Appendix for "Offshoring in a Ricardian World"

This Appendix presents the proofs of Propositions 1 - 6 and the derivations of the results in Section IV.

Proof of Proposition 1
We want to show that

\[
\left( \frac{T_m}{L_m} \right)^{\kappa} (T_m w_m^{-\theta} + \Phi_m)^{1/\theta} \geq \left( \frac{T_1}{L_1} \right)^{\kappa} (T_1 w_1^{-\theta} + T_2 w_2^{-\theta} + \Phi_m)^{1/\theta}
\]

Since \( \left( \frac{T_1}{L_1} \right)^{\kappa} > \left( \frac{T_m}{L_m} \right)^{\kappa} \), it is enough to prove the inequality for \( \Phi_m = 0 \). Thus, using \( w_i = \delta (T_i/L_i)^{\kappa} \) for \( i = 1, 2, m \) then we need to show that

\[
\left( \frac{T_m}{L_m} \right)^{\kappa} \left( T_m \delta^{-\theta} \left( \frac{L_m}{T_m} \right)^{\kappa \theta} \right)^{1/\theta} \geq \left( \frac{T_1}{L_1} \right)^{\kappa} \left( T_1 \delta^{-\theta} \left( \frac{L_1}{T_1} \right)^{\kappa \theta} + T_2 \delta^{-\theta} \left( \frac{L_2}{T_2} \right)^{\kappa \theta} \right)^{1/\theta}
\]

This implies that \( w_0 \) is decreasing, while \( \delta (T_2/L_2)^{\kappa} \) is increasing in \( \alpha \). To determine the sign of \( w_1'(\alpha) \) on \([0, \alpha]\) we should then compare

\[
w_1'(\alpha) = \delta \left( \frac{T_1}{L_1} \right)^{\kappa} \frac{(1 - \kappa)}{(1 + \alpha)^{\kappa}} - \delta \left( \frac{T_2}{L_2} \right)^{\kappa} \frac{\alpha}{(L_2 - \alpha L_1)^{\kappa \theta}} \frac{\alpha L_1}{L_2 - \alpha L_1}
\]
$w'_1(0)$ and $w'_1(\tilde{\alpha})$ with zero. Focusing first on $w'_1(\tilde{\alpha})$, using the definition of $\tilde{\alpha}$, we get

$$w'_1(\tilde{\alpha}) = \frac{\kappa T_2 \delta (1 + \eta L_1/L_2)^{\kappa}}{(L_2 + L_1)^{\kappa}} \left( -1 - \frac{(\eta - 1) L_1}{L_2 + L_1} \right) < 0$$

Turning to $w'_1(0)$, note that

$$w'_1(0) = \frac{\delta}{T_1} \left( \frac{T_1}{L_1} \right)^{\kappa} (1 - \kappa) - \frac{\delta T_2^{\kappa}}{L_2^{\kappa}} = \delta \left( \frac{T_2}{L_2} \right)^{\kappa} (\eta^{\kappa} (1 - \kappa) - 1) > 0 \iff \eta > (1 - \kappa)^{-1/\kappa}$$

Thus, if $\eta \leq (1 - \kappa)^{-1/\kappa}$, then $w_1(\alpha)$ is always decreasing on $[0, \tilde{\alpha}]$. If $\eta > (1 - \kappa)^{-1/\kappa}$, then $w_1(\alpha)$ is shaped like an inverted $U$ on $[0, \tilde{\alpha}]$. \textbf{Q.E.D.}

**Proof of Proposition 3**

Recall that $\Phi \equiv \sum_k T_k c^{-\theta}_k$. Thus, it is useful to use

$$\Phi = T_1 c^{-\theta}_1 + T_2 w^{-\theta}_2 + \Phi_{-m}$$

where $\Phi_{-m}$ is not affected by $\alpha$. We know that $c_1 = \delta \left( \frac{T_1}{L_1} \right)^{\kappa}$ and $w_2 = \delta \left( \frac{T_2}{L_2} \right)^{\kappa}$, so

$$T_1 c^{-\theta}_1 + T_2 w^{-\theta}_2 = \delta^{-\theta} \left( T_1^{\kappa} L_1^{\theta \kappa} (1 + \alpha)^{-\kappa} + T_2^{\kappa} (L_2 - \alpha L_1)^{\theta \kappa} \right)$$

This implies that

$$\left( T_1 c^{-\theta}_1 + T_2 w^{-\theta}_2 \right)' = \delta^{-\theta} \theta \kappa L_1 \left[ (T_1^{\kappa} L_1^{\theta \kappa} (1 + \alpha)^{-\kappa} - T_2^{\kappa} (L_2 - \alpha L_1)^{-\kappa} \right]$$

We need to compare $f(\alpha) = \left( \frac{T_1}{T_1^{\kappa}} \right)^{\kappa} (1 + \alpha)^{-\kappa} - \left( \frac{T_2}{T_2^{\kappa}} \right)^{\kappa} > 0$, while simple algebra reveals that $f(\tilde{\alpha}) = 0$. Since $f'(\alpha) < 0$, then $f(\tilde{\alpha}) = 0$ implies that $f'(\alpha) > 0$ for any $\alpha \in [0, \tilde{\alpha}]$. This means that $(T_1 c^{-\theta}_1 + T_2 w^{-\theta}_2)' > 0$, or $\Phi'_\alpha > 0$. But given $P = \gamma \Phi^{-1/\theta}$ then this implies that $P'_\alpha < 0$. \textbf{Q.E.D.}

**Proof of Proposition 4**

We know that the sign of $\left( \frac{w'_1}{w_1} \right)'_\alpha$ is the same as the sign of $\frac{w'_1}{w_1} - \frac{P'}{P}$. But simple differentiation and simplification reveals that

$$\frac{w'_1}{w_1} = G(x, \alpha) \equiv \frac{x(1 - \kappa) \left( \frac{f(\alpha)}{1 + \alpha} \right)^{\kappa} - \left( 1 + \frac{\alpha L_1/L_2}{f(\alpha)} \right)}{x(1 + \alpha) \left( \frac{f(\alpha)}{1 + \alpha} \right)^{\kappa} - \alpha}$$

$$\frac{P'}{P} = F(x, \alpha) \equiv -\kappa \frac{x^{\frac{1}{1+\alpha}} - \left( \frac{1}{f(\alpha)} \right)^{\kappa}}{x(1 + \alpha)^{\theta \kappa} + \frac{L_2}{L_1} (f(\alpha))^{\theta \kappa} + \frac{\delta \kappa}{L_1}$$
where \( x = \eta^\kappa, \beta/\alpha = 1 - \alpha L_1/L_2 \). Let \( x_F(\alpha) \) and \( x_G(\alpha) \) be defined implicitly by \( F(x,\alpha) = 0 \) and \( G(x,\alpha) = 0 \), respectively. The following lemma, whose proof is simple and therefore omitted, summarizes a number of properties of these functions:

**Lemma 1** \( F(x,\alpha) \) is decreasing in \( x \), \( G(x,\alpha) \) is increasing in \( x \),

\[
x_F(\alpha) = \left(1 + \frac{\alpha}{f(\alpha)}\right)^\kappa > 1, \text{ and } x_G(\alpha) = \frac{1 + \frac{\alpha L_1/L_2}{f(\alpha)}}{(1 - \kappa) \left(\frac{f(\alpha)}{1 + \alpha}\right)^\kappa} > 1.
\]

Also, \( x_F(\alpha) < x_G(\alpha), x'_F(\alpha) > 0, x'_G(\alpha) > 0 \).

Let \( x_M(\alpha) \) be defined implicitly by \( G(x,\alpha) = F(x,\alpha) \). Such a solution necessarily exists since \( x_F(\alpha) < x_G(\alpha) \) and \( F(x,\alpha) \) is decreasing in \( x \) and \( G(x,\alpha) \) is increasing in \( x \). Also, it is clear that \( 1 < x_F(\alpha) < x_M(\alpha) < x_G(\alpha) \). Since \( x > x_M(\alpha) \) implies \( G > F \) then it also implies that \( w_1/P \) is increasing. Similarly, \( x < x_M(\alpha) \) implies that \( w_1/P \) is decreasing. The following lemma (whose proof is long and tedious and therefore provided at the end of this Appendix) is critical:

**Lemma 2** \( x_M(\alpha) \) is increasing

Let \( \hat{\eta} \) be equal to \( x_M(0)^{1/\kappa} \). If \( \eta \leq \hat{\eta} \), then \( x = \eta^\kappa \leq x_M(0) \leq x_M(\alpha) \) for any \( \alpha \). This implies that \( F(x) > G(x) \) (except the case when \( x = \hat{\eta}^\kappa \) and \( \alpha = 0 \)), so \( w_1/P \) is decreasing. This establishes the first part of the proposition. To establish the second part, we need the following lemma:

**Lemma 3** For any \( \eta > \hat{\eta} = (x_M(0))^{1/\kappa} \) we have \( x = \eta^\kappa < x_M(\bar{\alpha}(\eta)) \).

**Proof.** The proof relies on showing that \( F(\eta^\kappa, \bar{\alpha}(\eta)) = 0 \), which implies that \( \eta^\kappa = x_F(\bar{\alpha}(\eta)) \). If this is true then \( x_M(\bar{\alpha}(\eta)) > \eta^\kappa \), because since \( x_F(\alpha) < x_M(\alpha) \) for all \( \alpha \) then \( \eta^\kappa = x_F(\bar{\alpha}(\eta)) < x_M(\bar{\alpha}(\eta)) \), which establishes the result. But from the definition of \( \bar{\alpha} \) we see that

\[
\eta = \frac{1 + \bar{\alpha}}{f(\bar{\alpha})}
\]

and plugging this into \( F(\eta^\kappa, \bar{\alpha}(\eta)) \) shows that \( F(\eta^\kappa, \bar{\alpha}(\eta)) = 0 \). \( \square \)

This lemma implies that if \( \eta > \hat{\eta} \) then \( w_1/P \) is increasing for \( \alpha = 0 \) and decreasing just before \( \alpha = \bar{\alpha}(\eta) \), with a unique point \( \alpha \) for which \( x_M(\alpha) = x \) at which \( G = F \) and hence \( (w_1/P)'_\alpha = 0 \). This implies that the curve \( w_1/P \) as a function of \( \alpha \) in the interval \( \alpha \in [0, \bar{\alpha}] \) is shaped like an inverted \( U \).

**Proof of Proposition 5**

The only thing left to show is that steady state \( P \) is decreasing in \( \alpha \). It is sufficient to show that \( \Phi_{mt} = T_{1t} e^{-\theta t} + T_{2t} w_{2t}^{-\theta} \) is decreasing in \( \alpha \). But

\[
\Phi_{mt} = \frac{1/L_{GL}}{(\bar{\phi}_1 r_1 \varphi c_{1t}^{-\theta} + \bar{\phi}_2 r_2 w_{2t}^{-\theta})}
\]
Using $c_1^{1-\kappa} = \left(\frac{\phi_1/\phi_N}{w_1}\right)^\kappa$ and $w_2 = (\phi_2/\phi_N)^\kappa$ then

$$
\Phi_{mt} = \left(\frac{1}{L_{2t}^m t}\right) \left(\phi_1 r_1 \varphi \left(\frac{\phi_1/\phi_N}{w_{1t}}\right)^{-\kappa \theta/(1-\kappa)} + \phi_2 r_2 \left(\phi_2/\phi_N\right)^{-\kappa \theta}\right)
$$

But plugging in from the equations (24) and (25) we get that

$$
\varphi r_1 w_1 + w_2 r_2 = \varphi w_1 + w_2
$$

which is increasing in $\alpha$. Q.E.D.

**Proof of Proposition 6**

I first show that $x = \alpha w_2/w_1$ is increasing in $\alpha$. From (29) we get $(\phi_1/\phi_N)^\kappa = z (1 + x)^{1-\kappa} = z(1 + x)^{1-\kappa} w_1$. Since $x$ is increasing in $\alpha$ then $z$ must be increasing in $\alpha$. In turn, this implies that $x$ must be increasing in $\alpha$.

Now, recall that $r_1$ is determined as the solution of $r_1 = r (1 + \alpha (1 - r_1) w_2/w_1)$. Both the LHS and the RHS are linear functions in $r_1$, with the LHS increasing and the RHS decreasing. An increase in $\alpha$ moves the RHS schedule upward because $\alpha w_2/w_1$ increases with $\alpha$, while the LHS schedule remains the same. This implies that $r_1$ increases.

In the text, before stating proposition 6 I also stated that $\alpha (1 - r_1) w_2/w_1$ is increasing in $\alpha$. To see this, note that since $r_1$ is increasing in alpha, then the RHS of $r_1 = r (1 + \alpha (1 - r_1) w_2/w_1)$ must be increasing in $\alpha$, so $\alpha (1 - r_1) w_2/w_1$ is increasing in $\alpha$.

Finally, to prove that $r_2$ is decreasing in $\alpha$, from (25) I need to show that $\alpha (1 - r_1)$ is increasing in $\alpha$. But we know that $\alpha (1 - r_1) w_2/w_1$ is increasing in $\alpha$ while $w_2$ is constant and $w_1$ is increasing. This implies that $\alpha (1 - r_1)$ must be increasing in $\alpha$. Q.E.D.

**Equilibrium with offshore costs: 2 countries**

Let

$$
C(w_1, w_2, \lambda) \equiv w_2 F(1, \lambda) + \int_{w_2}^{w_1} x dF(x, \lambda/w_2) dx + w_1 (1 - F(w_1, \lambda/w_2))
$$

Integration and simplification yields

$$
C(w_1, w_2, \lambda) = w_2 - (w_2/\lambda) \exp(-\lambda w_1/w_2) + (w_2/\lambda) \exp(-\lambda)
$$

Letting $s(w_1, w_2, \lambda) \equiv F(w_1/w_2, \lambda)$ and $\sigma(w_1, w_2, \lambda) = \Gamma(w_1, w_2, \lambda)s(w_1, w_2, \lambda)$, or

$$
\sigma(w_1, w_2, \lambda) \equiv 1 - (1/\lambda) \exp(-\lambda w_1/w_2) - w_1/w_2 \exp(-\lambda w_1/w_2) + (1/\lambda) \exp(-\lambda)
$$

then an equilibrium for $\lambda < \lambda_m$ is determined by the following equations:
\[ C(w_1, w_2, \lambda) = \delta \left( \frac{T_1}{L_1/(1-s(w_1, w_2))} \right)^\kappa \]  

(1)

and

\[ w_2 = \delta \left( \frac{T_2}{L_2 - \sigma(w_1, w_2, \lambda)L_1/(1-s(w_1, w_2, \lambda))} \right)^\kappa \]  

(2)

Totally differentiating above yields equation (1) yields

\[
\frac{dC}{d\lambda} = \frac{\partial C}{\partial w_1} \frac{dw_1}{d\lambda} + \frac{\partial C}{\partial w_2} \frac{dw_2}{d\lambda} + \frac{\partial C}{\partial \lambda} = -\delta(T_1/L_1)^\kappa (1-s)^{\kappa-1} \frac{ds}{d\lambda}
\]

As \( \lambda \to 0 \) then \( s \to 0 \), hence \( \partial C/\partial w_2 \to 0 \) and

\[
\frac{\partial C}{\partial w_1} \frac{dw_1}{d\lambda} + \frac{\partial C}{\partial \lambda} = -\delta(T_1/L_1)^\kappa \frac{ds}{d\lambda}
\]

Now,

\[
\frac{ds}{d\lambda} = \frac{\partial s}{\partial w_1} \frac{dw_1}{d\lambda} + \frac{\partial s}{\partial w_2} \frac{dw_2}{d\lambda} + \frac{\partial s}{\partial \lambda} = (\lambda/w_2) \exp(-\lambda w_1/w_2) - (\lambda w_1/w_2^2) \exp(-\lambda w_1/w_2) + (w_1/w_2) \exp(-\lambda w_1/w_2)
\]

When \( \lambda \to 0 \) then \( ds/d\lambda \to w_1/w_2 \). Noting that \( w_1 \to \delta(T_1/L_1)^\kappa \) as \( \lambda \to 0 \), we see that

\[
\frac{\partial C}{\partial w_1} \frac{dw_1}{d\lambda} = -\kappa \frac{w_1^2}{w_2} - \frac{\partial C}{\partial \lambda}
\]

Simple differentiation reveals that

\[
\frac{\partial C}{\partial \lambda} = \frac{w_2 \exp(-\lambda w_1/w_2) - w_2 \exp(-\lambda) + w_1 \lambda \exp(-\lambda w_1/w_2) - w_2 \lambda \exp(-\lambda)}{\lambda^2}
\]

Since both the numerator and denominators converge to 0 as \( \lambda \to 0 \) then we can use L’Hopital’s Theorem to find that

\[
\lim_{\lambda \to 0} \frac{\partial C}{\partial \lambda} = \lim_{\lambda \to 0} \frac{w_1(-\lambda)(w_1/w_2) \exp(-\lambda w_1/w_2) + w_2 \lambda \exp(-\lambda)}{2\lambda} = \frac{w_2 - w_1^2/w_2}{2}
\]

Thus,

\[
\lim \left( -\kappa \frac{w_1^2}{w_2} \frac{\partial C}{\partial \lambda} \right) = -\kappa \frac{w_1^2}{w_2} \frac{(w_1/w_2)^2 - (w_2/2) \left( 1 - \left( \frac{w_1}{w_2} \right)^2 \right)}{2}
\]

\[
= \frac{(w_2/2) \left( (w_1/w_2)^2 (1 - 2\kappa) - 1 \right)}{2}
\]

This implies that \( \lim_{\lambda \to 0} dw_1/d\lambda > 0 \) if and only if \( (w_1/w_2)^2 > 1/(1-2\kappa) \).
To show that $\partial C/\partial \lambda < 0$, note that this is equivalent to

$$(1 + w_1 \lambda/w_2) \exp(-w_1/w_2) < (1 + \lambda) \exp(-\lambda)$$

This inequality holds since the function $f(x) = (1 + x) \exp(-x)$ is clearly decreasing.

**Equilibrium with offshoring costs: 3 countries**

I first characterize an equilibrium in which $w_1 > w_3 > w_2$. In this case the distribution of $c_1$ is

$$\Pr(C_1 \leq c_1) = \begin{cases} 0 & \text{if } c_1 < w_2 \\ F(c_1, \lambda/w_2) & \text{if } c_1 \in [w_2, w_3] \\ F(c_1, \varphi) & \text{if } c_1 \in [w_3, w_1] \\ 1 & \text{if } c_1 \geq w_1 \end{cases}$$

where $\varphi \equiv \lambda/w_2 + \lambda/w_3$, and unit cost of the common input in country 1 is

$$c_1(w_1, w_2, w_3) = w_2 F(1, \lambda) + \int_{w_2}^{w_3} dF(x, \lambda/w_2) + w_3 \left( F(w_3, \varphi) - F(1, \lambda) \right)$$

The equilibrium condition for country 1 is

$$c_1(w_1, w_2, w_3) = \delta \left( \frac{T_1 (1 - s)}{L_1} \right)^\kappa$$

where $s = 1 - \exp(-\varphi w_3)$ is the total share of services offshored. This share is distributed between countries 2 and 3 as follows:

$$s_{12} = F(w_3, \lambda/w_2) + [F(w_1, \varphi) - F(w_3, \varphi)] \left( \frac{\lambda/w_2}{\varphi} \right)$$

and

$$s_{13} = F(w_3, \lambda/w_2 + \lambda/w_3) - F(w_3, \lambda/w_2) + [F(w_1, \varphi) - F(w_3, \varphi)] \left( \frac{\lambda/w_3}{\varphi} \right)$$

Note that $s = s_{12} + s_{13}$.

On the other hand, we have that

$$\Pr(C_3 \leq c_3) = \begin{cases} 0 & \text{if } c_3 < w_2 \\ F(c_3, \lambda/w_2) & \text{if } c_3 \in [w_2, w_3] \\ 1 & \text{if } c_3 \geq w_3 \end{cases}$$

and

$$c_3(w_1, w_2, w_3) = w_2 F(1, \lambda) + \int_{w_2}^{w_3} x dF(x, \lambda_2/w_2) + w_3 (1 - F(w_3, \lambda_2/w_2))$$
and
\[ s_{32}(w_2, w_3) = F(w_3, \lambda_2/w_2) \]

The equilibrium conditions for countries 2 and 3 are \( c_i(w_1, w_2, w_3) = \delta \left( T_i/\bar{L}_i \right) \) for \( i = 2, 3 \), with \( c_2(w_1, w_2, w_3) = w_2 \). To derive \( \bar{L}_3 \), note that country 1 uses \( L_1/(1-s) \) of every service, and if the offshoring cost of a service in country 3 is \( \zeta_3 \) then it takes \( \zeta_3 L_1/(1-s) \) units of labor to produce \( L_1/(1-s) \) units of a service delivered in country 1. The expectation of \( \zeta_3 \) for services offshored by country 1 to country 3 is

\[ \sigma_{13}(w_1, w_2, w_3) = F(w_3, \varphi) - F(w_3, \lambda/w_2) + \left( \frac{\lambda/w_3}{\varphi} \right) \int_{w_3}^{w_1} x F(x, \varphi) \]

This implies that \( \bar{L}_3 = \left( \frac{\sigma_{13}}{1-s} \right) L_1 \) units of labor are left in country 3 for final good production. But this country will offshore \( s_{32} \) services to country 2, so it will use \( \left( \frac{\sigma_{13}}{1-s} \right) L_1 \left( \frac{1}{1-s_{32}} \right) \) of each service for domestic production. This implies that

\[ \bar{L}_3 = \left( L_3 - \left( \frac{\sigma_{13}}{1-s} \right) L_1 \right) \left( \frac{1}{1-s_{32}} \right) \]

On the other hand, country 2 export services to countries 1 and 3. The labor it takes to export services to country 1 is \( \left( \frac{\sigma_{12}}{1-s} \right) L_1 \), where \( \sigma_{12} \) is the expectation of \( \zeta_2 \) for services offshored by country 1 to country 2, and is given by

\[ \sigma_{12}(w_1, w_2, w_3) = (1 - \exp(-\lambda)) + \left( \frac{1}{w_2} \right) \int_{w_2}^{w_3} x F(x, \lambda/w_2) \]

\[ + \left( \frac{\lambda/w_2}{\varphi} \right) \int_{w_3}^{w_1} x F(x, \varphi) \]

Country 2 also exports services to country 3, and by the reasoning above, we know that it takes \( \left( L_3 - \left( \frac{\sigma_{13}}{1-s} \right) L_1 \right) \left( \frac{\sigma_{32}}{1-s_{32}} \right) \) units of labor to do so, with \( \sigma_{32} \) representing the expectation of \( \zeta_2 \) for services offshored by country 3 to country 2, given by

\[ \sigma_{32}(w_1, w_2, w_3) = (1 - \exp(-\lambda)) + \left( \frac{1}{w_2} \right) \int_{w_2}^{w_3} x F(x, \lambda/w_2) \]

Thus, we find that

\[ \bar{L}_2 = L_2 - \left( \frac{\sigma_{12}}{1-s} \right) L_1 - \left( L_3 - \left( \frac{\sigma_{13}}{1-s} \right) L_1 \right) \left( \frac{\sigma_{32}}{1-s_{32}} \right) \]

If the previous system yields wages that do not respect \( w_1 > w_3 > w_2 \) then it is not an equilibrium. Other possible equilibrium configurations have
\( w_1 > w_2 = w_3, \ w_1 = w_3 > w_2, \text{ and } w_1 = w_2 = w_3. \) The last case entails full offshoring. Such an equilibrium satisfies

\[
\frac{T_m}{L_m} = \frac{T_1(1 - s)}{L_1} = \frac{T_2}{L_2 - \left( \frac{s_{12}}{1 - s} \right) L_1 - \left( L_3 - \left( \frac{s_{13}}{1 - s} \right) L_1 \right) \left( \frac{s_{32}}{1 - s_{32}} \right)} = \frac{T_3(1 - s_{32})}{L_3 - \left( \frac{s_{13}}{1 - s} \right) L_1}
\]

with \( s = s_{12} + s_{13}. \) For every \( s_{32} \) these equations determine \( s_{12} \) and \( s_{13}, \) so these variables are not uniquely pinned down in equilibrium: there is no uniqueness because of the absence of offshoring costs for the services that are traded, but all equilibria entail the same wages.\(^1\) If one can find a solution with \( s < F(1, 2\lambda) \) and \( s_{12}, s_{13}, s_{32} < F(1, \lambda), \) then this solution corresponds to an equilibrium with full offshoring. Since \( F(1, 2\lambda) \) and \( F(1, \lambda) \) both converge to \( 1 \) as \( \lambda \to \infty \) then necessarily there is some critical value of \( \lambda \) such that for higher values of \( \lambda \) the equilibrium entails full offshoring.

I now establish the equilibrium conditions when wages entail \( w_1 > w_2 = w_3 \) and \( w_1 = w_3 > w_2. \) If \( w_1 > w_2 = w_3 \) then countries 2 and 3 are integrated and their wage should be the same as the one we would would get in a two country system, with

\[
w_2 = w_3 = w_{23} = \delta \left( \frac{T_2 + T_3}{L_2 + L_3 - \left( \frac{s_{13}}{1 - s} \right) L_1} \right)^\kappa
\]

but with countries 2 and 3 drawing their offshoring cost from \( F(\zeta, 2\lambda), \) and with

\[
s = F(w_1/w_{23}, 2\lambda) \text{ and } \sigma = F(1, 2\lambda) + \int_1^{w_1/w_{23}} x \text{d}F(x, 2\lambda)
\]

This equilibrium has \( \sigma_{12} \) and \( \sigma_{13} \) and \( s_{32} \) such that

\[
w_{23} = \delta \left( \frac{T_2}{L_2 - \left( \frac{s_{12}}{1 - s} \right) L_1 - \left( L_3 - \left( \frac{s_{13}}{1 - s} \right) L_1 \right) \left( \frac{s_{32}}{1 - s_{32}} \right)} \right)^\kappa
\]

and

\[
w_{23} = \delta \left( \frac{T_3}{L_3 - \left( \frac{s_{13}}{1 - s} \right) L_1 \left( \frac{1 - s_{32}}{1 - s} \right)} \right)^\kappa
\]

with the restriction that \( \sigma_{32} = s_{32} \leq F(1, \lambda) \) and

\[
\sigma_{12}, \sigma_{13} \leq F(1, \lambda) + (1/2) \int_1^{w_1/w_{23}} x \text{d}F(x, 2\lambda)
\]

\(^1\)Although there are 3 equations for 3 unknowns \( (s_{12}, s_{13} \text{ and } s_{32}), \) these equations are linearly dependent, so they determine only two unknowns.
The first term on the RHS is the measure of services for which $\zeta_2$ or $\zeta_3$ are equal to 1, while the second term is the offshoring cost for services with $\zeta_i \in [1, w_i/w_3]$.

Now consider the case with $w_1 = w_3 > w_2$. This entails

$$w_1 = w_3 = w_{13} = \delta \left( \frac{(T_1 + T_3)(1-s)}{L_1 + L_3} \right)^\kappa$$

and

$$w_2 = \delta \left( \frac{T_2}{L_2 - \frac{\sigma}{1-s}(L_1 + L_3)} \right)^\kappa$$

where $s = F(w_{13}/w_3, \lambda)$ and

$$\sigma = F(1, \lambda) + \int_1^{w_{13}/w_2} x DF(x, \lambda)$$

For this to be an equilibrium, we need that $\sigma_{13} = s_{13} \leq F(1, \lambda)$, have $s_{32} = s_{12} = F(w_{13}/w_2, \lambda)$ and

$$\sigma_{32} = \sigma_{12} = F(1, \lambda) + \int_1^{w_{13}/w_2} x DF(x, \lambda)$$

and the equations of the full system.

For the long run equilibrium, we have $c_i(w_1, w_2, w_3)^{1-\kappa} = \left( \frac{\phi_i/\phi_N}{w_i} \right)^\kappa$ for $i = 1, 2, 3$, with $c_2(w_1, w_2, w_3) = w_2$. This equation for $i = 2$ can be solved directly to yield $w_2 = (\phi_2/\phi_N)^\kappa$, as in (28). Plugging this into the equation for $i = 3$ yields an equation that can be solved for the equilibrium wage in country 3, $w_3$. We can easily check that $w_3$ is increasing in $\lambda$. Finally, plugging the solution $w_3(\lambda)$ into the equation for $i = 1$ yields the equilibrium $w_1(\lambda)$. I have not been able to show that $w_1(\lambda)$ is increasing.

**Proof of Lemma 2**

Lemma 2 above establishes that $x_M(\alpha)$ is increasing. To prove this lemma, I first introduce some notation. Let

$$H(x, \alpha) = x^2 + \frac{Bx}{A} - C/A^2$$

$$J(x, \alpha) = \left[ \left( 1 + \frac{\alpha b L_1/L_2}{f(\alpha)} \right) - Ax(1-b) \right] \frac{\text{const}}{(1+\alpha)A^2}$$

where

$$A = \left( \frac{f(\alpha)}{1+\alpha} \right)^b, \text{ const } = \delta^\alpha (f(\alpha))^b \Phi_{-m}/L_1$$

$$B = (1-b)C - \left( 1 + \frac{\alpha b L_1/L_2}{f(\alpha)} \right) - b - \frac{\alpha}{(1+\alpha)}$$

$$C = \left( \frac{L_2/L_1}{(1+\alpha)} \right) \frac{f(\alpha)}{(1+\alpha)}$$

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Simple algebra shows that \( G(x, \alpha) = F(x, \alpha) \Leftrightarrow H(x, \alpha) = J(x, \alpha) \), so \( x_M(\alpha) \) solves
\[
H(x, \alpha) = J(x, \alpha)
\]
The proof that \( x_M(\alpha) \) is increasing includes three steps:

1) First, I prove that the solution \( x^0_M(\alpha) \) of \( H(x, \alpha) = 0 \) is increasing in alpha. Since \( J(x, \alpha) \) is flat in \( x \) if \( \Phi^- = 0 \) (since \( \text{const} = 0 \)) then this implies that if \( \Phi^- = 0 \) then \( x_M(\alpha) = x^0_M(\alpha) \) is increasing in \( \alpha \). The rest of the proof extends this to \( \Phi^* > 0 \).

2) Next, I prove that if \( \alpha_2 > \alpha_1 \) then \( H(x, \alpha_2) < H(x, \alpha_1) \) for any \( x \geq x^0_M(\alpha_1) \).

3) Finally, I prove that the solution of \( J(x, \alpha_2) = J(x, \alpha_1) \), where \( \alpha_2 \) is greater and close to \( \alpha_1 \), is less than \( x^0_M(\alpha_1) \).

Thus, given that the slope of \( J \) w.r.t. \( x \) increases (declines in absolute value) as \( \alpha \) increases, then the three steps above are sufficient to prove that \( x_M(\alpha) \) is increasing in alpha, since the shift of \( J(x, \alpha) \) with an increase in alpha amplifies the effect of increasing \( \alpha \) on \( x^0_M(\alpha) \).

**First step:** We want to prove that \( x^0_M(\alpha) \) is increasing in alpha. This is done by solving explicitly for the highest solution to \( H(x, \alpha) = 0 \) and then differentiating w.r.t. \( \alpha \) and showing that the result is positive. Given the expression for \( H(x, \alpha) = 0 \) then \( x^0_M(\alpha) \) is determined by the positive solution of
\[
A^2x^2 + ABx - C = 0,
\]
or
\[
x_M(\alpha) = \frac{-B + \sqrt{B^2 + 4C}}{2A}
\]
Differentiation yields:
\[
\frac{dx_M(\alpha)}{d\alpha} = A \left( \frac{2BB' + 4C'}{2\sqrt{B^2 + 4C}} - B' \right) - A' \left( \sqrt{B^2 + 4C} - B \right) \frac{2B}{2A^2}.
\]
It is easy to show that this is positive if and only if
\[
(A'B - AB') \left( \sqrt{B^2 + 4C} - B \right) > A'4C - 2C'A
\]
Differentiating to get \( A' \) and \( C' \) and then plugging in and simplifying reveals that
\[
A'4C - 2C'A = 2A \frac{1 + L_2/L_1}{(1 + \alpha)^2} (1 - 2b).
\]
Hence, we want to show that
\[
\left( \frac{A'}{A}B - B' \right) \left( \sqrt{B^2 + 4C} - B \right) > 2 \frac{1 + L_2/L_1}{(1 + \alpha)^2} (1 - 2b)
\]
Now,

\[ \frac{A'}{A} = -b \left( \frac{f(\alpha)}{(1+\alpha)} \right)^{b-1} \frac{1+L_1/L_2}{(1+\alpha)^2} = -b \frac{1+L_1/L_2}{f(\alpha)(1+\alpha)} \]

and

\[ -B' = (1-b)(L_2/L_1) \frac{1+L_1/L_2}{(1+\alpha)^2} + bL_1/L_2 \frac{1}{(f(\alpha))^2} + \frac{b}{(1+\alpha)^2}. \]

Consider \( \sqrt{B^2+4C} - B \) as a function of \( b \in (0, 1/2) \). We have

\[ \left( \sqrt{B^2+4C} - B \right)' = \frac{2BB'}{2\sqrt{B^2+4C}} - B' \]

\[ = B' \left( \frac{B - \sqrt{B^2+4C}}{\sqrt{B^2+4C}} \right) > 0, \]

as \( B - \sqrt{B^2+4C} < 0 \) and \( B' < 0 \). Thus, it is sufficient to show that

\[ \left( \frac{A'}{B} - B' \right) \left( \sqrt{B^2+4C} - B \right)_{b=0} > 2 \frac{1+L_2/L_1}{(1+\alpha)^2}(1-2b), \]

But

\[ \left( \sqrt{B^2+4C} - B \right)_{b=0} = \sqrt{\left( \frac{L_2/L_1}{1+\alpha} - 1 \right)^2 + 4(L_2/L_1)\frac{f(\alpha)}{(1+\alpha)^2}} - \left( \frac{L_2/L_1}{f(\alpha)} - 1 \right) \]

\[ = (L_2/L_1)\frac{f(\alpha)}{(1+\alpha)^2} + 1 - \left( \frac{L_2/L_1}{f(\alpha)} - 1 \right) = 2. \]

So, we want to prove that

\[ \left( \frac{A'}{A} - B' \right) > \frac{1+L_2/L_1}{(1+\alpha)^2}(1-2b) \]

Some manipulation reveals that

\[ \frac{A'}{A} - B' = (1-b)^2 \frac{1+L_2/L_1}{(1+\alpha)^2} + bL_1/L_2 \frac{1}{(f(\alpha))^2} \]

\[ + \frac{b}{(1+\alpha)^2} + b \frac{1+L_1/L_2}{f(\alpha)(1+\alpha)} \left( 1 + \frac{\alpha bL_1/L_2}{f(\alpha)^2} + b + \frac{\alpha}{(1+\alpha)^2} \right) \]

But it is trivial to establish that this is positive.

**Second step:** Consider equation \( H(x, \alpha_1) = H(x, \alpha_2) \) for any \( \alpha_1 : \alpha_2 > \alpha_1 \).

It is a linear equation so it has a unique solution. Moreover, so

\[ \left( \frac{(L_2/L_1)f(\alpha)}{(1+\alpha)A^2} \right)' = \frac{L_2}{L_1} \left( \frac{f(\alpha)}{(1+\alpha)} \right)^{1-2b} \left( \frac{f(\alpha)}{(1+\alpha)} \right)'_{\alpha}. \]
Since \( \theta > 1 \) (an assumption in EK 2002) \( b < 1/2 \). That is, \( 1 - 2b > 0 \). This means that 
\[
\left( \frac{L_2/L_1}{(1 + \alpha)A^2} \right) \alpha < 0 \quad \text{or} \quad \left( \frac{L_2/L_1}{(1 + \alpha)A^2} \right) \alpha > 0.
\]
That is, the intercept of \( H(x, \alpha) \) with vertical axis is always negative and increasing in \( \alpha \). Thus, \( 0 > H(0, \alpha_2) > H(0, \alpha_1) \). Since \( H \) is U-shaped and \( x_M^0(\alpha_2) > x_M^0(\alpha_1) > 0 \) (see 2) then 
\[
H(x_M^0(\alpha_1), \alpha_2) < H(x_M^0(\alpha_1), \alpha_1) = 0.3
\]
By continuity, there must exist \( x^* \in (0, x_M^0(\alpha_1)) \) such that \( H(x^*, \alpha_1) = H(x^*, \alpha_2) \). Since there is a unique solution to this equation, it follows that \( H(x, \alpha_2) < H(x, \alpha_1) \) for all \( x \geq x_M^0(\alpha_1) \).

**Third step:** It is obvious if \( J(x, \alpha) \) is fixed and does not change with an increase in \( \alpha \), then from the previous two steps we can say that \( x_M(\alpha) \) is increasing in \( \alpha \). However, with an increase in \( \alpha \) the curve \( J(x, \alpha) \) pivots around some point, with the slope becoming higher or less negative. If we prove that the solution to \( J(x, \alpha_2) = J(x, \alpha_1) \) with \( \alpha_2 \) just higher than \( \alpha_1 \) is less than \( x_M^0(\alpha_1) \), then we are done with the proof because the change in \( J(x, \alpha) \) amplifies the overall effect on \( x_M(\alpha) \). We have

\[
J(x, \alpha) = D(\alpha) - F(\alpha)x
\]

where

\[
D(\alpha) = \left( 1 + \frac{abL_1/L_2}{f(\alpha)} \right) \frac{\text{const}}{(1 + \alpha)A^2}
\]

\[
F(\alpha) = A(1-b)\frac{\text{const}}{(1 + \alpha)A^2}
\]

Then,

\[
J(x, \alpha_2) = J(x, \alpha_1) \iff x = \frac{D(\alpha_1) - D(\alpha_2)}{F(\alpha_1) - F(\alpha_2)}
\]

If we take the limit \( \alpha_2 \to \alpha_1 \), then

\[
x = \frac{D'(\alpha)}{F'(\alpha)}
\]

Tedious algebra shows that

\[
\frac{D'(\alpha)}{F'(\alpha)} = \frac{1}{1-b} \left( \frac{(1 + \alpha)}{f(\alpha)} \right)^b \left( 1 + \frac{abL_1/L_2}{f(\alpha)} \right) - \frac{(1 + \alpha) \left\{ bL_1/L_2 \left( f(\alpha) \right)^2 + \left( 1 + \frac{abL_1/L_2}{f(\alpha)} \right) \right\}}{(1 + \alpha) f(\alpha)}
\]

\( ^2 \)The last inequality comes from \( x_M(\alpha) = \frac{-B+\sqrt{B^2+4C}}{2A} \) and noting that \( -B + \sqrt{B^2+4C} > -B + \sqrt{B^2} = -B + |B| > -B + B = 0. \)

\( ^3 \)To see this, recall that \( x_M^0(\alpha) \) is the highest solution to \( H(x, \alpha) = 0 \) so that \( H(x_M^0(\alpha, \alpha) > 0 \). Thus, it must be the case that \( H(x_M^0(\alpha_1), \alpha_2) < 0 \), for otherwise the curve \( H(x, \alpha_2) \) would have its lower solution to \( H(x, \alpha_2) = 0 \) for a level of \( x \) higher than \( x_M^0(\alpha_1) \) and hence given the U-shape form of \( H \) it would follow that \( H(0, \alpha_2) > 0 \), which is a contradiction.
Next, we compare $D^0_F$ with $x_F(\alpha) = \left(\frac{1+\alpha}{f(\alpha)}\right)^b < x_M^0(\alpha)$ (this last inequality follows because $x_F(\alpha) < x_M(\alpha)$ for all $\Phi_m$ including $\Phi_m = 0$, but $x_M(\alpha; \Phi_m = 0) = x_M^0(\alpha)$). Algebra shows that this is equivalent to

$$
\frac{(1+\alpha) \left\{\frac{L_1/L_2}{f(\alpha)^2} + \left(1 + \frac{\alpha b L_1/L_2}{(1+\alpha)/f(\alpha)}\right) / (1 - b) \right\}}{\frac{\alpha L_1/L_2}{f(\alpha)}} > 1
$$

The left side of the inequality positively depends on $b$. Thus, to prove the inequality we can take $b = 0$, and then simple algebra reveals that the inequality holds. Thus, we proved that the solution of $J(x, \alpha_2) = J(x, \alpha_1)$ for $\alpha_2$ higher but close to $\alpha_1$ is strictly less than $x_F(\alpha_1) < x_M^0(\alpha_1)$. Q.E.D.