Online Appendix

Common Ownership and the Secular Stagnation Hypothesis

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PROOF OF PROPOSITION 1: The objective function of the firm is strictly concave. The second derivative of the objective function with respect to labor is:

\[ F_{LL} - 2\omega' - \omega'' \cdot (L_j + \lambda L_{-j}) < 0 \]

since \( F_{LL} < 0 \) and \(-2\omega' - \omega'' \cdot (L_j + \lambda L_{-j}) < 0\) because we are assuming that labor supply is constant elasticity. The second derivative of the objective function with respect to capital is

\[ F_{KK} - 2\rho' - \rho'' \cdot (K_j + \lambda K_{-j}) < 0 \]

since \( F_{KK} < 0 \) and \(-2\rho' - \rho'' \cdot (K_j + \lambda K_{-j}) < 0\). The latter inequality follows because \(-2\rho' - \rho'' \cdot (K_j + \lambda K_{-j}) = -\rho'(K) \left[ 2 + \rho''(K)K/\rho'(K)(s^K_j + \lambda(1-s^K_j)) \right]\), where \(s^K_j\) is firm \(j\)'s share of capital and the expression in brackets is positive because \(\rho''(K)K/\rho'(K) \geq -1\). To see this, note that \(\rho'(K) = \frac{\gamma}{1-\gamma} \frac{E}{E-K} \frac{\rho(K)}{K}\) and \(\rho''(K) = \frac{\gamma}{1-\gamma} \frac{\rho'(K)E-K}{(E-K)^2} \left[ \frac{K}{E-K} + \frac{\rho'(K)}{\rho(K)} - 1 \right]\). Thus, \(\rho''(K)K/\rho'(K) = K/(E-K) + \rho'(K)/\rho(K) - 1 \geq -1\).

The fact that \(F_{LL} \cdot F_{KK} - F_{LK}^2\) is positive (since \(F\) is concave) implies that the determinant of the matrix of second derivatives is positive, which is the last condition we needed to establish strict concavity of the objective function. From the first-order conditions, it is then clear that the reaction functions are continuous, and therefore a Nash equilibrium exists.

To prove that there is a unique symmetric equilibrium, we consider the system of FOCs when employment and capital are symmetric across firms, and show that there is a unique solution. From
the FOC for labor, we can solve for labor as a function of capital, obtaining:

\[
L = \left[ \frac{A\alpha}{\chi^{\frac{1}{1-\sigma}} (1 + \frac{H}{\eta})} \right]^{\frac{1}{1-\alpha+\frac{1}{\eta}}} \frac{1}{K^{1-\alpha+\frac{1}{\eta}}}. 
\]

Replacing this in the FOC for capital, we obtain an implicit equation for capital:

\[
A(1-\alpha) \left[ \frac{A\alpha}{\chi^{\frac{1}{1-\sigma}} (1 + \frac{H}{\eta})} \right]^{\frac{1}{1-\alpha+\frac{1}{\eta}}} \frac{1}{K^{1-\alpha+\frac{1}{\eta}}} - \frac{\rho(K^*(H))}{\chi^{\frac{1}{1-\sigma}} (1 + \frac{H}{\eta})} - \rho(K^*(H)) (1 + \frac{H/\varepsilon(K^*)}{1 - \delta}) = 0.
\]

The limit when \( K \to 0^+ \) of this expression is \(+\infty\), while the limit when \( K \to E^- \) is \(-\infty\). The derivative of this expression with respect to \( K \) is negative, which implies that there is a unique solution to the equation. The two-equation characterization of the equilibrium obtains directly from imposing symmetry in the FOCs of the firm.

**PROOF OF PROPOSITION 2:**

(a) We start by noting that the number of firms \( J \) and the common ownership parameter \( \phi \) enter the equilibrium equation for capital only through market concentration \( H \). We then use the equilibrium equation for capital to define capital as an implicit function of \( H \in (0, 1] \):

\[
A(1-\alpha) \left[ \frac{A\alpha}{\chi^{\frac{1}{1-\sigma}} (1 + \frac{H}{\eta})} \right]^{\frac{1}{1-\alpha+\frac{1}{\eta}}} \frac{1}{K^{1-\alpha+\frac{1}{\eta}}} \equiv \rho(K^*(H)) (1 + \frac{H/\varepsilon(K^*)}{1 - \delta}).
\]

Taking log and derivative with respect to \( \log H \) yields

\[
-\frac{\alpha}{1-\alpha+\frac{1}{\eta}} \left( \frac{H}{\eta} + \frac{1}{\eta} \frac{d \log K^*}{d \log H} \right) = \frac{\rho \cdot (1 + H/\varepsilon)}{\varepsilon \cdot (1 + H/\varepsilon) - (1 - \delta)} \left[ \frac{1}{\varepsilon} \frac{d \log K^*}{d \log H} + \frac{H}{\varepsilon} \left( 1 + \frac{d \log K^*}{d \log H} \right) \right].
\]
Solving for $\varepsilon_{KH} \equiv \frac{d \log K^*}{d \log H}$:

$$\varepsilon_{KH} = -\frac{\alpha + \frac{H}{\eta}}{1 - \alpha + \frac{1}{\eta}} + \frac{\rho (1 + H / \varepsilon)}{1 + \frac{H}{\varepsilon}} \frac{\eta (1 + H / \varepsilon) - (1 - \delta) \frac{H}{\varepsilon} + \frac{H}{\varepsilon} s}{1 - \alpha + \frac{1}{\eta}} < 0.$$

(b) We know that

$$L^* = \left[ \frac{A \alpha}{\chi^{1/\sigma} \left( 1 + \frac{H}{\eta} \right)} \right]^{\frac{1}{1-\alpha+\frac{1}{\eta}}} K^{* \frac{1}{1-\alpha+\frac{1}{\eta}}}.$$

which is decreasing in $H$ and increasing in $K$. Since $H$ increases when the number of firms decreases or common ownership increases, and $K$ decreases with them, $L$ must decline with both lower $J$ and higher $\phi$.

(c), (d), and (e) Since the equilibrium real wage and real interest rates are increasing in $L$ and $K$, they also must decline when the number of firms decreases or common ownership increases. A lower level of employment and capital also implies lower output.

(f) The labor share of income is $\frac{\omega (L)}{F(K,L)} = \frac{\alpha}{1 + H / \eta}$. A decrease in the number of firms or an increase in the common ownership parameter $\phi$ increases $H$ and therefore decreases the labor share.

(g) We can obtain:

$$\text{sgn} \left\{ \frac{d \log \mu^*_K}{d \log H} \right\} = \text{sgn} \left\{ \frac{\eta + H}{s (1 - s)} \left[ \rho (K^*) - \left( \frac{\gamma}{1 - \gamma} + 1 \right) (1 - \delta) \right] \varepsilon_{KH} + \rho (K^*) - (1 - \delta) \right\}.$$

All else equal, given that $\varepsilon_{KH} < 0$ the expression above is minimized for $\gamma = 0$ for which it becomes:

$$\text{sgn} \left\{ \frac{d \log \mu^*_K}{d \log H} \right\} = \text{sgn} \left\{ \frac{1 - s}{s} + \varepsilon_{KH} \right\}.$$

Thus, a sufficient condition for the real interest rate markup $\mu^*_K$ to be increasing in $H$ is that the
elasticity of (equilibrium) capital with respect to \( H \) be low enough:

\[
|\varepsilon_{KH}| < \frac{1 - s}{s}.
\]

**PROOF OF PROPOSITION 3:**

As in Azar and Vives (2018), the competitive equilibrium relative price of sector \( n \)'s good is

\[
\frac{p_n}{P} = \left( \frac{1}{N} \right)^{1/\theta} \left( \frac{c_n}{c} \right)^{-1/\theta},
\]

where \( P \) is the price index \( \left( \frac{1}{N} \sum_{n=1}^{N} p_n^{1-\theta} \right)^{1/(1-\theta)} \). The competitive equilibrium relative price of sector \( n \) is

\[
\psi_n(K, L) = \left( \frac{1}{N} \right)^{1/\theta} \left( \frac{\sum_{j=1}^{J} F(K_{jn}, L_{jn})}{\left[ \sum_{m=1}^{N} \left( \frac{1}{N} \right)^{1/\theta} \left( \frac{\sum_{j=1}^{J} F(K_{jn}, L_{jn})}{(\theta^{-1})/(\theta^{-1})} \right)^{(\theta-1)/\theta} \right]} \right)^{-1/\theta}.
\]

The derivative with respect to \( L_{jn} \) is, as in Azar and Vives (2018):

\[
\frac{\partial \psi_n}{\partial L_{jn}} = -\frac{1}{\theta} \psi_n \left( 1 - \frac{p_n c_n}{PC} \right) \frac{F_L(K_{jn}, L_{jn})}{c_n} < 0.
\]

The derivative with respect to \( K_{jn} \) is similar:

\[
\frac{\partial \psi_n}{\partial K_{jn}} = -\frac{1}{\theta} \psi_n \left( 1 - \frac{p_n c_n}{PC} \right) \frac{F_K(K_{jn}, L_{jn})}{c_n} < 0.
\]

Also similarly to Azar and Vives (2018), the derivatives of the relative price in other sectors \( m \neq n \) are given by:

\[
\frac{\partial \psi_m}{\partial L_{jn}} = \frac{1}{\theta} \psi_n \frac{p_m c_m F_L(K_{jn}, L_{jn})}{PC} c_m > 0
\]

and

\[
\frac{\partial \psi_m}{\partial K_{jn}} = \frac{1}{\theta} \psi_n \frac{p_m c_m F_K(K_{jn}, L_{jn})}{PC} c_m > 0.
\]
The first-order condition of firm $j$ with respect to $L_{jn}$ is

$$
\psi_n F_L(K_{jn}, L_{jn}) - \omega - \omega' \left( L_{jn} + \lambda_{\text{intra}} \sum_{k \neq j} L_{kn} + \lambda_{\text{inter}} \sum_{m \neq n} \sum_{k=1}^J L_{km} \right) + \frac{\partial \psi_n}{\partial L_{jn}} \left( F(K_{jn}, L_{jn}) + \lambda_{\text{intra}} \sum_{k \neq j} F(K_{kn}, L_{kn}) \right) + \lambda_{\text{inter}} \sum_{m \neq n} \frac{\partial \psi_m}{\partial L_{jn}} \sum_{k=1}^J F(K_{km}, L_{km}) = 0.
$$

The first-order condition with respect to $K_{jn}$ is

$$
\psi_n F_K(K_{jn}, L_{jn}) - \rho - \rho' \left( K_{jn} + \lambda_{\text{intra}} \sum_{k \neq j} K_{kn} + \lambda_{\text{inter}} \sum_{m \neq n} \sum_{k=1}^J K_{km} \right) + (1 - \delta) + \frac{\partial \psi_n}{\partial K_{jn}} \left( F(K_{jn}, L_{jn}) + \lambda_{\text{intra}} \sum_{k \neq j} F(K_{kn}, L_{kn}) \right) + \lambda_{\text{inter}} \sum_{m \neq n} \frac{\partial \psi_m}{\partial K_{jn}} \sum_{k=1}^J F(K_{km}, L_{km}) = 0.
$$

In a symmetric equilibrium, similarly to Azar and Vives (2018), the first-order condition with respect to $L_{nj}$ simplifies to

$$
\frac{F_L(K_{jn}, L_{jn}) - \omega(L)}{\omega(L)} = \frac{\omega'(L) L}{\omega(L)} \left[ s_{jn}^L + \lambda_{\text{intra}} s_{jn}^L + \lambda_{\text{inter}} (1 - s_{jn}^L - s_{jn}^L) \right] + \frac{1}{\theta} \left( 1 - \frac{1}{N} \right) \frac{F_L(K_{jn}, L_{jn})}{\omega(L)} [s_{jn} + \lambda_{\text{intra}}(1 - s_{jn}) - \lambda_{\text{inter}}],
$$

where $s_{jn}^L \equiv L_{jn}/L$ is the labor market share of firm $j$ in sector $n$, $s_{jn}^L \equiv \sum_{k \neq j} L_{kn}/L$ is the combined labor market share of the other firms in sector $n$, and $s_{jn} \equiv F(K_{jn}, L_{jn})/c_n$ is the product market share of firm $j$ in sector $n$.

Analogously, the first-order condition with respect to $K_{jn}$ simplifies to

$$
\frac{F_K(K_{jn}, L_{jn}) - \rho(K)}{\rho(K)} + 1 - \delta = \frac{\rho'(K) K}{\rho(K) - 1 + \delta} \left[ s_{jn}^K + \lambda_{\text{intra}} s_{jn}^K + \lambda_{\text{inter}} (1 - s_{jn}^K - s_{jn}^K) \right] + (1 - \delta) + \frac{1}{\theta} \left( 1 - \frac{1}{N} \right) \frac{F_K(K_{jn}, L_{jn})}{\rho(K) - 1 + \delta} [s_{jn} + \lambda_{\text{intra}}(1 - s_{jn}) - \lambda_{\text{inter}}],
$$

where $s_{jn}^K \equiv K_{jn}/K$ is the capital market share of firm $j$ in sector $n$, $s_{jn}^K \equiv \sum_{k \neq j} K_{kn}/L$ is the combined capital market share of the other firms in sector $n$. 

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In a symmetric equilibrium the labor market share of firm \( j \) in sector \( n \) is \( \frac{1}{JN} \), its capital market share is also \( \frac{1}{JN} \), and its product market share is \( \frac{1}{J} \). Since \( \frac{\omega'(L) L}{\omega(L)} = \frac{1}{\eta} \), and defining \( \mu_L = F_L / \omega - 1 \), the first-order condition with respect to \( L_{jn} \) can be written as

\[
\mu_L^* = \frac{1}{\eta} \left[ \frac{\lambda_{intra}}{JN} + \frac{J - 1}{JN} - \frac{N - 1}{N} \right] + \frac{1 + \mu_L}{\theta} \left( \frac{1}{J_n} + \frac{\lambda_{intra}}{JN} - \frac{\lambda_{inter}}{N} \right) \left[ \frac{1}{J} + \frac{\lambda_{intra}}{J} - \frac{\lambda_{inter}}{N} \right].
\]

Similarly, since \( \frac{\rho'(K) K}{\rho(K) - 1 + \delta} = \frac{1}{\epsilon(K)} \left( \frac{1}{\rho(K)} - 1 \right) \), and defining \( \mu_K = F_K / (\rho - 1 + \delta) - 1 \), the first-order condition with respect to capital can be written as

\[
\mu_K^* = \frac{1}{\epsilon(K)} \left( \frac{1}{\rho(K) - 1 + \delta} \right) \left[ \frac{\lambda_{intra}}{JN} + \frac{J - 1}{JN} - \frac{N - 1}{N} \right] + \frac{1 + \mu_K}{\theta} \left( \frac{1}{J} \right) \left[ \frac{1}{J} + \frac{\lambda_{intra}}{J} - \frac{\lambda_{inter}}{N} \right].
\]

Solving for \( 1 + \mu_L^* \) and \( 1 + \mu_K^* \), we obtain

\[
1 + \mu_L^* = \frac{1 + H_{labor}/\eta}{1 - (H_{product} - \lambda_{inter}) (1 - 1/N) / \theta}
\]

\[
1 + \mu_K^* = \frac{1 + H_{capital}/(\epsilon(K) (1 - (1 - \delta) / \rho(K)))}{1 - (H_{product} - \lambda_{inter}) (1 - 1/N) / \theta},
\]

which are the expressions for the markdowns in the proposition.