Online Appendix for:

Testing Ambiguity Models through the Measurement of Probabilities for Gains and Losses

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A − Test of models

A.1. Maxmin expected utility (MEU) and Maxmax expected utility

Lemma 1: For all disjoint events E and F, $\inf(I_E) + \inf(I_F) \leq \inf(I_{E\cup F})$ and $\sup(I_E) + \sup(I_F) \geq \sup(I_{E\cup F})$.

Proof: This can be deduced from the definition of I_E (see main text).

Result 1. MEU predicts $BC^-(E) \leq 0 \leq BC^+(E) = -BC^-(E)$.

Proof:

Under MEU, $BC^-(E) = 1 - \sup(I_E) - \sup(I_{S-E})$ and $BC^+(E) = 1 - \inf(I_E) - \inf(I_{S-E})$

Lemma 1 (with $\sup(I_S)=\inf(I_S)=1$) implies that $BC^-(E) \leq 0$, $BC^+(E) \geq 0$, and hence, $BC^-(E) \leq BC^+(E)$. Moreover, $\sup(I_E) = 1 - \inf(I_{S-E})$ and $\sup(I_{S-E}) = 1 - \inf(I_E)$ implies $BC^+(E) = -BC^-(E)$.

Result 2. MEU predicts $LA^-(E_i,E_j) \geq 0 \geq LA^+(E_i,E_j)$.

Proof: Under MEU, $LA^-(E_i,E_j) = \sup(I_E) + \sup(I_F) - \sup(I_{E\cup F})$ and $LA^+(E_i,E_j) = \inf(I_E) + \inf(I_F) - \inf(I_{E\cup F})$. Hence, according to Lemma 1, $LA^-(E_i,E_j) \geq 0$, $LA^+(E_i,E_j) \leq 0$, and, $LA^-(E_i,E_j) \geq LA^+(E_i,E_j)$.

Result 3. MEU predicts $UA^{-}(E) \leq 0 \leq UA^{+}(E)$.

Proof: Under MEU, $UA^{-}(E_i) = \inf(I_{Ej}) + \inf(I_{Ek}) - \inf(I_{Ejk})$ and $UA^{+}(E_i) = \sup(I_{Ej}) + \sup(I_{Ek}) - \sup(I_{Ejk})$. Hence, according to Lemma 1, $UA^{-}(E_i) \leq 0 \leq UA^{+}(E_i)$.

Result 4. Under MEU, $TA^{+} = ITA^{-} \leq 0 \leq TA^{-} = ITA^{+}$

Proof: Under MEU, $TA^{+} = \inf(I_{E1}) + \inf(I_{E2}) + \inf(I_{E3}) - 1$ and $ITA^{-} = 2 - \sup(I_{E12}) - \sup(I_{E23}) - \sup(I_{E13}) = \inf(I_{E1}) + \inf(I_{E2}) + \inf(I_{E3}) - 1$ because $\sup(I_{E12}) = 1 - \inf(I_{E3})$.

Hence, $TA^{+} = ITA^{-}$

Moreover, Lemma 1 implies: $1 = \inf(I_S) \geq \inf(I_{E1}) + \inf(I_{E23}) \geq \inf(I_{E1}) + \inf(I_{E2}) + \inf(I_{E3})$, and therefore, $TA^{+} = ITA^{-} \leq 0$.

Similarly, $TA^{-} = \sup(I_{E1}) + \sup(I_{E2}) + \sup(I_{E3}) - 1$ and $ITA^{-} = 2 - \inf(I_{E12}) - \inf(I_{E23}) - \inf(I_{E13}) = \sup(I_{E1}) + \sup(I_{E2}) + \sup(I_{E3}) - 1$ because $\inf(I_{E12}) = 1 - \sup(I_{E3})$.

Hence, $TA^{-} = ITA^{+}$

Moreover, Lemma 1 implies: $1 = \sup(I_S) \leq \sup(I_{E1}) + \sup(I_{E23}) \leq \sup(I_{E1}) + \sup(I_{E2}) + \sup(I_{E3})$, and therefore, $0 \leq TA^{-} = ITA^{+}$.

All proofs for maxmax follow exactly the same steps, but replacing inf by sup and reciprocally. We omit them for the sake of brevity.

A.2. α-Maxmin

Result 5. Under α-Maxmin with I_E not being a singleton, $BC^{-}(E) \geq 0 \geq BC^{+}(E) \Leftrightarrow \alpha \leq ½$ and $BC^{-}(E) \leq 0 \leq BC^{+}(E) \Leftrightarrow \alpha \geq ½$. Moreover, $BC^{+}(E) = -BC^{-}(E)$.

Proof: Under α-Maxmin:

$BC^{-}(E) \geq 0$

$\Leftrightarrow \alpha\ (\sup(I_E) + \sup(I_{S-E})) + (1-\alpha)\ (\inf(I_E) + \inf(I_{S-E})) \leq 1$

$\Leftrightarrow (2\alpha - 1)\ ((\sup(I_E) - \inf(I_E)) \leq 0$

(because $\sup(I_{S-E}) = (1 - \inf(I_E))$ and $\inf(I_{S-E}) = (1 - \sup(I_E))$)

$\Leftrightarrow \alpha \geq ½$ (because I_E not being a singleton implies $\sup(I_E) > \inf(I_E)$).

Reversing all inequalities in the proof demonstrates $BC^-(E) \leq 0 \Leftrightarrow \alpha \geq ½$.

Similarly: $BC^+(E) \geq 0$

$\Leftrightarrow \alpha\ (\inf(I_E) + \inf(I_{S-E})) + (1-\alpha)\ (\sup(I_E) + \sup(I_{S-E})) \leq 1$

$\Leftrightarrow (2\alpha - 1)\ ((\inf(I_E) - \sup(I_E)) \leq 0$

(because $\sup(I_{S-E}) = (1 - \inf(I_E))$ and $\inf(I_{S-E}) = (1 - \sup(I_E))$)

$\Leftrightarrow \alpha \leq ½$ (because I_E not being a singleton implies $\sup(I_E) > \inf(I_E)$).

Reversing all inequalities in the proof demonstrates $BC^+(E) \leq 0 \Leftrightarrow \alpha \leq ½$

Moreover, it follows from $\sup(I_E) = 1 - \inf(I_{S-E})$ and $\sup(I_{S-E}) = 1 - \inf(I_E)$ that $BC^+(E) = -BC^-(E)$.

Result 6. Under α-Maxmin, $LA^+(E_i,E_j) \leq LA^-(E_i,E_j) \Leftrightarrow \alpha \geq ½$ and $LA^+(E_i,E_j) \geq LA^-(E_i,E_j) \Leftrightarrow \alpha \leq ½$. No further restriction on sign.

Proof:

Set of priors	$m^-(I_{Ei}) = m^-(I_{Ej})$	$m^-(I_{Eij})$	$LA^-(E_i,E_j)$	$m^+(I_{Ei}) = m^+(I_{Ej})$	$m^+(I_{Eij})$	$LA^+(E_i,E_j)$
$I_{Ei} = I_{Ej} = [0; 1]$ and $I_{Eij} = [0; 1]$	α	α	$\alpha > 0$	$(1-\alpha)$	$(1-\alpha)$	$(1-\alpha) > 0$
$I_{Ei} = I_{Ej} = [0; ½]$ and $I_{Eij} = [¼; 1]$	$½\ \alpha$	$\alpha + (1-\alpha)\ ¼$	$-¼\ (1-\alpha) < 0$	$(1-\alpha)\ ½$	$¼\ \alpha + (1-\alpha)$	$-¼\ \alpha < 0$
$I_{Ei} = I_{Ej} = [0; ½]$ and $I_{Eij} = [¼; ¾]$	$½\ \alpha$	$¾\ \alpha + (1-\alpha)\ ¼$	$½\ \alpha - ¼$ > 0 if $\alpha > ½$ < 0 if $\alpha < ½$	$(1-\alpha)\ ½$	$¼\ \alpha + ¾\ (1-\alpha)$	$¼ - ½\ \alpha$ < 0 if $\alpha > ½$ > 0 if $\alpha < ½$

$LA^+(E_i,E_j) \geq LA^-(E_i,E_j) \Leftrightarrow$

$\alpha\,(\inf(I_{Ei}) + \inf(I_{Ej}) - \inf(I_{Eij})) + (1-\alpha)\,(\sup(I_{Ei}) + \sup(I_{Ej}) - \sup(I_{Eij})) \geq$
$\alpha\,(\sup(I_{Ei}) + \sup(I_{Ej}) - \sup(I_{Eij})) + (1-\alpha)\,(\inf(I_{Ei}) + \inf(I_{Ej}) - \inf(I_{Eij}))$
$\Leftrightarrow (2\alpha - 1)\,(\inf(I_{Ei}) + \inf(I_{Ej}) - \inf(I_{Eij}) - \sup(I_{Ei}) - \sup(I_{Ej}) + \sup(I_{Eij})) \geq 0$
$\Leftrightarrow (2\alpha - 1) \leq 0$ (by Lemma 1) $\Leftrightarrow \alpha \leq ½$.
Reversing all inequalities demonstrate $LA^{+}(E_i,E_j) \leq LA^{-}(E_i,E_j) \Leftrightarrow \alpha \geq ½$.

Result 7. Under α-Maxmin, $UA^{+}(E_i) \leq UA^{-}(E_i) \Leftrightarrow \alpha \leq ½$ and $UA^{+}(E_i) \geq UA^{-}(E_i) \Leftrightarrow \alpha \geq ½$. No further restriction on signs.

Proof:

Under α-Maxmin, $UA^{+}(E_i) = 1 - \inf(I_{Eik}) - \inf(I_{Eij}) + \inf(I_{Ei}) = \sup(I_{Ej}) + \sup(I_{Ek}) - \sup(I_{Ejk}) = LA^{-}(E_j,E_k)$ and $UA^{-}(E_i) = 1 - \sup(I_{Eik}) - \sup(I_{Eij}) + \sup(I_{Ei}) = \inf(I_{Ej}) + \inf(I_{Ek}) - \inf(I_{Ejk}) = LA^{+}(E_j,E_k)$. The result follows from the previous one.

Result 8. Under α-Maxmin,
$\alpha \leq ⅓ \Rightarrow TA^{-} = ITA^{+} \leq 0 \leq ITA^{-} = TA^{+}$.
$⅓ \leq \alpha \leq ½ \Rightarrow TA^{-} = ITA^{+} \leq ITA^{-} = TA^{+}$ (no further restriction on sign).
$½ \leq \alpha \leq ⅔ \Rightarrow TA^{+} = ITA^{-} \leq ITA^{+} = TA^{-}$ (no further restriction on sign).
$\alpha \geq ⅔ \Rightarrow TA^{+} = ITA^{-} \leq 0 \leq ITA^{+} = TA^{-}$.

Proof:

Under α-Maxmin, $TA^{+} = \alpha\,(\inf(I_{E1}) + \inf(I_{E2}) + \inf(I_{E3})) + (1-\alpha)\,(\sup(I_{E1}) + \sup(I_{E2}) + \sup(I_{E3})) - 1 = 2 - \alpha\,(\sup(I_{E12}) + \sup(I_{E23}) + \sup(I_{E13})) + (1-\alpha)\,(\inf(I_{E12}) + \inf(I_{E23}) + \inf(I_{E13})) = ITA^{-}$.
Moreover, $TA^{-} = (1-\alpha)\,(\inf(I_{E1}) + \inf(I_{E2}) + \inf(I_{E3})) + \alpha\,(\sup(I_{E1}) + \sup(I_{E2}) + \sup(I_{E3})) - 1 = 2 - (1-\alpha)\,(\sup(I_{E12}) + \sup(I_{E23}) + \sup(I_{E13})) + \alpha\,(\inf(I_{E12}) + \inf(I_{E23}) + \inf(I_{E13})) = ITA^{+}$.
$TA^{+} > 0$
$\Rightarrow \alpha\,(\inf(I_{E1}) + \inf(I_{E2}) + \inf(I_{E3})) + (1-\alpha)\,(\sup(I_{E1}) + \sup(I_{E2}) + \sup(I_{E3})) > 1$

$\Rightarrow (3\alpha - 2)\ (\inf(I_{E1}) + \inf(I_{E2}) + \inf(I_{E3})) > (3\alpha - 2)$

(Using that Lemma 1 implies $\sup(I_{E1}) \leq 1 - \inf(I_{E2}) - \inf(I_{E3})$, $\sup(I_{E2}) \leq 1 - \inf(I_{E1}) - \inf(I_{E3})$, and $\sup(I_{E3}) \leq 1 - \inf(I_{E1}) - \inf(I_{E2})$)

$\Rightarrow (3\alpha - 2) < 0$ (because $0 \leq \inf(I_{E1}) + \inf(I_{E2}) + \inf(I_{E3}) < 1$)

$\Rightarrow \alpha < ⅔$.

$TA^{+} < 0$

$\Rightarrow \alpha\ (\inf(I_{E1}) + \inf(I_{E2}) + \inf(I_{E3})) + (1 - \alpha)\ (\sup(I_{E1}) + \sup(I_{E2}) + \sup(I_{E3})) < 1$

$\Rightarrow (1 - 3\alpha)\ (\sup(I_{E1}) + \sup(I_{E2}) + \sup(I_{E3})) < (1 - 3\alpha)$

(Using that Lemma 1 implies $\inf(I_{E1}) \geq 1 - \sup(I_{E2}) - \sup(I_{E3})$, $\inf(I_{E2}) \geq 1 - \sup(I_{E1}) - \sup(I_{E3})$, and $\inf(I_{E3}) \geq 1 - \sup(I_{E1}) - \sup(I_{E2})$)

$\Rightarrow (1 - 3\alpha) < 0$ (because $1 \leq \sup(I_{E1}) + \sup(I_{E2}) + \sup(I_{E3})$)

$\Rightarrow ⅓ < \alpha$.

Note that TA^{-} is equal to TA^{+} for which we would have exchanged α and $(1 - \alpha)$. Hence, $TA^{-} > 0 \Rightarrow (1 - \alpha) < ⅔ \Rightarrow ⅓ < \alpha$ and

$TA^{-} < 0 \Rightarrow ⅓ < (1 - \alpha) \Rightarrow \alpha < ⅔$.

As a consequence:

$\alpha \geq ⅔ \Rightarrow TA^{+} = ITA^{-} \leq 0 \leq ITA^{+} = TA^{-}$

and

$\alpha \leq ⅓ \Rightarrow TA^{-} = ITA^{+} \leq 0 \leq ITA^{-} = TA^{+}$.

Moreover, $TA^{-} - TA^{+} = (1 - 2\alpha)\ (\inf(I_{E1}) + \inf(I_{E2}) + \inf(I_{E3}) - \sup(I_{E1}) - \sup(I_{E2}) - \sup(I_{E3}))$. Then $TA^{-} \leq TA^{+} \Leftrightarrow \alpha \leq ½$.

Assume $⅓ < \alpha < ⅔$:

Set of priors	$m^{-}(I_{Ei})$	$TA^{-} = ITA^{+}$	$m^{+}(I_{Ei})$	$ITA^{-} = TA^{+}$
$I_{Ei} = [0; 1]$ for all i	α	$3\alpha - 1 > 0$	$(1 - \alpha)$	$3(1 - \alpha) - 1 > 0$
$I_{Ei} = [0; ½]$ for all i	$½\ \alpha$	$3/2\ \alpha - 1 < 0$	$(1 - \alpha)\ ½$	$3/2\ (1 - \alpha) - 1 < 0$
$I_{Ei} = [0; 2/3]$ for all i	$2\alpha/3$	$2\ \alpha - 1$	$2(1 - \alpha)/3$	$1 - 2\alpha$

A.3. Variational model (VM)

In this section, we consider 3 given outcomes: x, 0, and −x (where $x>0$).

We assume (without loss of generality) $U(x) = 1$, $U(0) = 0$, and $U(-x) = z$.

with $z < 0$.

Result 9. VM predicts $BC^{-}(E) \leq 0 \leq BC^{+}(E)$.

Proof: From Eq.(7) in Maccheroni et al. (2006), the set $\{P\in\Delta: c(P)=0\}$ is nonempty. Let P^* be such that $c(P^*) = 0$. Let P be the probability measure that minimizes, for event E, $P(E)z+c(P)$. Then $P(E)z + c(P) \leq P^*(E)z$. Let P′ be the probability measure that minimizes, for event, E^c $P'(E^c)z + c(P')$. Then also $P'(E^c)z + c(P') \leq P^*(E^c)z$.

Because P^* is a probability measure, $P^*(E) + P^*(E^c) =1$. Consequently,

$P(E)z + c(P) + P'(E^c)z + c(P') \leq z$.

Dividing by z, which is negative, we obtain that the sum of the matching probabilities for E and E^c in the loss domain should be at least 1, and thus $BC^{-}(E) \leq 0$.

Now, let P be the probability measure that minimizes, for event E, $P(E) + c(P)$. Then $P(E) + c(P) \leq P^*(E)$. Let P′ be the probability measure that minimizes, for event E^c, $P'(E^c) + c(P')$. Then also $P'(E^c) + c(P') \leq P^*(E^c)$.

Because P^* is a probability measure, $P^*(E) + P^*(E^c) =1$. Consequently,

$P(E) + c(P) + P'(E^c) + c(P') \leq 1$.

We thus obtain that the sum of the matching probabilities for E and E^c in the gain domain should not be more than 1; therefore, $0 \leq BC^{+}(E)$.

Result 10. VM predicts TA^{+} , $ITA^{-} \leq 0 \leq TA^{-}$, ITA^{+}

Proof: Let us define $P^*\in\{P\in\Delta: c(P)=0\}$.

We must have for any event E: $\min_{P\in\Delta}(P(E)+c(P)) \leq P^*(E)$,

which implies

$\min_{P\in\Delta}(P(E_1)+c(P)) + \min_{P\in\Delta}(P(E_2)+c(P)) + \min_{P\in\Delta}(P(E_3)+c(P)) \leq 1$,

and therefore $TA^+ \leq 0$.

Similarly, it implies

$\min_{P\in\Delta}(P(E_{12})+c(P)) + \min_{P\in\Delta}(P(E_{23})+c(P)) + \min_{P\in\Delta}(P(E_{13})+c(P)) \leq 2$,

and therefore $0 \leq ITA^+$.

For losses, we must have for any event E: $\min_{P\in\Delta}(P(E)z+c(P)) \leq P^*(E)z$,

which implies

$[\min_{P\in\Delta}(P(E_1)z + c(P))]/z + [\min_{P\in\Delta}(P(E_2)z + c(P))]/z + [\min_{P\in\Delta}(P(E_3)z + c(P))]/z \geq 1$,

and

$[\min_{P\in\Delta}(P(E_1)z + P(E_2)z +c(P))]/z + [\min_{P\in\Delta}(P(E_1)z + P(E_3)z + c(P))]/z + [\min_{P\in\Delta}(P(E_2)z + P(E_3)z + c(P))]/z \geq 2$,

and therefore $ITA^- \leq 0 \leq TA^-$.

A.4. The smooth model for ambiguity (KMM)

In this section, we again consider 3 outcomes: x, 0, and −x (where x > 0).

We also assume that $U(x) = 1$, $U(0) = 0$, and $U(-x) = z$. with $z < 0$.

Lemma 2: If φ is concave, $x_E0 \sim x_p0$ implies $p \leq \int_\Pi P(E)d\mu$ and $-x_E0 \sim -x_q0$ implies

$q \geq \int_\Pi P(E)d\mu$. If φ is convex, $x_E0 \sim x_p0$ implies $p \geq \int_\Pi P(E)d\mu$ and $-x_E0 \sim -x_q0$ implies $q \leq \int_\Pi P(E)d\mu$.

Proof:

$x_E0 \sim x_p0$ implies $\varphi(p) = \int_\Pi \varphi(P(E))d\mu$.

Concavity of φ implies $\int_\Pi \varphi(P(E))d\mu \leq \varphi(\int_\Pi P(E)d\mu)$.

Therefore $\varphi(p) \leq \varphi(\int_\Pi P(E)d\mu)$.

φ is strictly increasing. As a consequence, $p \leq \int_\Pi P(E)d\mu$.

On the contrary, convexity of φ implies $\int_\Pi \varphi(P(E))d\mu \geq \varphi(\int_\Pi P(E)d\mu)$.

Therefore $\varphi(p) \geq \varphi(\int_\Pi P(E)d\mu)$.

φ is strictly increasing. As a consequence, $p \geq \int_\Pi P(E)d\mu$.

Moreover, $-x_E0 \sim -x_q0$ implies $\varphi(qz) = \int_\Pi \varphi(P(E)z)d\mu$.

Concavity of φ implies $\int_\Pi \varphi(P(E)z)d\mu \leq \varphi(\int_\Pi P(E)zd\mu)$.

Therefore $\varphi(qz) \leq \varphi(\int_\Pi P(Ez)d\mu)$.

φ is strictly increasing. As a consequence, $qz \leq \int_\Pi P(E)zd\mu$.

Dividing by $z<0$ gives $q \geq \int_\Pi P(E)d\mu$.

On the contrary, convexity of φ implies $\int_\Pi \varphi(P(E)z)d\mu \geq \varphi(\int_\Pi P(E)zd\mu)$.

Therefore $\varphi(qz) \geq \varphi(\int_\Pi P(Ez)d\mu)$.

φ is strictly increasing. As a consequence, $qz \geq \int_\Pi P(E)zd\mu$.

Dividing by $z<0$ gives $q \leq \int_\Pi P(E)d\mu$.

Result 11. φ concave implies $BC^-(E) \leq 0 \leq BC^+(E)$ and φ convex implies $BC^+(E) \leq 0 \leq BC^-(E)$.

Proof:

Define p, s, q and r by $x_E0 \sim x_p0$, $x_{E^c}0 \sim x_s0$, $-x_E0 \sim -x_q0$ and $-x_{E^c}0 \sim -x_r0$. According to Lemma 2, φ concave implies $q \geq \int_\Pi P(E)d\mu$ and $r \geq \int_\Pi (1 - P(E))d\mu$ and therefore implies $q + r \geq 1$. It also implies $p \leq \int_\Pi P(E)d\mu$ and $s \leq \int_\Pi (1 - P(E))d\mu$ and therefore $p + s \leq 1$. $BC^-(E) \leq 0 \leq BC^+(E)$ follows.

Similarly, according to Lemma 2, φ convex implies $q \leq \int_\Pi P(E)d\mu$ and $r \leq \int_\Pi (1 - P(E))d\mu$ and therefore implies $q + r \leq 1$. It also implies $p \geq \int_\Pi P(E)d\mu$ and $s \geq \int_\Pi (1 - P(E))d\mu$ and therefore $p + s \geq 1$. $BC^-(E) \geq 0 \geq BC^+(E)$ follows.

Result 12. φ concave implies TA^+, $ITA^- \leq 0 \leq TA^-$, ITA^+ but φ convex implies TA^-, $ITA^+ \leq 0 \leq TA^+$, ITA^-.

Proof:

If p, q and r are such that $x_E0 \sim x_p0$, $x_F0 \sim x_q0$, and $x_G0 \sim x_r0$ (with E, F, and G a partition of S), then if φ is convex, Lemma 2 implies that

$p + q + r \geq \int_\Pi P(E)d\mu + \int_\Pi P(F)d\mu + \int_\Pi P(G)d\mu = 1$, and hence, $TA^+ \geq 0$.

But if φ is concave, Lemma 2 implies that

$p + q + r \leq \int_\Pi P(E)d\mu + \int_\Pi P(F)d\mu + \int_\Pi P(G)d\mu = 1$, and hence, $TA^+ \leq 0$.

If p, q and r are such that $-x_E0 \sim -x_p0$, $-x_F0 \sim -x_q0$, and $-x_G0 \sim -x_r0$ (with E, F, and G a partition of S), then if φ is convex, Lemma 2 implies that

$p + q + r \leq \int_\Pi P(E)d\mu + \int_\Pi P(F)d\mu + \int_\Pi P(G)d\mu = 1$, and hence, $TA^- \leq 0$.

But if φ is concave, Lemma 2 implies that

$p + q + r \geq \int_\Pi P(E)d\mu + \int_\Pi P(F)d\mu + \int_\Pi P(G)d\mu = 1$, and hence, $TA^- \geq 0$.

If p, q and r are such that $x_{E^c}0 \sim x_p0$, $x_{F^c}0 \sim x_q0$, and $x_{G^c}0 \sim x_r0$ (with E, F, and G a partition of S), then if φ is convex, Lemma 2 implies that

$p + q + r \geq \int_\Pi P(F\cup G)d\mu + \int_\Pi P(E\cup G)d\mu + \int_\Pi P(E\cup F)d\mu = 2$, and hence, $ITA^+ \leq 0$.

But if φ is concave, Lemma 2 implies that

$p + q + r \leq \int_\Pi P(F\cup G)d\mu + \int_\Pi P(E\cup G)d\mu + \int_\Pi P(E\cup F)d\mu = 2$, and hence, $ITA^+ \geq 0$.

If p, q and r are such that $-x_{E^c}0 \sim -x_p0$, $-x_{F^c}0 \sim -x_q0$, and $-x_{G^c}0 \sim -x_r0$ (with E, F, and G a partition of S), then if φ is convex, Lemma 2 implies that

$p + q + r \leq \int_\Pi P(F\cup G)d\mu + \int_\Pi P(E\cup G)d\mu + \int_\Pi P(E\cup F)d\mu = 2$, and hence, $ITA^- \geq 0$.

But if φ is concave, Lemma 2 implies that

$p + q + r \geq \int_{\Pi} P(F \cup G)d\mu + \int_{\Pi} P(E \cup G)d\mu + \int_{\Pi} P(E \cup F)d\mu = 2$, and hence, $ITA^{-} \leq 0$.

A.5. Choquet expected utility (CEU)

Result 13. CEU predicts $BC^{-}(E) = BC^{+}(E)$. No further restrictions on sign.

Proof: Let p, s, q and r be defined by $x_E0 \sim x_p0$, $x_{E^c}0 \sim x_s0$, $-x_E0 \sim -x_q0$ and $-x_{E^c}0 \sim -x_r0$. Under CEU, this implies $p=w^{-1}(W(E))$, $s=w^{-1}(W(E^c))$, $r=w^{-1}(W(E))$, $q=w^{-1}(W(E^c))$. Therefore $q + r - 1= p + s - 1$. It is straightforward that with no further conditions on W and w^{-1} than that they are increasing, $BC^{-}(E)$ and $BC^{+}(E)$ can be of any sign.

Result 14. CEU predicts $LA^{+}(E_i,E_j) + LA^{-}(E_i,E_j) \leq 1$. No further restriction on signs.

Proof:

$LA^{+}(E_i,E_j) = w^{-1}\circ W(E_i) + w^{-1}\circ W(E_j) - w^{-1}\circ W(E_{ij})$ and $LA^{-}(E_i,E_j) = 1 - w^{-1}\circ W(E_{kj}) - w^{-1}\circ W(E_{ik}) + w^{-1}\circ W(E_k)$ (for $k \neq i, j$).

The only property that we can use is that $w^{-1}\circ W$ is increasing. As a consequence, there is no restriction on the sign of $LA^{+}(E_i,E_j)$ and $LA^{-}(E_i,E_j)$ but:

$LA^{+}(E_i,E_j) + LA^{-}(E_i,E_j) = w^{-1}\circ W(E_i) + w^{-1}\circ W(E_j) - w^{-1}\circ W(E_{ij}) + 1 - w^{-1}\circ W(E_{kj}) - w^{-1}\circ W(E_{ik}) + w^{-1}\circ W(E_k) = 1 - (w^{-1}\circ W(E_{ij}) - w^{-1}\circ W(E_i) + w^{-1}\circ W(E_{kj}) - w^{-1}\circ W(E_j) + w^{-1}\circ W(E_{ik}) - w^{-1}\circ W(E_k)) \leq 1$.

Result 15. CEU predicts $UA^{+}(E_i) + UA^{-}(E_i) \leq 1$. No further restriction on signs.

Proof:

$UA^{+}(E_k) = 1 - w^{-1}\circ W(E_{kj}) - w^{-1}\circ W(E_{ik}) + w^{-1}\circ W(E_k)$ and $UA^{-}(E_k) = w^{-1}\circ W(E_i) + w^{-1}\circ W(E_j) - w^{-1}\circ W(E_{ij})$ (for $k\neq i, j$).

From the preceding result, we conclude that there is no restriction on the sign of $UA^{+}(E_i)$ and $UA^{-}(E_i)$ but $UA^{+}(E_i) + UA^{-}(E_i) \leq 1$.

Result 16. CEU implies $TA^{+} = ITA^{-}$, $TA^{-} = ITA^{+}$, and $TA^{+} + ITA^{+} \leq 1$

Proof: Under CEU, $x_{E1}0 \sim x_{p1}0$, $x_{E2}0 \sim x_{p2}0$, $x_{E3}0 \sim x_{p3}0$, $-x_{E23}0 \sim -x_{q1}0$, $-x_{E13}0 \sim -x_{q2}0$, $-x_{E12}0 \sim -x_{q3}0$, imply $p_i = 1 - q_i = w^{-1}(W(E_i))$ for $i\in\{1,2,3\}$.

Therefore $TA^{+} = ITA^{-}$.

Similarly, $-x_{E1}0 \sim -x_{p1}0$, $-x_{E2}0 \sim -x_{p2}0$, $-x_{E3}0 \sim -x_{p3}0$, $x_{E23}0 \sim x_{q1}0$, $x_{E13}0 \sim x_{q2}0$, $x_{E12}0 \sim x_{q3}0$, imply $p_i = 1 - q_i = w^{-1}(W(S - E_i))$ for $i\in\{1,2,3\}$.

Therefore $TA^{-} = ITA^{+}$.

Moreover, $TA^{+} + ITA^{+} = 1 - (w^{-1}(W(E_{12})) - w^{-1}(W(E_1)) + w^{-1}(W(E_{23})) - w^{-1}(W(E_2)) + w^{-1}(W(E_{13})) - w^{-1}(W(E_3))) \leq 1$ (because $w^{-1}\circ W$ is increasing).

It is straightforward that this only property of $w^{-1}\circ W$ does not restrict the sign of TA^{+} and ITA^{+} any further.

A.6. Prospect theory (PT)

Result 17. PT does not predict anything about $BC^{-}(E)$ and $BC^{+}(E)$.

$BC^{+}(E) = (w^{+})^{-1}(W^{+}(E)) + (w^{+})^{-1}(W^{+}(S - E))$ and

$BC^{-}(E) = (w^{-})^{-1}(W^{-}(E)) + (w^{-})^{-1}(W^{-}(S - E))$.

Proof: Ambiguity-generated insensitivity does not predict anything about the relationship between the weight assign to an event and to its complement. And PT does not assume any special link between the weighting functions for losses than for gains.

Result 18. PT with ambiguity-generated insensitivity predicts that $LA^{+}(Ei,Ej)$, $LA^{-}(E_i,E_j)$, $UA^{+}(E_i)$, and $UA^{-}(E_i)$ are all positive but does not predict any specific link between the positive indexes and the respective indexes for losses.

Proof: Recall that a decision maker exhibits *ambiguity-generated insensitivity* if for s = +,–, (i) $W^{s}(E_i) = w^{s}(p_i)$ and $W^{s}(E_j) = w^{s}(q_j)$ imply that $W^{s}(E_{ij}) \leq w^{s}(p_i+p_j)$ provided that $w^{s}(p_i+p_j)$ is bounded away from 1, and (ii) $W^{s}(E_{ij}) = w^{s}(p_{ij})$ and $W^{s}(E_{ik}) = w^{s}(p_{ik})$ imply that $W^{s}(E_i) \geq w^{s}(p_{ij} + p_{ik} - 1)$ provided that $W^{s}(E_i)$ is bounded away from 0.

The following proof is the same for all sign s.

From (i), we can derive that the matching probabilities for E_i and E_j are p_i and p_j respectively but the matching probability p_{ij} of E_{ij} must be less than p_i+p_j (because $W^{s}(E_{ij}) \leq w^{s}(p_i+p_j)$). It follows that $LA^{s}(E_i,E_j)= p_i+p_j-p_{ij}$ must be positive.

From (ii), we can derive that the matching probabilities of E_{ij} and E_{ik} are p_{ij} and p_{ik} respectively (for any sign s) but the matching probability p_i for E_i must be more than $p_{ij} + p_{ik} - 1$ (because $W^{s}(E_i) \geq w^{s}(p_{ij} + p_{ik} - 1)$). It follows that $UA^{s}(E_i) = 1 - p_{ij} - p_{ik} + p_i \geq 0$.

Result 19. PT with ambiguity-generated insensitivity predicts that $0 \leq TA^{+} + ITA^{+} \leq 1$ and $0 \leq ITA^{-} + TA^{-} \leq 1$ but does not predict any specific link between the positive indexes and the respective indexes for losses.

Proof: $TA^{s} + ITA^{s} = p_i + p_j + p_k - 1 + 2 - p_{ij} - p_{ik} - p_{jk} \geq p_k + 1 - p_{ik} - p_{jk} \geq 0$ where the first inequality comes from (i) and the second from (ii).

Moreover, the increasingness of W^{s} and w^{s} implies $TA^{s} + ITA^{s} = 1 + p_i + p_j + p_k - p_{ij} - p_{ik} - p_{jk} \leq 1$.

A.7. Vector expected utility (VEU)

We consider two nonzero outcomes x and −x and we take U(0) = 0, U(x) = 1, and U(−x) = z with z < 0. Under VEU, $x_E0 \sim x_p0$ and $-x_E0 \sim -x_q0$ imply p = P(E) + $A((\zeta_i(E)P(E))_{0\le i<n})$ and q = P(E) + $A((\zeta_i(E)P(E)z)_{0\le i<n})/z$ respectively.

Result 20. If A is negative, then $BC^-(E) \le 0 \le BC^+(E)$. If A is positive, then $BC^+(E) \le 0 \le BC^-(E)$.

Proof: Under VEU, $BC^-(E) = -A((\zeta_i(E)P(E)z)_{0\le i<n})/z - A((\zeta_i(E)(1-P(E))z)_{0\le i<n})/z$.

$BC^+(E) = - A((\zeta_i(E)P(E)z)_{0\le i<n}) - A((\zeta_i(E)(1-P(E))z)_{0\le i<n})$.

Result 21. If A is negative, then TA^+ , $ITA^- \le 0 \le TA^-$, ITA^+. If A is positive, then TA^-, $ITA^+ \le 0 \le TA^+$, ITA^-.

Proof: Consider a partition of S into E_1, E_2, and E_3. Under VEU,

$TA^+ = A((\zeta_i(E_1)P(E_1))_{0\le i<n}) + A((\zeta_i(E_2)P(E_2))_{0\le i<n}) + A((\zeta_i(E_3)P(E_3))_{0\le i<n})$.

$TA^- = A((\zeta_i(E_1)(1-P(E_1))z)_{0\le i<n})/z + A((\zeta_i(E_2)(1-P(E_2))z)_{0\le i<n})/z$
$+ A((\zeta_i(E_3)(1-P(E_3))z)_{0\le i<n})/z$.

$ITA^+ = -A((\zeta_i(E_1)(1-P(E_1)))_{0\le i<n}) - A((\zeta_i(E_2)(1-P(E_2)))_{0\le i<n}) - A((\zeta_i(E_3)(1-P(E_3)))_{0\le i<n})$

$ITA^- = -A((\zeta_i(E_1)(1-P(E_1))z)_{0\le i<n})/z - A((\zeta_i(E_2)(1-P(E_2))z)_{0\le i<n})/z$
$- A((\zeta_i(E_3)(1-P(E_3))z)_{0\le i<n})/z$.

The result follows.

A.8. Additivity indexes

Result 22. Violations of any 4 of the 5 types of additivity do not imply a violation of the 5^{th} one.

Proof: Let us consider a nonzero outcome x and a partition of S in 3 events E_1, E_2, E_3. The third column of Table A.1 is an example of an additive situation. The others are examples of violations of 4 types of additivity, not implying a violation of the 5th one.

Matching probabilities	**m(E_1,**x)	**⅓**	⅖	⅕	⅗	**⅓**	½
	m(E_2,x)	**⅓**	⅖	⅕	⅗	**⅓**	½
	m(E_3,x)	**⅓**	⅖	⅕	⅗	**⅓**	½
	m(E_{12},x)	⅔	⅗	⅖	⅘	½	⅔
	m(E_{13},x)	⅔	⅗	⅖	⅘	½	⅔
	m(E_{23},x)	⅔	⅗	⅖	⅘	½	⅔
Binary complementarity	**BC(E_1)**	**0**	**0**	⅖	−⅖	⅙	−⅙
	BC(E_2)	**0**	**0**	⅖	−⅖	⅙	−⅙
	BC(E_3)	**0**	**0**	⅖	−⅖	⅙	−⅙
Lower additivity	**LA(E_1,E_2)**	**0**	⅕	**0**	⅖	⅙	**⅓**
	LA(E_1,E_3)	**0**	⅕	**0**	⅖	⅙	**⅓**
	LA(E_2,E_3)	**0**	⅕	**0**	⅖	⅙	**⅓**
Upper additivity	**UA(E_1)**	**0**	⅕	⅖	**0**	**⅓**	⅙
	UA(E_2)	**0**	⅕	⅖	**0**	**⅓**	⅙
	UA(E_3)	**0**	⅕	⅖	**0**	**⅓**	⅙
Direct and indirect ternary additivity	**TA**	**0**	⅕	−⅖	⅘	**0**	½
	ITA	**0**	⅕	⅘	−⅖	½	**0**

Table A.1: Example of violations of none or all but one types of additivity

B – Proportion of subjects satisfying each prediction

	Experiment 1						Experiment 2					
	AEX			SENSEX			AEX			TOP40		
$BC^{-}(E) = BC^{+}(E) = 0$	E_1: 49%	E_2: 46%	E_3: 32%	E_1: 49%	E_2: 43%	E_3: 41%	E_1: 52%	E_2: 45%	E_3: 53%	E_1: 45%	E_2: 37%	E_3: 54%
$BC^{-}(E) \leq 0 \leq BC^{+}(E)$	E_1: 62%	E_2: 62%	E_3: 51%	E_1: 62%	E_2: 54%	E_3: 51%	E_1: 59%	E_2: 52%	E_3: 61%	E_1: 56%	E_2: 50%	E_3: 59%
$BC^{-}(E) \geq 0 \geq BC^{+}(E)$	E_1: 76%	E_2: 81%	E_3: 68%	E_1: 78%	E_2: 78%	E_3: 78%	E_1: 84%	E_2: 82%	E_3: 84%	E_1: 83%	E_2: 79%	E_3: 86%
$BC^{-}(E) = BC^{+}(E)$	E_1: 76%	E_2: 78%	E_3: 65%	E_1: 78%	E_2: 78%	E_3: 73%	E_1: 86%	E_2: 79%	E_3: 80%	E_1: 72%	E_2: 76%	E_3: 84%
$-BC^{-}(E) = BC^{+}(E)$	E_1: 81%	E_2: 76%	E_3: 70%	E_1: 68%	E_2: 70%	E_3: 68%	E_1: 85%	E_2: 78%	E_3: 82%	E_1: 86%	E_2: 77%	E_3: 82%
$LA^{-}(E_i,E_j) = LA^{+}(E_i,E_j) = 0$	E_{12}: 27%	E_{23}: 24%	E_{13}: 24%	E_{12}: 24%	E_{23}: 24%	E_{13}: 22%	E_{12}: 26%	E_{23}: 22%	E_{13}: 29%	E_{12}: 30%	E_{23}: 28%	E_{13}: 28%
$LA^{-}(E_i,E_j) \geq 0 \geq LA^{+}(E_i,E_j)$	E_{12}: 46%	E_{23}: 49%	E_{13}: 51%	E_{12}: 51%	E_{23}: 46%	E_{13}: 46%	E_{12}: 49%	E_{23}: 40%	E_{13}: 47%	E_{12}: 47%	E_{23}: 50%	E_{13}: 41%
$LA^{-}(E_i,E_j) \geq LA^{+}(E_i,E_j)$	E_{12}: 97%	E_{23}: 92%	E_{13}: 97%	E_{12}: 97%	E_{23}: 95%	E_{13}: 92%	E_{12}: 99%	E_{23}: 96%	E_{13}: 97%	E_{12}: 97%	E_{23}: 96%	E_{13}: 95%
$LA^{-}(Ei,Ej) \leq LA^{+}(Ei,Ej)$	E_{12}: 76%	E_{23}: 76%	E_{13}: 86%	E_{12}: 76%	E_{23}: 81%	E_{13}: 81%	E_{12}: 79%	E_{23}: 80%	E_{13}: 77%	E_{12}: 81%	E_{23}: 83%	E_{13}: 79%
$LA^{-}(E_i,E_j) \leq 0 \leq LA^{+(}E_i,E_j)$	E_{12}: 46%	E_{23}: 46%	E_{13}: 49%	E_{12}: 41%	E_{23}: 43%	E_{13}: 43%	E_{12}: 42%	E_{23}: 51%	E_{13}: 48%	E_{12}: 51%	E_{23}: 47%	E_{13}: 47%
$LA^{+(}(E_i,E_j) + LA^{-}(E_i,E_j) \leq 1$	E_{12}: 97%	E_{23}: 95%	E_{13}:100%	E_{12}: 95%	E_{23}:100%	E_{13}: 97%	E_{12}: 99%	E_{23}:100%	E_{13}:100%	E_{12}:100%	E_{23}:100%	E_{13}: 99%
$LA^{+}(E_i,E_j)$, $LA^{-}(E_i,E_j) \geq 0$	E_{12}:100%	E_{23}: 95%	E_{13}: 95%	E_{12}:100%	E_{23}: 95%	E_{13}:100%	E_{12}:100%	E_{23}: 99%	E_{13}: 99%	E_{12}: 98%	E_{23}: 98%	E_{13}: 98%
$UA^{-}(E) = UA^{+}(E) = 0$	E_1: 22%	E_2: 22%	E_3: 27%	E_1: 27%	E_2: 22%	E_3: 27%	E_1: 27%	E_2: 21%	E_3: 29%	E_1: 26%	E_2: 30%	E_3: 26%
$UA^{-}(E) \leq 0 \leq UA^{+}(E)$	E_1: 24%	E_2: 27%	E_3: 43%	E_1: 32%	E_2: 27%	E_3: 43%	E_1: 35%	E_2: 30%	E_3: 36%	E_1: 38%	E_2: 38%	E_3: 37%
$UA^{-}(E) \leq UA^{+}(E)$	E_1: 89%	E_2: 92%	E_3: 84%	E_1: 92%	E_2: 97%	E_3: 95%	E_1: 93%	E_2: 97%	E_3: 96%	E_1: 93%	E_2: 94%	E_3: 92%

$UA^{-}(E) \geq UA^{+}(E)$	E_1: 97%	E_2: 95%	E_3: 97%	E_1: 95%	E_2: 97%	E_3: 95%	E_1: 99%	E_2: 99%	E_3: 97%	E_1: 97%	E_2: 98%	E_3: 93%
$UA^{-}(E) \geq 0 \geq UA^{+}(E)$	E_1: 59%	E_2: 51%	E_3: 57%	E_1: 51%	E_2: 54%	E_3: 54%	E_1: 57%	E_2: 57%	E_3: 62%	E_1: 49%	E_2: 61%	E_3: 54%
$UA^{+}((E) + UA^{-}(E) \leq 1$	E_1: 92%	E_2: 95%	E_3: 95%	E_1: 89%	E_2: 95%	E_3: 100%	E_1: 99%	E_2: 99%	E_3: 99%	E_1: 97%	E_2: 99%	E_3: 98%
$UA^{+}((E), UA^{-}(E) \geq 0$	E_1: 97%	E_2: 97%	E_3: 97%	E_1: 97%	E_2: 100%	E_3: 100%	E_1: 100%	E_2: 98%	E_3: 100%	E_1: 98%	E_2: 99%	E_3: 97%
$TA^{+} = TA^{-} = ITA^{-} = ITA^{+} = 0$	8%			11%			8%			7%		
$TA^{+}, ITA^{-} \leq 0 \leq TA^{-}, ITA^{+}$	19%			24%			13%			10%		
$TA^{+} = ITA^{-} \leq 0 \leq TA^{-} = ITA^{+}$	16%			22%			12%			9%		
$TA^{+} = ITA^{-} \leq TA^{-} = ITA^{+}$	32%			51%			56%			54%		
$TA^{+} = ITA^{-} \geq TA^{-} = ITA^{+}$	57%			54%			77%			70%		
$TA^{+} = ITA^{-} \geq 0 \geq TA^{-} = ITA^{+}$	22%			22%			40%			35%		
$TA^{+}, ITA^{-} \geq 0 \geq TA^{-}, ITA^{+}$	32%			30%			43%			41%		
$TA^{+} = ITA^{-}$ & $TA^{-} = ITA^{+}$ & $TA^{+}+ITA^{+} \leq 1$ & $TA^{-}+ITA^{-} \leq 1$	57%			57%			78%			73%		
$0 \leq TA^{+} + ITA^{+} \leq 1$ & $0 \leq ITA^{-} + TA^{-} \leq 1$	89%			92%			99%			97%		

Table B.1: Proportion of subjects satisfying each prediction