simple form such that it is possible to calculate the integrals in the demand function. (ii) When a firm deviates from the equilibrium prices, the resulted reservation frontier $\phi_\Delta(\cdot)$ will shift around. If $\bar{u}$ or $\underline{u}$ or both are finite, then $\phi_\Delta(\cdot)$ may hit the horizontal or vertical axis (when the firm reduces its prices and the stopping region expands sufficiently), or it may degenerate (when the firm raise its prices sufficiently). This will give rise to boundary constraints which we should take into account in demand calculation.

The second issue can be avoided if we consider the distributions with unbounded supports (i.e., with $\bar{u} = \infty$ and $\underline{u} = -\infty$). But among those distributions, $\phi(\cdot)$ is usually analytically unsolvable. (In simulations, we thus need to approximate it numerically.)⁴ In the exponential example, $\phi(\cdot)$ takes a simple form ($e^{-\phi(u)} = s - e^{-u}$). But we need to

³An alternative method is to show that $\pi(x, kx)$ is quasi-concave in x for any k (i.e., the profit function $\pi(x, y)$ is single-peaked at $(0, 0)$ along each line which passes through the original point). This will reduce a two-dimensional problem to a one-dimensional problem.

⁴The only exception I notice is the logistic distribution. In that case $\phi(\cdot)$ solves $(1+e^{-u})(1+e^{-\phi(u)}) = e^s$. But it turns out that the demand function $Q(x, y)$ is still complicated such that no progress can be made.

deal with the boundary constraints caused by $\underline{u} = 0$. As we will see below, to characterize demand we need to divide the space of (x, y) into 10 regions. In the uniform example, $\phi(\cdot)$ also takes a relatively simple form ($\phi(u) = 1 - \sqrt{2s - (1 - u)^2}$). But there are more boundary constraints in that case (as both $\bar{u}$ and $\underline{u}$ are finite), and we actually need to divide the space of (x, y) into 20 regions. Moreover, it turns out that in some regions the demand function does not have a simple expression due to integration problem.

1.1 The exponential example

With the exponential distribution, I can show that $\pi(x, y)$ is maximized at $x = y = 0$. (So the price p defined by the first-order condition is indeed an equilibrium price.) The logic of the argument is basically as follows: on a certain area around $(0, 0)$, I can show that $\pi(x, y)$ is actually concave; and beyond that area I can directly verify $\pi(x, y) < \pi(0, 0)$. (In the latter case, $\pi(x, y)$ can be complicated and hard to deal with. I will then make use of a simpler auxiliary function $\hat{\pi}(x, y)$ such that $\pi(x, y) \leq \hat{\pi}(x, y) < \pi(0, 0)$.)

In the following, I first derive $Q(x, y)$, the demand function for product 1. (The demand function for product 2 is then $Q(y, x)$.) As we will see below, $Q(x, y)$ takes different forms on different regions of (x, y) . In some regions, it can be complicated. In that case, I derive some “amplified” demand functions which will be used to construct $\hat{\pi}(x, y)$ later.

Demand functions. When the support of the match utility distribution is $[0, \infty)$, we have

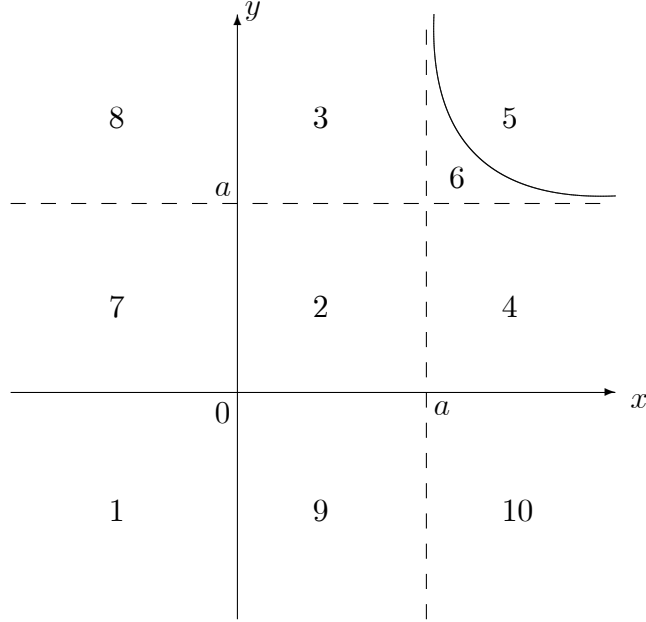
$$\begin{aligned} Q(x, y) &= \int_0^\infty [1 - F(\phi(u + x) - y)(1 - F(u + x))]dF(u) + \int_0^\infty F(\phi(u))(1 - F(u - x))dF(u) \\ &= \int_0^{a-x} F(u + x)f(u)du + \int_a^\infty [1 - F(\phi(u) - y)(1 - F(u))]f(u - x)du \\ &\quad + \int_0^a (1 - F(u - x))f(u)du + \int_a^\infty F(\phi(u))(1 - F(u - x))f(u)du . \end{aligned}$$

(The second step is because $\phi(u)$ is only defined on (a, ∞) and for $u \leq a$ we have stipulated $\phi(u) = \infty$. We also changed the integral variable in the second term.) In the exponential case, $f(u) = e^{-u}$ for $u \geq 0$ and equals zero otherwise, and $F(u) = 1 - e^{-u}$ for $u \geq 0$ and equals zero otherwise. Taking into account these boundary conditions yields

$$\begin{aligned} Q(x, y) &= \int_0^{\max\{0, a-x\}} (1 - e^{-\max\{0, u+x\}}) e^{-u} du \\ &\quad + \int_a^\infty [1 - (1 - e^{-\max\{0, \phi(u)-y\}})e^{-u}]e^{-(u-x)}\mathbb{I}_{u-x \geq 0} du \\ &\quad + \int_0^a e^{-\max\{0, u-x\}} e^{-u} du + \int_a^\infty (1 - e^{-\phi(u)})e^{-\max\{0, u-x\}} e^{-u} du , \end{aligned} \quad (1)$$

where $\mathbb{I}_{u-x \geq 0}$ is an indicator function (i.e., it equals 1 if $u - x \geq 0$ and 0 otherwise). Two useful facts in the exponential example are: $e^{-a} = s$ and $e^{-\phi(u)} = s - e^{-u}$.

To deal with the boundary conditions in (1), it is helpful to divide the space of (x, y) into ten regions as indicated in the graph below (where the curve in the northeast corner is $y = \phi(x)$).



- Region 1: $x, y < 0$

In this region, only the boundary constraint $\max\{0, u + x\}$ in the first term in (1) matters. By changing the integral variable, one can check that

$$Q(x, y) = e^x \left(1 + \frac{s^3}{6} (e^y - 1) \right) .$$

- Region 2: $0 < x, y < a$

In this region, only the boundary constraint $\max\{0, u - x\}$ in the third term matters. One can check that

$$Q(x, y) = 2 - e^{-x} + \frac{s^3}{6} e^x (e^y - 1) .$$

- Region 3: $0 < x < a < y$

In this region, as well as $\max\{0, u - x\}$ in the third term, the boundary constraint $\max\{0, \phi(u) - y\}$ in the second term in (1) also matters. Using the fact that $\phi(u) - y > 0 \Leftrightarrow u < \phi^{-1}(y) = \phi(y)$, one can check that

$$Q(x, y) = 2 - e^{-x} + \frac{1}{6}e^x (3s^2 - s^3 - 3se^{-y} + e^{-2y}) .$$

Notice that if we ignore the boundary constraint $\max\{0, \phi(u) - y\}$, $1 - e^{-(\phi(u) - y)}$ can take negative values for some u . This will increase the demand artificially, and the amplified demand has the same expression as in Region 2. So in Region 3,

$$Q(x, y) \leq 2 - e^{-x} + \frac{s^3}{6}e^x(e^y - 1) . \quad (2)$$

- Region 4: $0 < y < a < x$

In this region, the first term in (1) equals zero and the third term equals $\int_0^a e^{-u} du = 1 - s$. One can then check that

$$Q(x, y) = 2 - \frac{1}{2}s^2 - e^{-x} + \frac{1}{2}se^{-x}(1 + e^y) - \frac{1}{6}e^{-2x}(1 + 2e^y) .$$

Notice that ignoring the constraints in the second and fourth terms in (1) will amplify the demand. So in Region 4, one can check that

$$Q(x, y) \leq 1 - s + se^x + \frac{s^3}{6}e^x(e^y - 1) . \quad (3)$$

- Regions 5 and 6: $x, y > a$

In Region 5, we have $y > \phi(x)$ and so one can check that

$$Q(x, y) = 2 - \frac{1}{2}s^2 - \frac{1}{2}(1 - s)e^{-x} - \frac{1}{6}e^{-2x} .$$

In Region 6, we have $y < \phi(x)$ and so

$$Q(x, y) = 2 - \frac{1}{2}s^2 - \frac{1}{2}(1 - s)e^{-x} - \frac{1}{6}e^{-2x} - \frac{1}{6}e^{x+y} (e^{-x} - s + e^{-y})^2 (s + 2e^{-x} - e^{-y}) .$$

However, as we did in Region 4, if we ignore the boundary constraints in the second and fourth terms in (1), we have

$$Q(x, y) \leq 1 - s + se^x + \frac{s^3}{6}e^x(e^y - 1) . \quad (4)$$

- Region 7: $x < 0 < y < a$

In this region, the boundary constraints are the same as in the case of $x, y < 0$. So the demand is the same as in Region 1:

$$Q(x, y) = e^x \left(1 + \frac{s^3}{6} (e^y - 1) \right) .$$

- Region 8: $x < 0$ and $y > a$

In this region, the demand is

$$Q(x, y) = e^x + \frac{1}{6}e^x (3s^2 - s^3 - 3se^{-y} + e^{-2y}) .$$

But if we ignore the boundary constraint $\max\{0, \phi(u) - y\}$ in the second term in (1), the demand will be amplified and it will have the same expression as in Region 7. Hence, in Region 8,

$$Q(x, y) \leq e^x \left(1 + \frac{s^3}{6} (e^y - 1) \right) . \quad (5)$$

- Region 9: $y < 0 < x < a$

In this region, the boundary constraints are the same as in the case of $0 < x, y < a$, and so

$$Q(x, y) = 2 - e^{-x} + \frac{s^3}{6}e^x(e^y - 1) .$$

- Region 10: $y < 0$ and $x > a$

In this region, the boundary constraints are the same as in Region 4, so

$$Q(x, y) = 2 - \frac{1}{2}s^2 - e^{-x} + \frac{1}{2}se^{-x}(1 + e^y) - \frac{1}{6}e^{-2x}(1 + 2e^y) .$$

However, if we ignore the boundary constraints in the second and fourth terms as we did in Region 4, we have

$$Q(x, y) \leq 1 - s + se^x + \frac{s^3}{6}e^x(e^y - 1) . \quad (6)$$

Single-peakedness of the profit function. Remember that when firm I deviates to $(p - x, p - y)$, its profit is

$$\pi(x, y) = (p - x)Q(x, y) + (p - y)Q(y, x) . \quad (7)$$

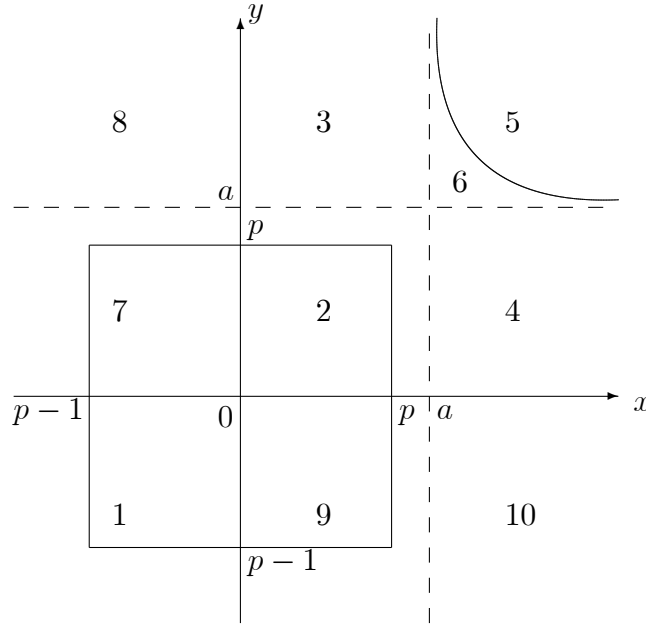
We have known that $\pi(x, y)$ has a critical point $(0, 0)$ given p solves the first-order condition. At $x = y = 0$, the demand for each product is $Q(0, 0) = 1$ and the corresponding profit is $\pi(0, 0) = 2p$. In the following, I aim to show that $\pi(x, y) \leq \pi(0, 0)$ for any (x, y) when the

search cost condition $s \leq 1$ is satisfied, and the equality holds only at $x = y = 0$. Recall that in the exponential example,

$$p = \frac{1}{1 + s^3/6} \in [\frac{6}{7}, 1] \text{ for } s \in [0, 1] . \quad (8)$$

I deal with two cases separately.

Case I: Let us first consider the case with $p \leq a$. (Using (8) and $a = -\ln s$, this requires s be less than about 0.371.) In this case, the following argument basically shows that $\pi(x, y)$ is concave for $(x, y) \in (p-1, p)^2$, and beyond that area $\pi(x, y) < \pi(0, 0)$.



The first simple observation is that in the positive quadrant $(x, y) \in (0, \infty)^2$ we can focus on $(x, y) \in (0, p)^2$. Notice that the demand for each product must be non-negative but cannot exceed 2. In the positive quadrant, if $x \geq p$ or $y \geq p$ or both, then (7) implies that $\pi(x, y)$ must be less than $\pi(0, 0) = 2p$.

Another useful observation in this case is that $\pi(x, y)$ must be less than $\pi(0, 0) = 2p$ in Regions 8 and 10. Let us consider Region 8 with $x < 0$ and $y > a$. Then

$$\begin{aligned} \pi(x, y) = & (p-x)e^x \left(1 + \frac{1}{6} (3s^2 - s^3 - 3se^{-y} + e^{-2y}) \right) \\ & + (p-y) \left(2 - e^{-y} - \frac{1}{2}s^2 + \frac{1}{2}se^{-y}(1+e^x) - \frac{1}{6}e^{-2y}(1+2e^x) \right) . \end{aligned}$$

(By using the symmetry, the demand function $Q(y, x)$ in this case takes the form in Region 10.) Given $p \leq a < y$, the second component must be negative. In the same time, in the first term we have $(p - x)e^x \leq e^{p-1} < 1$,⁵ and

$$1 + \frac{1}{6} (3s^2 - s^3 - 3se^{-y} + e^{-2y}) < 1 + \frac{1}{6} (3s^2 - s^3) \leq \frac{4}{3}.$$

(The first inequality used $e^{-2y} - 3se^{-y} < 0$ for $y > a$, and the second inequality used $3s^2 - s^3 \leq 2$ for $s \leq 1$.) Therefore, $\pi(x, y) < \frac{4}{3} < 2p$ since $p = \frac{1}{1+s^3/6} \geq \frac{6}{7}$ for $s \leq 1$. Region 10 can be treated similarly.

Now there are only four regions—1, 2, 7, 9—remained to deal with.

- Region 1: $x, y < 0$

In this region, the deviation profit function is

$$\begin{aligned} \pi(x, y) &= (p - x)e^x \left(1 + \frac{s^3}{6}(e^y - 1)\right) + (p - y)e^y \left(1 + \frac{s^3}{6}(e^x - 1)\right) \\ &= \left(1 - \frac{s^3}{6}\right)(p - x)e^x + \left(1 - \frac{s^3}{6}\right)(p - y)e^y + \frac{s^3}{6}e^{x+y}(2p - x - y). \end{aligned}$$

We first argue that we can focus on $(x, y) \in (p - 1, 0)^2$. Note that the partial derivative with respect to x is

$$\pi_1(x, y) = \left(1 - \frac{s^3}{6}\right)(p - x - 1)e^x + \frac{s^3}{6}e^{x+y}(2p - x - y - 1).$$

Since $2p - 1 > 0$, the second term must be positive for $x, y < 0$. Hence, the profit strictly increases with x if $x \leq p - 1$ for any given $y < 0$. Similarly, the profit also strictly increases with y if $y \leq p - 1$ for any given $x < 0$.

Second, we show that $\pi(x, y)$ is strictly concave on $(p - 1, 0)^2$. One can check that

$$\begin{aligned} \pi_{11}(x, y) &= \left(1 - \frac{s^3}{6}\right)(p - x - 2)e^x + \frac{s^3}{6}e^{x+y}(2p - x - y - 2), \\ \pi_{22}(x, y) &= \left(1 - \frac{s^3}{6}\right)(p - y - 2)e^y + \frac{s^3}{6}e^{x+y}(2p - x - y - 2), \\ \pi_{12}(x, y) &= \frac{s^3}{6}e^{x+y}(2p - x - y - 2). \end{aligned}$$

Since $p - x - 1 < 0$ and $p - y - 1 < 0$ for $(x, y) \in (p - 1, 0)^2$, it is ready to see that each term in $\pi_{ii}(x, y)$ is negative and $\pi_{11}(x, y)\pi_{22}(x, y) > \pi_{12}(x, y)^2$. This proves the claim.

Since $\pi(x, y)$ has a critical point at $x = y = 0$, we conclude that $\pi(x, y) < \pi(0, 0) = 2p$ for $x, y < 0$.

⁵Notice that $(p - x)e^x$ is single-peaked at $x = p - 1$.

- Region 2: $0 < x, y < p$

Given $p < a$, the profit function in this region is

$$\pi(x, y) = (p - x) \left(2 - e^{-x} + \frac{s^3}{6} e^x (e^y - 1) \right) + (p - y) \left(2 - e^{-y} + \frac{s^3}{6} e^y (e^x - 1) \right) .$$

We show that $\pi(x, y)$ is strictly concave on $(0, p)^2$. One can check that

$$\begin{aligned} \pi_{11}(x, y) &= -2 \left(e^{-x} - \frac{s^3}{6} e^x \right) - (p - x) \left(e^{-x} + \frac{s^3}{6} e^x \right) + \frac{s^3}{6} e^{x+y} (2p - x - y - 2) , \\ \pi_{22}(x, y) &= -2 \left(e^{-y} - \frac{s^3}{6} e^y \right) - (p - y) \left(e^{-y} + \frac{s^3}{6} e^y \right) + \frac{s^3}{6} e^{x+y} (2p - x - y - 2) , \\ \pi_{12}(x, y) &= \frac{s^3}{6} e^{x+y} (2p - x - y - 2) . \end{aligned}$$

Since $p < 1$ and $e^{-x} > \frac{s^3}{6} e^x$ for $x \leq a$, each term in $\pi_{ii}(x, y)$ is negative for $(x, y) \in (0, p)^2$ and $\pi_{11}(x, y)\pi_{22}(x, y) > \pi_{12}(x, y)^2$. This proves the claim.

Since $\pi(x, y)$ has a critical point at $x = y = 0$, we conclude that $\pi(x, y) < \pi(0, 0)$ for $0 < x, y < p$.

- Region 7: $x < 0$ and $0 < y < a$, and Region 9: $0 < x < a$ and $y < 0$

I only report the details for Region 9. (Region 7 can be treated similarly.) In Region 9, the profit function is

$$\pi(x, y) = (p - x) \left(2 - e^{-x} + \frac{s^3}{6} e^x (e^y - 1) \right) + (p - y) e^y \left(1 + \frac{s^3}{6} (e^x - 1) \right) .$$

(Notice that $Q(y, x)$ takes the form in Region 7.)

I first argue that we can focus on $(x, y) \in (0, p) \times (p - 1, 0)$. (i) $\pi(x, y)$ is decreasing in $x \in [p, a]$ for any given $y < 0$. Note that

$$\pi_1(x, y) = - \left(2 - e^{-x} + \frac{s^3}{6} e^x (e^y - 1) \right) + (p - x) \left(e^{-x} + \frac{s^3}{6} e^x (e^y - 1) \right) + \frac{s^3}{6} (p - y) e^{x+y} .$$

The first term in $\pi_1(x, y)$ equals

$$- (2 - e^{-x}) + \frac{s^3}{6} e^x (1 - e^y) < -1 + \frac{s^2}{6} .$$

(The inequality used $e^{-x}, e^y \in (0, 1)$ for a positive x and a negative y , and $e^x < e^a = \frac{1}{s}$ for $x < a$.) The second term in $\pi_1(x, y)$ is negative because $x \geq p$ and $e^{-x} + \frac{s^3}{6} e^x (e^y - 1) > e^{-x} - \frac{s^3}{6} e^x > 0$ for $x < a$. The third term is smaller than $\frac{s^2}{6}$ by using $e^x < e^a = \frac{1}{s}$ and $(p - y)e^y \leq e^{p-1} \leq 1$. Therefore,

$$\pi_1(x, y) < -1 + \frac{s^2}{3} < 0 .$$

(ii) $\pi(x, y)$ is increasing in $y \leq p - 1$ for any $x < p$. This is because

$$\pi_2(x, y) = \frac{s^3}{6}(p - x)e^{x+y} + (p - y - 1)e^y \left(1 + \frac{s^3}{6}(e^x - 1)\right) > 0$$

for $x < p$ and $y \leq p - 1$.

Then I show $\pi(x, y)$ is concave for $(x, y) \in (0, p) \times (p - 1, 0)$. This is because

$$\pi_{11}(x, y) = -2 \left(e^{-x} - \frac{s^3}{6}e^x \right) - (p - x) \left(e^{-x} + \frac{s^3}{6}e^x \right) + \frac{s^3}{6}e^{x+y}(2p - x - y - 2) ,$$

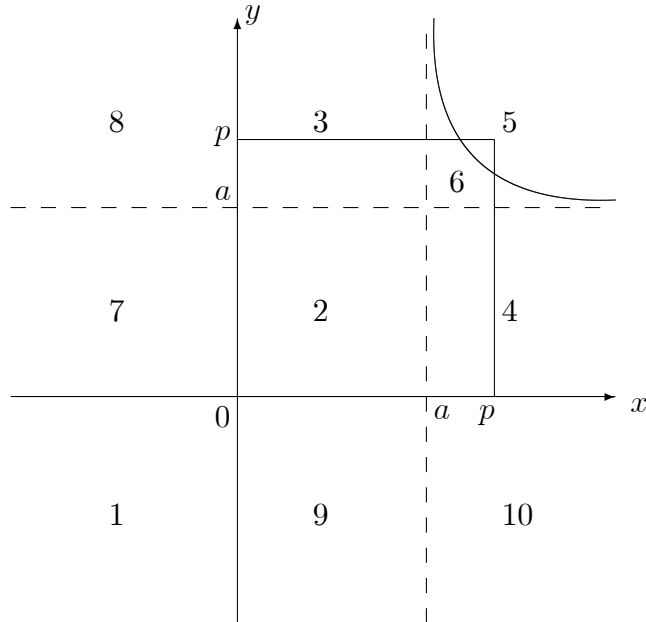
$$\pi_{22}(x, y) = \left(1 - \frac{s^3}{6} \right) (p - y - 2)e^y + \frac{s^3}{6}e^{x+y}(2p - x - y - 2) ,$$

$$\pi_{12}(x, y) = \frac{s^3}{6}e^{x+y}(2p - x - y - 2) .$$

Following a similar argument as in Regions 1 and 2, one can see that each term in $\pi_{ii}(x, y)$ is negative and $\pi_{11}(x, y)\pi_{22}(x, y) > \pi_{12}(x, y)^2$. This proves the claim.

Since $\pi(x, y)$ has a critical point at $x = y = 0$, we conclude that $\pi(x, y) < \pi(0, 0)$ in Region 9.

Case II: We now turn to the case with $p > a$ (i.e., when s is between about 0.371 and 1). This case is more complicated. For example, in the positive quadrant, although we can still focus on $(0, p)^2$ for the same reason as in Case I, we now need to deal with more cases as indicated in the figure below.



- Region 1: $x, y < 0$

In this negative quadrant, the profit function takes the form as in Case I, so the same analysis applies (i.e., $\pi(x, y)$ is concave for $(x, y) \in (p-1, 0)^2$, and beyond that area $\pi(x, y) < \pi(0, 0)$).

- Region 2: $0 < x, y < a$

In this region, the same analysis as in Case I also applies (i.e., $\pi(x, y)$ is concave).

- Region 3: $0 < x < a$ and $a < y < p$, and Region 4: $a < x < p$ and $0 < y < a$

I only report the details for Region 3. (Region 4 can be treated similarly.) In this region, the profit function is complicated. But we have known from (2) and (3) that

$$Q(x, y) \leq 2 - e^{-x} + \frac{s^3}{6}e^x(e^y - 1); \quad Q(y, x) \leq 1 - s + se^y + \frac{s^3}{6}e^y(e^x - 1) .$$

Since both $p - x$ and $p - y$ are positive, we have

$$\pi(x, y) \leq \hat{\pi}(x, y) \equiv (p-x) \left(2 - e^{-x} + \frac{s^3}{6}e^x(e^y - 1) \right) + (p-y) \left(1 - s + se^y + \frac{s^3}{6}e^y(e^x - 1) \right) .$$

First, we can show that $\hat{\pi}(x, y)$ is concave on $(0, p)^2$. This is because

$$\begin{aligned} \hat{\pi}_{11}(x, y) &= -2 \left(e^{-x} - \frac{s^3}{6}e^x \right) - (p-x) \left(e^{-x} + \frac{s^3}{6}e^x \right) + \frac{s^3}{6}e^{x+y}(2p-x-y-2) , \\ \hat{\pi}_{22}(x, y) &= \left(s - \frac{s^3}{6} \right) (p-y-2)e^y + \frac{s^3}{6}e^{x+y}(2p-x-y-2) , \\ \hat{\pi}_{12}(x, y) &= \frac{s^3}{6}e^{x+y}(2p-x-y-2) . \end{aligned}$$

One can check that $\frac{s^3}{3}e^x - 2e^{-x} < 0$ for $x < a$. So each term in the above second-order derivatives are negative. This implies that $\hat{\pi}_{ii}(x, y) < 0$ and $\hat{\pi}_{11}(x, y)\hat{\pi}_{22}(x, y) > \hat{\pi}_{12}(x, y)^2$.

Second, one can check that $\hat{\pi}_1(0, 0) = p(1 + \frac{s^3}{6}) - 1 = 0$ and $\hat{\pi}_2(0, 0) = p(s + \frac{s^3}{6}) - 1 \leq 0$. Third, we have $\hat{\pi}(0, 0) = 2p$. These three observations imply that in Region 3,

$$\pi(x, y) \leq \hat{\pi}(x, y) < \hat{\pi}(0, 0) + x\hat{\pi}_1(0, 0) + y\hat{\pi}_2(0, 0) \leq \hat{\pi}(0, 0) = \pi(0, 0) .$$

(The second step used the concavity of $\hat{\pi}(x, y)$.)

- Regions 5 and 6: $a < x < p$ and $a < y < p$

In this case, the actual profit function is again complicated. But we have known from (4) that $Q(x, y) \leq 1 - s + se^x + \frac{s^3}{6}e^x(e^y - 1)$. So

$$\pi(x, y) \leq \hat{\pi}(x, y) \equiv (p-x) \left(1 - s + se^x + \frac{s^3}{6}e^x(e^y - 1) \right) + (p-y) \left(1 - s + se^y + \frac{s^3}{6}e^y(e^x - 1) \right).$$

One can check that

$$\begin{aligned} \hat{\pi}_{11}(x, y) &= \left(s - \frac{s^3}{6} \right) (p - x - 2)e^x + \frac{s^3}{6}e^{x+y}(2p - x - y - 2), \\ \hat{\pi}_{22}(x, y) &= \left(s - \frac{s^3}{6} \right) (p - y - 2)e^y + \frac{s^3}{6}e^{x+y}(2p - x - y - 2), \\ \hat{\pi}_{12}(x, y) &= \frac{s^3}{6}e^{x+y}(2p - x - y - 2). \end{aligned}$$

Similarly as in Region 3, (i) $\hat{\pi}(x, y)$ is concave on $(0, p)^2$, (ii) $\hat{\pi}_1(0, 0) = \hat{\pi}_2(0, 0) = p(s + \frac{s^3}{6}) - 1 \leq 0$, and (iii) $\hat{\pi}(0, 0) = 2p$. Therefore, in Regions 5 and 6 we also have

$$\pi(x, y) \leq \hat{\pi}(x, y) < \hat{\pi}(0, 0) + x\hat{\pi}_1(0, 0) + y\hat{\pi}_2(0, 0) \leq \hat{\pi}(0, 0) = \pi(0, 0).$$

- Region 7: $x < 0$ and $0 < y < a$, and Region 9: $0 < x < a$ and $y < 0$

These two cases can be treated similarly as in Case I. For example, in Region 9, $\pi(x, y)$ is concave for $(x, y) \in (0, a) \times (p-1, 0)$, and beyond that area $\pi(x, y)$ is increasing in y .

- Region 8: $x < 0$ and $y > a$, and Region 10: $x > a$ and $y < 0$

I only report the details for Region 8. (Region 10 can be treated similarly.) First of all, using a similar argument as in Case I, one can show that if $x < 0$ and $y \geq p(> a)$, then $\pi(x, y) < 2p$. Thus, in Region 8 we can focus on $x < 0$ and $a < y < p$.

In this area, we already knew from (5) and (6) that

$$Q(x, y) \leq e^x \left(1 + \frac{s^3}{6}(e^y - 1) \right); \quad Q(y, x) \leq 1 - s + se^y + \frac{s^3}{6}e^y(e^x - 1).$$

Given both $p-x$ and $p-y$ are positive, we have

$$\pi(x, y) \leq \hat{\pi}(x, y) \equiv (p-x)e^x \left(1 + \frac{s^3}{6}(e^y - 1) \right) + (p-y) \left(1 - s + se^y + \frac{s^3}{6}e^y(e^x - 1) \right).$$

Notice that

$$\hat{\pi}_1(x, y) = (p-y)\frac{s^3}{6}e^{x+y} + (p-x-1)e^x \left(1 + \frac{s^3}{6}(e^y - 1) \right).$$

So $\hat{\pi}(x, y)$ is increasing in $x \leq p-1$ for any given $y \in (a, p)$.

Now we show that $\hat{\pi}(x, y)$ is concave for $(x, y) \in (p-1, 0) \times (0, p)$. This is because

$$\begin{aligned}\hat{\pi}_{11}(x, y) &= \left(1 - \frac{s^3}{6}\right) (p-x-2)e^x + (2p-x-y-2)\frac{s^3}{6}e^{x+y} , \\ \hat{\pi}_{22}(x, y) &= \left(s - \frac{s^3}{6}\right) (p-y-2)e^y + (2p-x-y-2)\frac{s^3}{6}e^{x+y} , \\ \hat{\pi}_{12}(x, y) &= (2p-x-y-2)\frac{s^3}{6}e^{x+y} .\end{aligned}$$

Each term is negative in the domain.

Finally, it is easy to verify that $\hat{\pi}(0, 0) = 2p$, $\hat{\pi}_1(0, 0) = -1 + p(1 + \frac{s^3}{6}) = 0$, and $\hat{\pi}_2(0, 0) = -1 + p(s + \frac{s^3}{6}) \leq 0$. All these observations imply that

$$\pi(x, y) \leq \hat{\pi}(x, y) < \hat{\pi}(0, 0) + x\hat{\pi}_1(0, 0) + y\hat{\pi}_2(0, 0) \leq \hat{\pi}(0, 0) = \pi(0, 0) .$$

1.2 Numerical results

I did numerical simulations for the following distributions (please find the Matlab code on the website: <https://sites.google.com/site/jidongzhou77/research>)⁶:

- Uniform distribution: $f(u) = 1$
- Exponential distribution: $f(u) = e^{-u}$
- Standard normal distribution: $f(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}$
- Logistic distribution: $f(u) = \frac{e^u}{(1+e^u)^2}$
- Standard Gumbel distribution: $f(u) = e^{-(u+e^{-u})}$

The graphs reported in the end of this appendix suggest that for the above distributions $\pi(x, y)$ is single-peaked at $x = y = 0$ and so the price p determined by the first-order condition is indeed an equilibrium price.

2 About Asymmetric Equilibrium

This part aims to show that when the two products are symmetric, if the density function f is logconcave, there are no asymmetric equilibria in which one firm, say, firm I, charges (p_L, p_H) and firm II charges (p_H, p_L) with $p_L < p_H$, and consumers search in a random

⁶We need to deal with the uniform and the exponential examples separately, because they have different boundary conditions. For all other distributions with an unbounded support, the code is general.

order. To this end, I will argue that starting from such a hypothetical asymmetric equilibrium, it is a profitable deviation for firm I to set prices $(p_L + \varepsilon, p_H - \varepsilon)$ with a small $\varepsilon > 0$, given that firm II sticks to (and consumers believe) the suggested equilibrium pricing strategy. For notational convenience, let $\Delta \equiv p_H - p_L > 0$.

Consider the hypothetical asymmetric equilibrium. Denote by $\phi_\Delta(u_1)$ the reservation frontier for those consumers who visit firm I first. It satisfies

$$\zeta(u_1 + \Delta) + \zeta(\phi_\Delta(u_1) - \Delta) = s .$$

(Charging a price lower/higher than its rival by Δ is equivalent to supplying a product with match utility higher/lower than its rival by Δ .) Then

$$\phi_\Delta(u_1) = \phi(u_1 + \Delta) + \Delta . \quad (9)$$

Relative to the reservation frontier $\phi(\cdot)$ in the symmetric equilibrium with $\Delta = 0$, the new reservation frontier $\phi_\Delta(\cdot)$ has shifted both leftward and upward by Δ . (See Figure A1(a) below where the thick curve is the new reservation frontier and the feint one is the reservation frontier in the symmetric equilibrium.) Similarly, for those consumers who visit firm II first, their stopping rule can be characterized by $\phi_\Delta(v_2)$ which solves $\zeta(\phi_\Delta(v_2) - \Delta) + \zeta(v_2 + \Delta) = s$. As we can see from Figure A1(b) below, relative to the reservation frontier in the symmetric equilibrium, it has shifted rightward and downward by Δ .

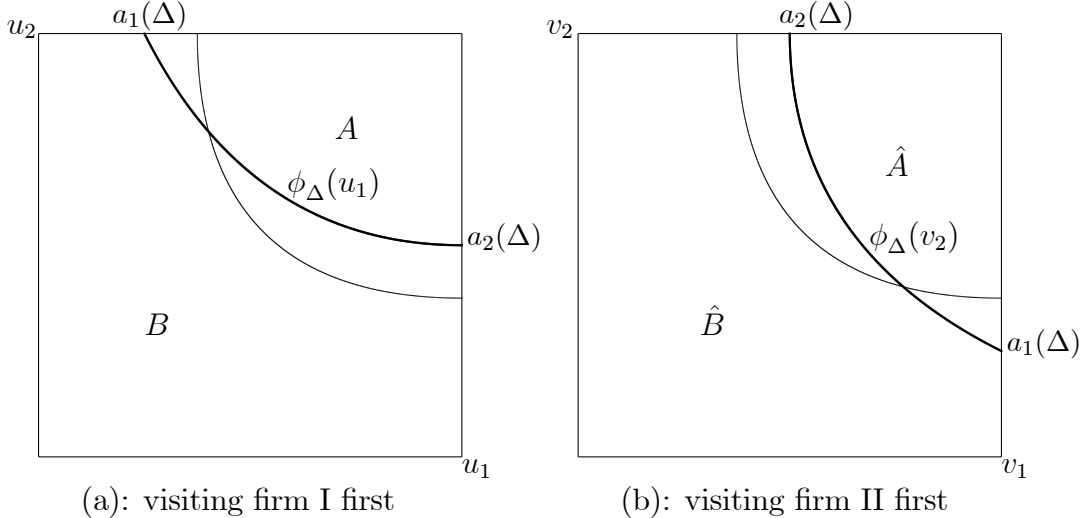


Figure A1: The stopping rule with asymmetric pricing

It is ready to see $a_1(\Delta) = \phi(\bar{u} - \Delta) - \Delta$ (if $\bar{u} = \infty$, $a_1(\Delta) = a - \Delta$) and $a_2(\Delta) = a + \Delta$. Notice that $\phi_\Delta(\cdot)$ is no longer symmetric around the 45-degree line, and if $\bar{u} < \infty$, it also has a flat segment on the interval $[\bar{u} - \Delta, \bar{u}]$ (which has not been precisely indicated in the

above graphs). In the following, for notational simplicity, I will restrict the discussion to the case with $\bar{u} = \infty$. (A slightly modified argument will work for $\bar{u} < \infty$.)

Now I derive the first-order effects if firm I deviates to $(p_L + \varepsilon, p_H - \varepsilon)$. Denote by Q_i the number of consumers who buy product i from firm I in the hypothetical equilibrium.

(i) The first-order effects of lowering p_H by ε (but keeping p_L unchanged). The direct loss from this deviation is εQ_2 . But it also leads to two gains. First, the reservation frontier in Figure A1(a) shifts downward by ε , so those consumers who sample firm I first will stop searching more likely. This generates a gain

$$\frac{\varepsilon}{2} p_H \int_{a-\Delta}^{\infty} [1 - F(\phi_{\Delta}(u_1) - \Delta)] f(u_1, \phi_{\Delta}(u_1)) du_1 + \frac{\varepsilon}{2} p_L \int_{a-\Delta}^{\infty} [1 - F(u_1 + \Delta)] f(u_1, \phi_{\Delta}(u_1)) du_1 .$$

Using (9) and changing the integral variable from u_1 to $u = u_1 + \Delta$, we can rewrite the above expression as

$$\frac{\varepsilon}{2} p_H \int_a^{\infty} [1 - F(\phi)] f(u - \Delta, \phi + \Delta) du + \frac{\varepsilon}{2} p_L \int_a^{\infty} [1 - F(u)] f(u - \Delta, \phi + \Delta) du . \quad (10)$$

(We have suppressed the independent variable u in $\phi(u)$.) Second, for those consumers who sample both firms, they will buy product 2 from firm I more likely, which generates a gain

$$\frac{\varepsilon}{2} p_H \left(\int_B f(u_2 - \Delta) dF(\mathbf{u}) + \int_{\hat{B}} f(v_2 + \Delta) dF(\mathbf{v}) \right) . \quad (11)$$

(Here $B(\hat{B})$ is the non-stopping region for those consumers who visit firm I(II) first as illustrated in Figure A1.)

(ii) The first-order effects of raising p_L by ε (but keeping p_H unchanged). The direct gain from this deviation is εQ_1 . But it also causes two losses. First, the reservation frontier in Figure A1(a) shifts rightward by ε , so those consumers who visit firm I first will continue to search more likely. This leads to a loss

$$\frac{\varepsilon}{2} p_H \int_{a+\Delta}^{\infty} [1 - F(u_2 - \Delta)] f(\phi_{\Delta}^{-1}(u_2), u_2) du_2 + \frac{\varepsilon}{2} p_L \int_{a+\Delta}^{\infty} [1 - F(\phi_{\Delta}^{-1}(u_2) + \Delta)] f(\phi_{\Delta}^{-1}(u_2), u_2) du_2 .$$

Changing the integral variable from u_2 to $u = \phi_{\Delta}^{-1}(u_2) + \Delta$ (and so $u_2 = \phi_{\Delta}(u - \Delta) = \phi(u) + \Delta$), we can rewrite it as

$$\frac{\varepsilon}{2} p_H \int_a^{\infty} [1 - F(\phi)] f(u - \Delta, \phi + \Delta) (-\phi') du + \frac{\varepsilon}{2} p_L \int_a^{\infty} [1 - F(u)] f(u - \Delta, \phi + \Delta) (-\phi') du . \quad (12)$$

Second, for those consumers who visit both firms, they will buy product 1 from firm I less likely, which leads to a loss

$$\frac{\varepsilon}{2} p_L \left(\int_B f(u_1 + \Delta) dF(\mathbf{u}) + \int_{\hat{B}} f(v_1 - \Delta) dF(\mathbf{v}) \right) . \quad (13)$$

The following claim completes the argument.

Claim 1 *Suppose the two products are symmetric, and the density function f is logconcave. Then the sum of all gains from the deviation, i.e., $\varepsilon Q_1 + (10) + (11)$, is greater than the sum of all losses, i.e., $\varepsilon Q_2 + (12) + (13)$.*

Proof. First, $Q_1 > Q_2$ since product 1 is cheaper but product 2 is more expensive at firm I. So the gain εQ_1 from raising p_L by ε is greater than the loss εQ_2 from lowering p_H by ε . Second, the symmetry of the setting implies

$$\int_B f(u_2 - \Delta) dF(\mathbf{u}) = \int_{\hat{B}} f(v_1 - \Delta) dF(\mathbf{v}) ; \quad \int_B f(u_1 + \Delta) dF(\mathbf{u}) = \int_{\hat{B}} f(v_2 + \Delta) dF(\mathbf{v}) .$$

Thus, the gain in (11) is greater than the loss in (13).

Finally, I show that the gain in (10) is also greater than the loss in (12) if f is logconcave. Notice that $-\phi' = \frac{1-F(u)}{1-F(\phi)}$ implies that

$$\int_a^\infty [1 - F(\phi)] f(u - \Delta, \phi + \Delta) (-\phi') du = \int_a^\infty [1 - F(u)] f(u - \Delta, \phi + \Delta) du .$$

Then it suffices to show that

$$\begin{aligned} & p_H \int_a^\infty [F(u) - F(\phi)] f(u - \Delta, \phi + \Delta) du \\ & \geq p_L \int_a^\infty [1 - F(u)] f(u - \Delta, \phi + \Delta) (-\phi' - 1) du \\ & = p_L \int_a^\infty [F(\phi) - F(u)] f(u - \Delta, \phi + \Delta) (-\phi') du \\ & = p_L \int_a^\infty [F(u) - F(\phi)] f(\phi - \Delta, u + \Delta) du. \end{aligned} \tag{14}$$

(The last step is from changing the integral variable from u to $\phi(u)$.)

I first argue that $\int_a^\infty [F(u) - F(\phi)] f(u - \Delta) f(\phi + \Delta) du > 0$ if f is logconcave. Let $\hat{u}$ solve $u = \phi(u)$. Then the left-hand side equals

$$\begin{aligned} & \int_a^{\hat{u}} [F(u) - F(\phi)] f(u - \Delta) f(\phi + \Delta) du + \int_{\hat{u}}^\infty [F(u) - F(\phi)] f(u - \Delta) f(\phi + \Delta) du \\ & = \int_{\hat{u}}^\infty [F(\phi) - F(u)] f(\phi - \Delta) f(u + \Delta) (-\phi') du + \int_{\hat{u}}^\infty [F(u) - F(\phi)] f(u - \Delta) f(\phi + \Delta) du \\ & > \int_{\hat{u}}^\infty [F(\phi) - F(u)] f(\phi - \Delta) f(u + \Delta) du + \int_{\hat{u}}^\infty [F(u) - F(\phi)] f(u - \Delta) f(\phi + \Delta) du \\ & = \int_{\hat{u}}^\infty [F(u) - F(\phi)] [f(u - \Delta) f(\phi + \Delta) - f(\phi - \Delta) f(u + \Delta)] du . \end{aligned} \tag{15}$$

(The first equality is from changing the integral variable. The inequality is because $\phi < u$ and $-\phi' \in (0, 1)$ for $u \in (\hat{u}, \infty)$.) If f is logconcave, then we have

$$\ln f(\phi + \Delta) - \ln f(\phi - \Delta) \geq \ln f(u + \Delta) - \ln f(u - \Delta)$$

for $\phi < u$ and $\Delta > 0$, which implies

$$f(u - \Delta)f(\phi + \Delta) - f(\phi - \Delta)f(u + \Delta) \geq 0$$

and so (15) is positive.

Then to have (14), it remains to prove

$$\int_a^\infty [F(u) - F(\phi)][f(u - \Delta)f(\phi + \Delta) - f(\phi - \Delta)f(u + \Delta)]du \geq 0 . \quad (16)$$

We already knew that the integrand is positive for $u \in [\hat{u}, \infty]$. For $u \in [a, \hat{u}]$, we have $u < \phi$ and so the sign of each $[\cdot]$ term in the integrand is reversed. Then it is positive again. Therefore, (16) must be true. ■

3 About More Products

I first present the first-order conditions for the case with m products. Let $\mathbf{u}_{-i} \equiv (u_j)_{j \neq i} \in \mathbb{R}^{m-1}$. In a symmetric equilibrium, without loss of generality the reservation frontier can be defined as $u_m = \phi(\mathbf{u}_{-m})$, where $\phi(\mathbf{u}_{-m})$ satisfies

$$\sum_{i=1}^{m-1} \zeta_i(u_i) + \zeta_m(\phi(\mathbf{u}_{-m})) = s .$$

As in the two-product case, let A be the acceptance set and B be its complement. Following the same logic as in the two-product case, the first-order condition for p_m is

$$\begin{aligned} 1 = & \underbrace{2p_m \int_B f_m(u_m) dF(\mathbf{u}) + p_m \int_{A_{-m}} [1 - F_m(\phi(\mathbf{u}_{-m}))] f(\mathbf{u}_{-m}, \phi(\mathbf{u}_{-m})) d\mathbf{u}_{-m}}_{\text{standard effect}} \\ & + \underbrace{\sum_{i=1}^{m-1} p_i \int_{A_{-m}} [1 - F_i(u_i)] f(\mathbf{u}_{-m}, \phi(\mathbf{u}_{-m})) d\mathbf{u}_{-m}}_{\text{joint search effect}} , \quad (17) \end{aligned}$$

where A_{-m} is the projection of A on an $(m-1)$ -dimensional hyperplane with a fixed u_m .

Now consider the uniform case with m symmetric products and independent match utilities. Then the first integral in (17) measures the volume of solid B , and it equals one minus the volume of solid A . Since A is $1/2^m$ of an m -dimensional sphere with radius $\sqrt{2s}$, we get

$$1 - \frac{V_m(\sqrt{2s})}{2^m} ,$$

where

$$V_m(r) = \frac{(r\sqrt{\pi})^m}{\Gamma(1 + m/2)}$$

is the volume of an m -dimensional sphere with radius r . ($\Gamma(\cdot)$ is the Gamma function.) The second integral in (17) equals

$$\int_{A_{-m}} [1 - \phi(\mathbf{u}_{-m})] d\mathbf{u}_{-m} = \frac{V_m(\sqrt{2s})}{2^m},$$

since it just measures the volume of A . Finally, the third integral in (17) equals

$$\int_{A_{-m}} (1 - u_1) d\mathbf{u}_{-m} = \frac{V_m(\sqrt{2s})}{2^{m-1}\pi}. \quad (18)$$

(This equality has no straightforward geometric interpretation. See its proof below.) Then the price formula in the main text follows.

Proof of (18): For $m = 2$, $A_{-m} = [a, 1]$ and (18) is easy to be verified. Now consider $m \geq 3$. Let $A_{-1,m}(u_1)$ be a “slice” of A_{-m} at u_1 . Then we have

$$\int_{A_{-m}} (1 - u_1) d\mathbf{u}_{-m} = \int_a^1 (1 - u_1) \left(\int_{A_{-1,m}(u_1)} d\mathbf{u}_{-1,m} \right) du_1.$$

Since $A_{-1,m}(u_1)$ is $1/2^{m-2}$ of an $(m-2)$ -dimensional sphere with radius $r = \sqrt{2s - (1 - u_1)^2}$, the internal integral term equals

$$\frac{V_{m-2}(r)}{2^{m-2}} = \frac{\pi^{(m-2)/2} r^{m-2}}{2^{m-2} \Gamma(m/2)}.$$

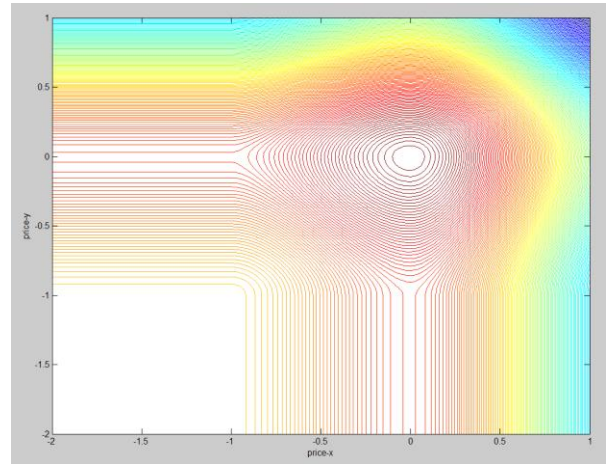
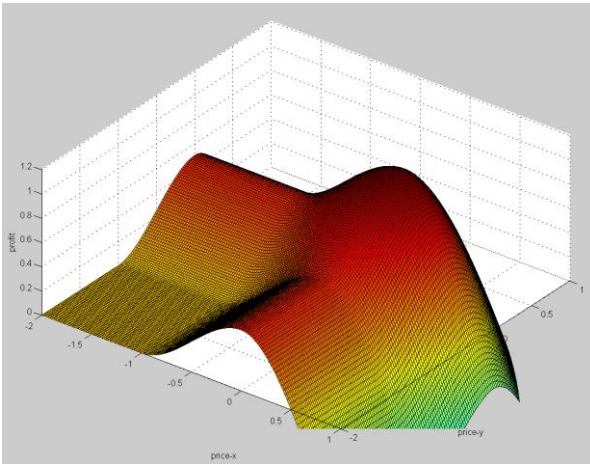
Hence,

$$\begin{aligned} \int_{A_{-m}} (1 - u_1) d\mathbf{u}_{-m} &= \frac{\pi^{(m-2)/2}}{2^{m-2} \Gamma(m/2)} \times \int_a^1 (1 - u_1) \left(\sqrt{2s - (1 - u_1)^2} \right)^{m-2} du_1 \\ &= \frac{\pi^{(m-2)/2}}{2^{m-2} \Gamma(m/2)} \times \frac{(\sqrt{2s})^m}{m} = \frac{V_m(\sqrt{2s})}{2^{m-1}\pi}. \end{aligned}$$

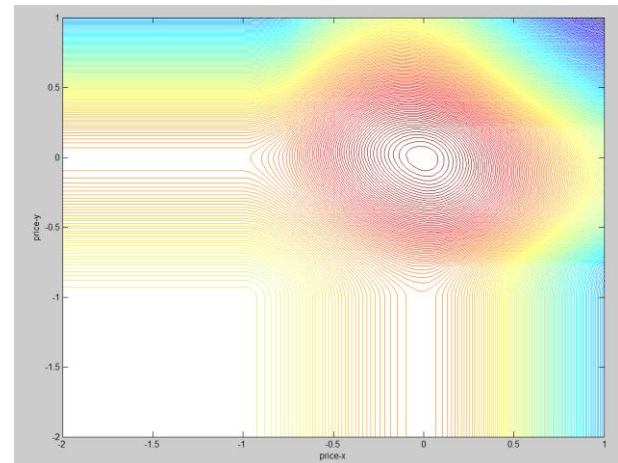
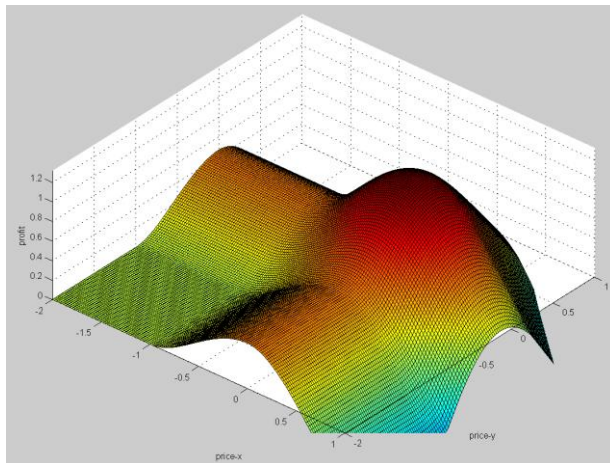
The second step used $a = 1 - \sqrt{2s}$ and the fact that the integrand is the derivative of $\frac{1}{m}(\sqrt{2s - (1 - u_1)^2})^m$ with respect to u_1 . The last step used the expression for $V_m(\cdot)$ and the Gamma function property $x\Gamma(x) = \Gamma(x + 1)$.

Uniform Distribution

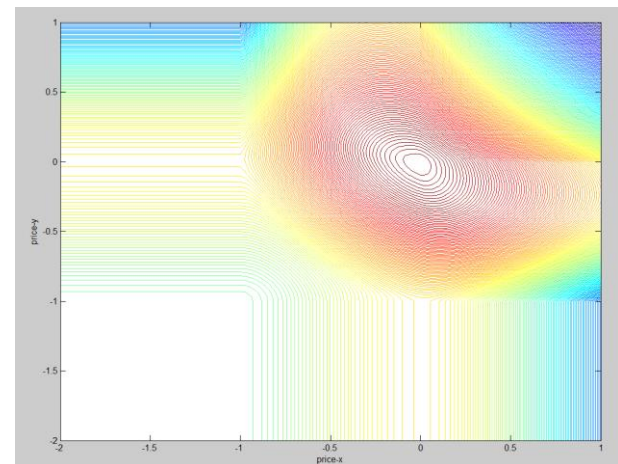
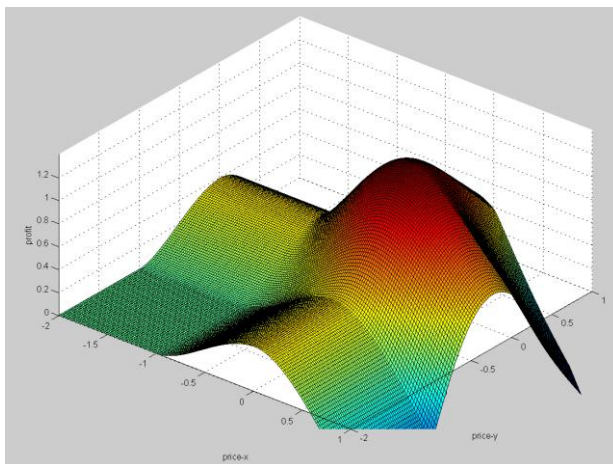
$s=0.01$:



$s=0.3$:

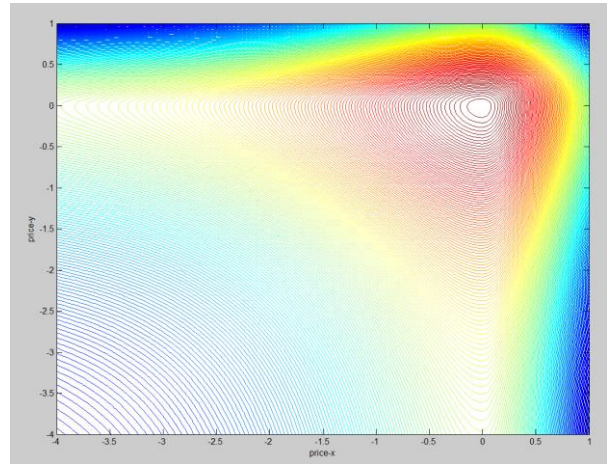
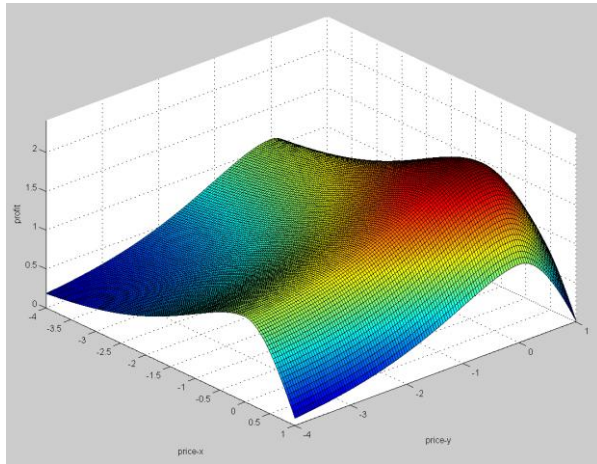


$s=0.5$:

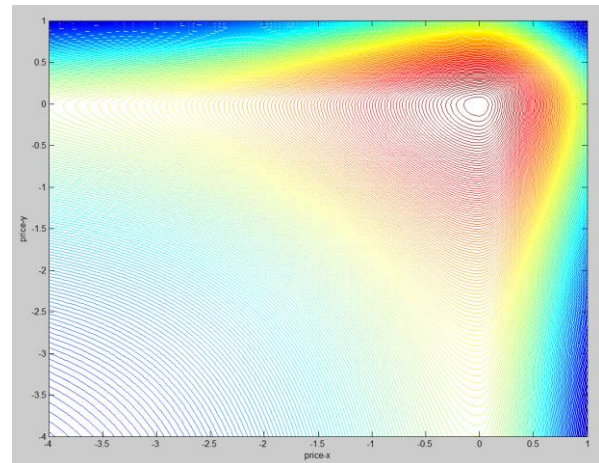
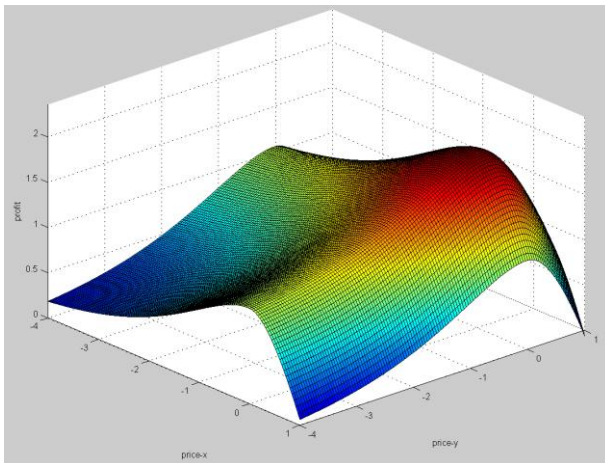


Exponential Distribution

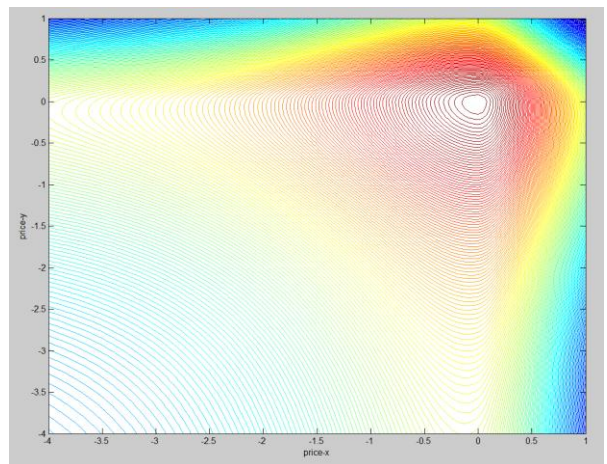
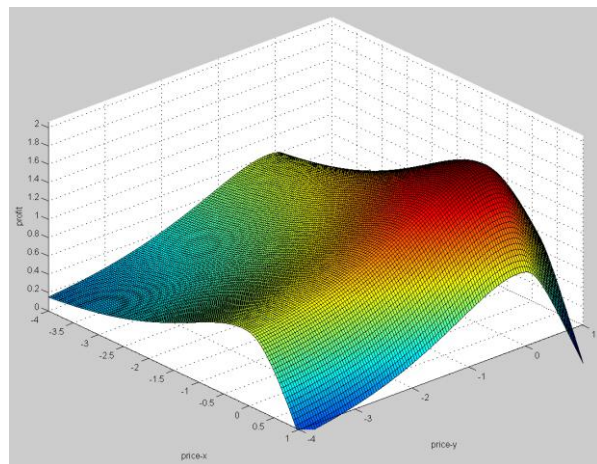
$s=0.01$:



$s=0.5$:

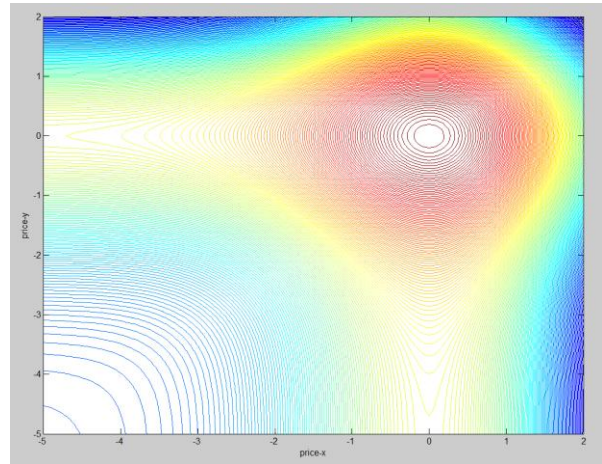
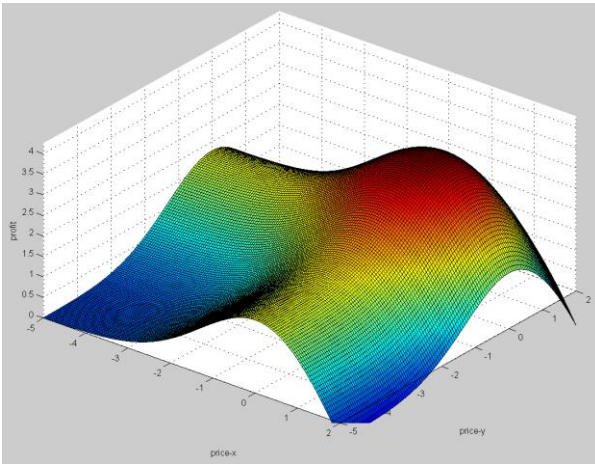


$s=1$:

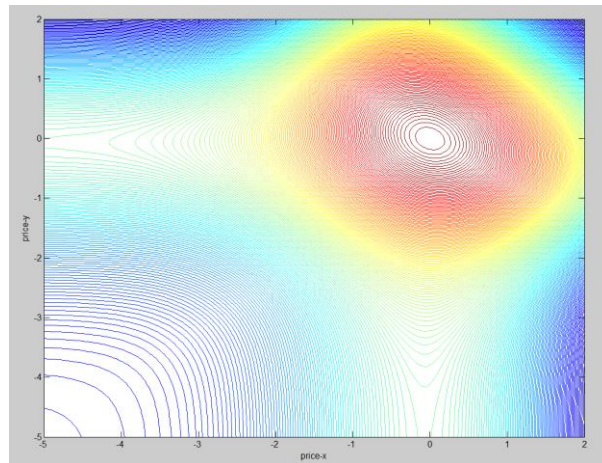
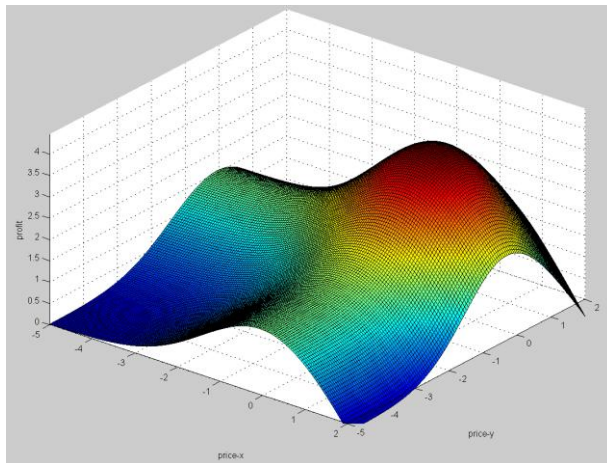


Standard Normal Distribution

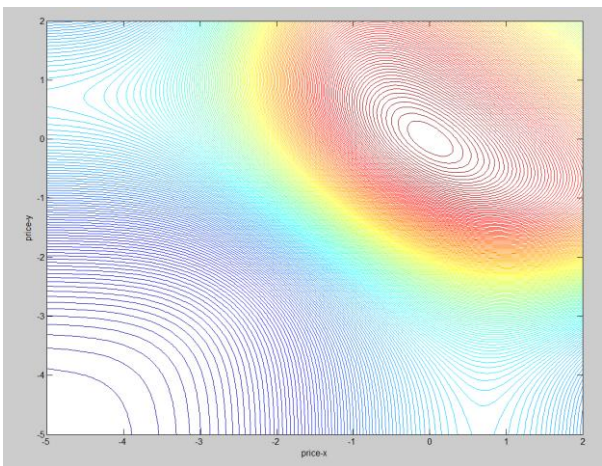
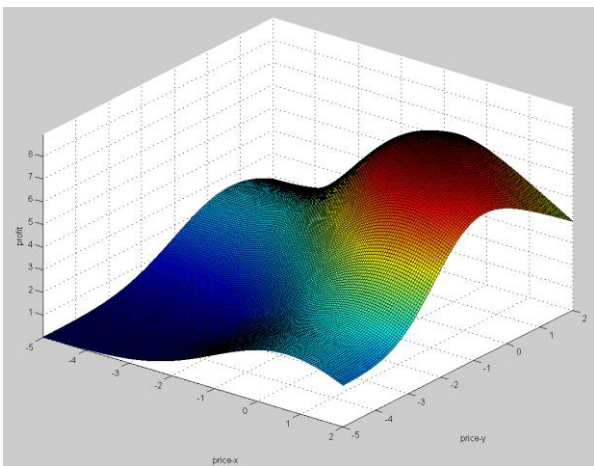
$s=0.01$:



$s=1$:

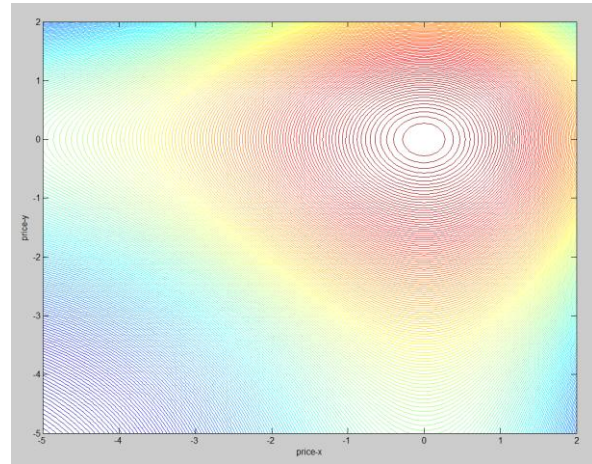
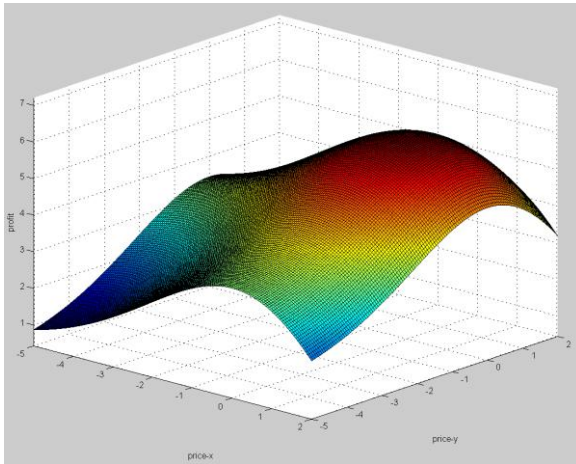


$s=2$:

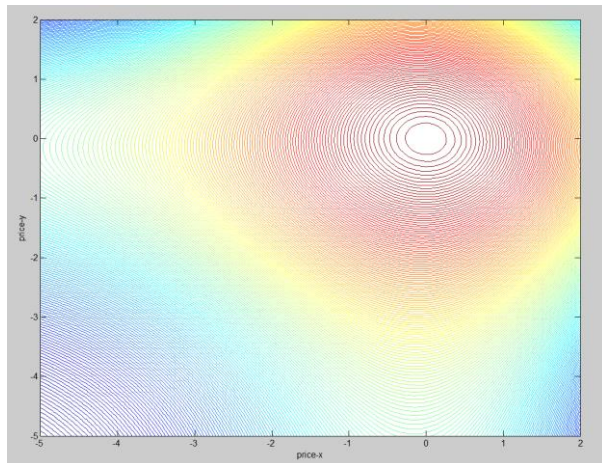
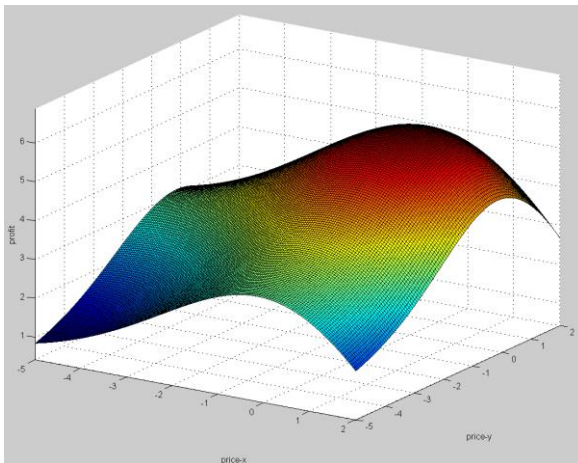


Logistic Distribution

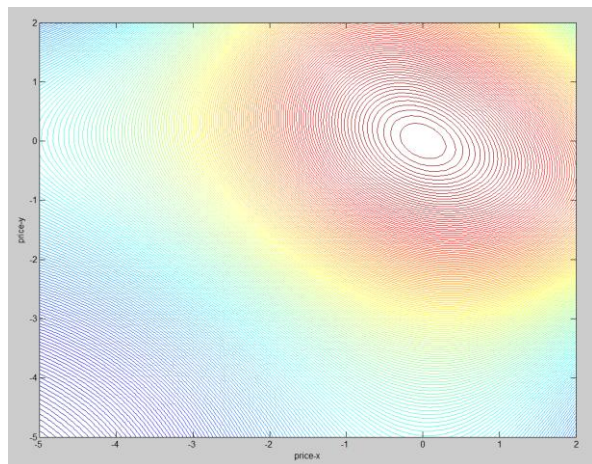
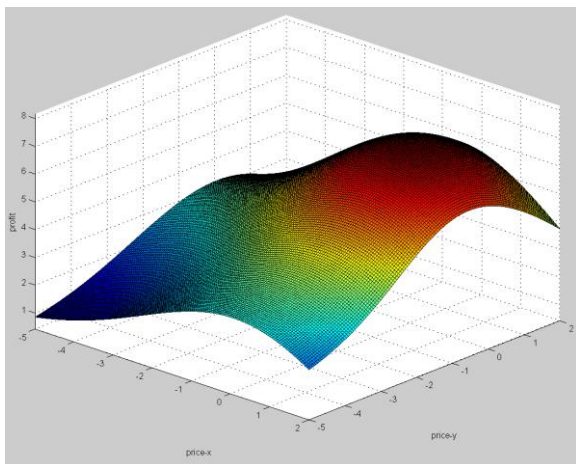
$s=0.01$:



$s=1$:

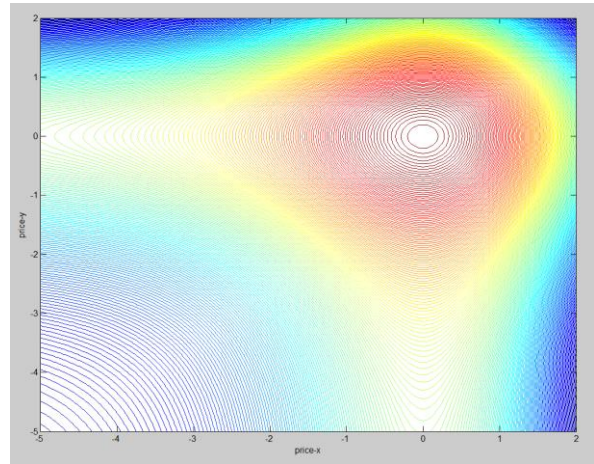
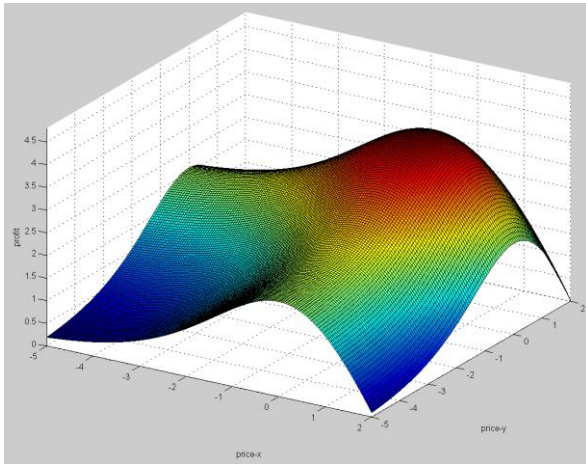


$s=2$:

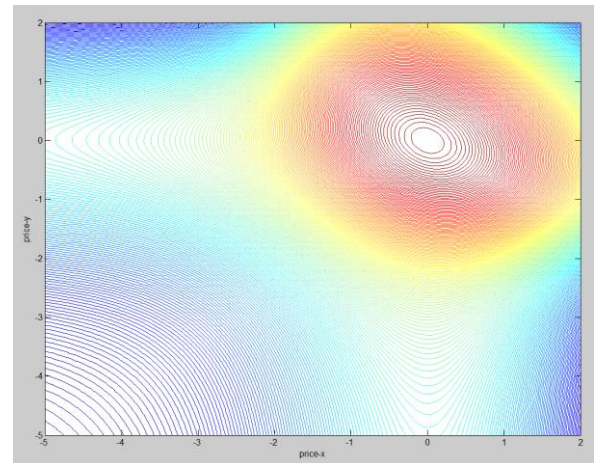
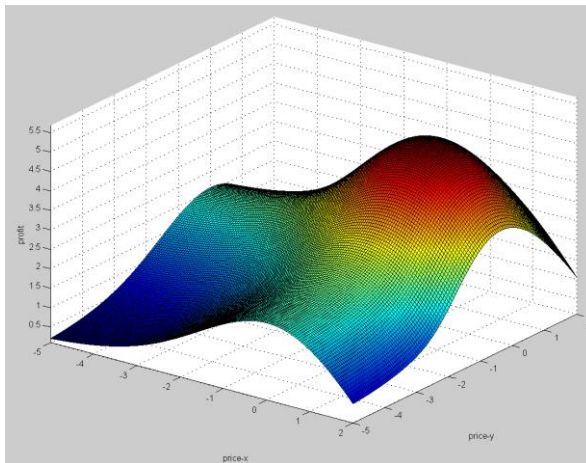


Standard Gumbel Distribution

$s=0.01$:



$s=1$:



$s=2$:

