Monetary Policy Frameworks and the Effective Lower Bound on Interest Rates

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Online Appendix

A1. Derivation of inflation equations

Combining equations (1) and (2) from the New Keynesian model yields the equation for inflation

$$\pi_t - \mathbb{E}_t\pi_{t+1} = \mu_t + \kappa(\epsilon_t - \alpha(i_t - \mathbb{E}_t\pi_{t+1} - r^*))) + \beta\mathbb{E}_t(\pi_{t+1} - \pi_{t+2}).$$

With the interest rate rules of the form (4), the final term is zero and inflation is determined by

$$\pi_t = (1+\alpha\kappa)\mathbb{E}_t\pi_{t+1} + \mu_t + \kappa\epsilon_t - \alpha\kappa(i_t - r^*).$$

Plugging in the interest rate rule from equation (4) for when the central bank is unconstrained, delivers inflation of the form

$$\pi_t = \alpha\kappa(r^* - \theta_0) + (1+\alpha\kappa(1-\theta_E))\mathbb{E}_t\pi_{t+1} + (1-\alpha\kappa\theta_\mu)\mu_t + \kappa(1-\alpha\theta_\epsilon)\epsilon_t. \tag{A1}$$

In the case where the interest rate rule would ask for nominal rates below the lower bound, the central bank sets the policy rate as low as possible, to i^{LB}

$$\pi_t = \mu_t + \kappa\epsilon_t - \alpha\kappa(i^{LB} - r^*) + (1+\alpha\kappa)\mathbb{E}_t\pi_{t+1}. \tag{A2}$$

These two equations in the appendix are used in the main body of the paper. Note that they imply the existence of two steady-state equilibria in the deterministic version of the model. In both equation, inflation expectations and inflation rates appear linearly. Hence each equation may be associated with a steady-state equilibrium. We refer to the equilibrium associated with the first equation to the "target equilibrium" and the one associated with the second equation as a "liquidity trap".

A2. Derivation of inflation expectations

With demand or supply shocks, the lower bound can become an occasionally binding constraint. In this case, both equations for inflation (A1) and (A2) have to be used to determine inflation expectations

$$\mathbb{E}\pi = \text{Prob}\left(i_t^{opt} < i^{LB}\right)\mathbb{E}\left[\pi_t | i_t^{opt} < i^{LB}\right] + \text{Prob}\left(i_t^{opt} \geq i^{LB}\right)\mathbb{E}\left[\pi_t | i_t^{opt} \geq i^{LB}\right]. \tag{A3}$$

For this equation, we drop the period t subscript from the expectations operator. In this model with the specified monetary policy rule, there is no information at time t that predicts period $t+1$ inflation and therefore conditional and unconditional expectations are identical.

For illustrative purposes, we drop the demand shock from the model, i.e., we set its variance to zero. Then the constraint on nominal interest rates binds when μ_t falls below a cutoff value $\bar{\mu} = \frac{1}{\theta_\mu}(i^{LB} - \theta_0 - \theta_E\mathbb{E}\pi_{t+1})$.

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There are three different cases: The cutoff value $\bar{\mu}$ can fall below, in, or above the support of the distribution for the supply shock. The probability of being constrained by the lower bound can thus be expressed as

$$\text{Prob}\left(i_t^{opt} < i^{LB}\right) = \begin{cases} 1 & \text{if} \quad -\bar{\mu} \leq -\hat{\mu} \\ \frac{1}{2\hat{\mu}}(\hat{\mu}+\bar{\mu}) & \text{if} \quad -\hat{\mu} < -\bar{\mu} < \hat{\mu} \\ 0 & \text{if} \quad -\bar{\mu} \geq \hat{\mu}. \end{cases}$$

If we plug this expression in equation (A3) and compute conditional expectations of inflation from equations (A1) and (A2) respectively, we get inflation expectations as

$$\mathbb{E}\pi_t = \begin{cases} -\alpha\kappa(i^{LB} - r^*) + (1+\alpha\kappa)\mathbb{E}\pi_{t+1} & \text{if} \quad -\bar{\mu} \leq -\hat{\mu} \\ -\frac{\alpha\kappa}{4\hat{\mu}}\frac{(\bar{\mu}-\hat{\mu})^2}{\theta_\mu} + (1+\alpha\kappa(1-\theta_E))\mathbb{E}\pi_{t+1} - \alpha\kappa(\theta_0 - r^*) & \text{if} \quad -\hat{\mu} < -\bar{\mu} < \hat{\mu} \\ -\alpha\kappa(\theta_0 - r^*) + (1+\alpha\kappa)\mathbb{E}\pi_{t+1} & \text{if} \quad -\bar{\mu} \geq \hat{\mu}. \end{cases}$$

The expression in the middle where the constraint is occasionally binding is of particular interest. The cutoff for the supply shock that depends linearly on inflation expectations appears quadratically. To find a steady-state equilibrium for inflation expectations, we need to solve a quadratic equation which results in two solutions for a parameter range.[5]

A3. *Inflation expectations in the presence of an upper bound*

The derivation of inflation expectations for the case where both a lower and upper bound are present follows the same steps as in Appendix A.A2. Due to the additional constraint, however, the list of distinct cases increases. With various conditions $\mathbb{C}^{\text{LB}}_{\cdot}$ and $\mathbb{C}^{\text{UB}}_{\cdot}$ on the lower and upper bounds, respectively, we distinguish the cases:

$$\mathbb{E}[\pi_t] = \begin{cases} (1+\alpha\kappa)\mathbb{E}[\pi_{t+1}] - \alpha\kappa(i^{\text{LB}} - r^*) & \text{if } \mathbb{C}^{\text{LB}}_c \\ \frac{\alpha\kappa}{4\theta_\mu\hat{\mu}}\left(i^{\text{UB}} + i^{\text{LB}} - 2\theta_0 - 2\theta_E\mathbb{E}[\pi_{t+1}]\right)\left((i^{\text{UB}} - i^{\text{LB}}) - 2\theta_\mu\hat{\mu}\right) + & \text{if } \mathbb{C}^{\text{LB}}_o \text{ and } \mathbb{C}^{\text{UB}}_o \\ \qquad + (1+\alpha\kappa(1-\theta_E))\mathbb{E}[\pi_{t+1}] + \alpha\kappa(r^* - \theta_0) & \\ -\frac{\alpha\kappa\theta_\mu}{4\hat{\mu}}\left(\hat{\mu} + \frac{1}{\theta_\mu}\left(i^{\text{LB}} - \theta_0 - \theta_E\mathbb{E}[\pi_{t+1}]\right)\right)^2 + & \text{if } \mathbb{C}^{\text{LB}}_o \text{ and } \mathbb{C}^{\text{UB}}_u \\ \qquad + (1+\alpha\kappa(1-\theta_E))\mathbb{E}[\pi_{t+1}] + \alpha\kappa(r^* - \theta_0) & \\ (1+\alpha\kappa)\mathbb{E}[\pi_{t+1}] - \alpha\kappa(i^{\text{UB}} - r^*) & \text{if } \mathbb{C}^{\text{UB}}_c \\ \frac{\alpha\kappa\theta_\mu}{4\hat{\mu}}\left(\hat{\mu} - \frac{1}{\theta_\mu}\left(i^{\text{UB}} - \theta_0 - \theta_E\mathbb{E}[\pi_{t+1}]\right)\right)^2 + & \text{if } \mathbb{C}^{\text{LB}}_u \text{ and } \mathbb{C}^{\text{UB}}_o \\ \qquad + (1+\alpha\kappa(1-\theta_E))\mathbb{E}[\pi_{t+1}] + \alpha\kappa(r^* - \theta_0) & \\ (1+\alpha\kappa(1-\theta_E))\mathbb{E}[\pi_{t+1}] + \alpha\kappa(r^* - \theta_0) & \text{if } \mathbb{C}^{\text{LB}}_u \text{ and } \mathbb{C}^{\text{UB}}_u \end{cases}$$

The various conditions determine whether a constraint never binds, $\mathbb{C}^{\cdot}_u$, always binds, $\mathbb{C}^{\cdot}_c$, or occasionally binds, $\mathbb{C}^{\cdot}_o$. The specific conditions on the lower bound are

$$\mathbb{C}^{\text{LB}}_u = \left\{\frac{1}{\theta_\mu}(i^{LB} - \theta_0 - \theta_E E[\pi_{t+1}]) < -\hat{\mu}\right\}$$

[5]For cases of a single equilibrium or non-existence, see Mertens and Williams (2018).

for the lower bound to never bind,

$$\mathbb{C}_o^{\text{LB}} = \left\{ -\hat{\mu} \leq \frac{1}{\theta_\mu}(i^{LB} - \theta_0 - \theta_E E[\pi_{t+1}]) \leq \hat{\mu} \right\}$$

for the lower bound to occasionally bind, and

$$\mathbb{C}_c^{\text{LB}} = \left\{ \frac{1}{\theta_\mu}(i^{LB} - \theta_0 - \theta_E E[\pi_{t+1}]) > \hat{\mu} \right\}$$

for the lower bound to always bind.

For the upper bound, the conditions are

$$\mathbb{C}_u^{\text{UB}} = \left\{ \frac{1}{\theta_\mu}(i^{UB} - \theta_0 - \theta_E E[\pi_{t+1}]) > \hat{\mu} \right\}$$

for the upper to never bind,

$$\mathbb{C}_o^{\text{UB}} = \left\{ -\hat{\mu} \leq \frac{1}{\theta_\mu}(i^{UB} - \theta_0 - \theta_E E[\pi_{t+1}]) \leq \hat{\mu} \right\}$$

for the upper bound to occasionally bind, and

$$\mathbb{C}_c^{\text{UB}} = \left\{ \frac{1}{\theta_\mu}(i^{UB} - \theta_0 - \theta_E E[\pi_{t+1}]) < -\hat{\mu} \right\}$$

for the upper bound to always bind.

When solving for inflation expectations, a third equilibrium besides the target equilibrium and the liquidity trap emerges. This equilibrium is associated with the upper bound on nominal interest rates. As in the case with only a lower bound, we restrict our analysis to the target equilibrium.

A4. Comparison of policies

Table A1 shows a comparison of various statistics from the simulations using the different policies.

Table A1—Comparison of policies

	Discretion		Dovish Policies		AIT		PLT
	No lower bound		Symmetric upper bound	Optimal θ_μ	$E\pi = 0$	Optimal θ_0	Optimal θ_p
$\mathbb{E}(\pi_t)$	0.000	-0.244	0.000	-0.085	0.000	0.104	0.000
$\mathbb{V}(\pi_t)$	0.287	0.675	0.642	0.668	0.501	0.446	0.190
$\mathbb{E}(x_t)$	0.000	-0.003	0.000	-0.001	0.000	0.001	0.002
$\mathbb{V}(x_t)$	2.934	2.053	2.035	1.912	2.381	2.501	2.788
$\mathbb{E}(\pi_t^2) + \lambda\mathbb{E}(x_t^2)$	1.020	1.248	1.150	1.153	1.096	1.082	0.887
$\mathbb{P}(i_t = i^{LB})$	—	0.273	0.184	0.172	0.205	0.178	0.076
$\mathbb{E}(\pi_t \vert i_t = i^{LB})$	—	-1.389	-1.193	-1.402	-1.124	-1.005	-0.652
$\mathbb{E}(\pi_t \vert i_t > i^{LB})$	0.000	0.185	0.269	0.189	0.290	0.344	0.054
θ_0	1.000	1.000	1.000	1.000	0.900	0.850	1.000
θ_μ	0.719	0.719	0.719	0.626	0.719	0.719	0.719
θ_E	1.722	1.722	1.722	1.722	1.722	1.722	1.722
θ_p	0.000	0.000	0.000	0.000	0.000	0.000	0.360

Note: The above table shows various statistics for the simulations discussed in the main body of the paper.