.

Differentiated Durable Goods Monopoly: A Robust Coase Conjecture

Online Appendix

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Abstract: The sections of the online appendix discuss: related product design exercises; the details of the example reported in Section 4 of the main text; proofs omitted from the main text; and proofs of results included in the appendix. Results on product design establish why horizontal product differentiation is profit-maximizing for the monopolist, and why niche products can be optimal when multiple varieties are for sale.

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1 Product Design Results

This section considers the product design problem faced by a durable goods monopolist. The monopolist assembles products from a basket of characteristics. For instance, a smartphone producer may decide which features to include in a new line of devices (such as screen resolution, battery size, memory, and camera), or an automobile manufacturer may choose how to blend performance, practicality, size, and fuel efficiency. The way these characteristics are assembled might generate dependence and dispersion in consumers' valuations for the varieties. Therefore, we ask how different product designs affect the profit of the monopolist. For instance, would the monopolist prefer vertically differentiated products or horizontally differentiated ones? Should the monopolist produce mass products with low volatility in valuations, or niche products with high dispersion in valuations? To tackle these questions, we study how optimal market-clearing profits are affected by the distribution of buyers' valuations. This approach relies implicitly on Proposition 3 and is legitimate for markets in which the monopolist can frequently revise prices. We refrain from presenting an explicit model in which products are constructed by bundling characteristics, for the sake of brevity.

The first result fixes the marginal distribution of buyers' valuations for each variety and asks what correlation structure between varieties maximizes the profit of the monopolist. This approach is valid whenever production costs depend only on the marginal distribution of valuations and not on their correlation structure. Results exploit classical contributions on copulas (Sklar 1959) to establish that the profit-maximizing design requires full horizontal product differentiation. Horizontal differentiation increases profits, as a strong preference for one variety tends to be associated with a low desire for the other. This favors market segmentation and increases profits by minimizing the value of units that are never purchased. One may conjecture that vertical product differentiation may then be the profit-minimizing design. However, we establish that this is not the case in general, as independent products occasionally generate smaller profits than vertically differentiated ones. The other substantive contribution analyzes how volatility in valuations affects the seller's profits. In particular, it asks whether the monopolist prefers a distribution of values or its mean. In a single-variety setting, the answer is trivial, as the monopolist always dislikes variance (in one case the product is sold at the minimum value, while in the other at the average value). However in multi-variety settings, two opposing forces are at play. Variance increases both the information rents of buyers, thereby hurting the monopolist, and the value of the durable good (that is, the value of the preferred variety), thereby benefitting the monopolist. The analysis establishes that either of the two forces can dominate, and it provides sufficient conditions for both scenarios. This shows why low-volatility mass products need not be an optimal design when multiple varieties can be sold.

Classical Results on Copulas: Recall that F_i denotes the marginal cumulative distribution of the consumers' valuations for variety i. For the rest of the analysis, let F_i be continuous, and let its support be the compact set $V_i = [0, \overline{v}_i] \subseteq [0,1]$.[1] Before proceeding to the design problem, we briefly review the notion of copula and some of its properties. For a more detailed discussion of these topics, we refer to Nelsen 2006.

A function $C : [0,1]^2 \to [0,1]$ is said to be a *copula* if

(1) $C(k,0) = C(0,k) = 0$ for every $k \in [0,1]$;

(2) $C(k,1) = C(1,k) = k$ for every $k \in [0,1]$;

(3) $C(x) + C(y) - C(x_a, y_b) - C(y_a, x_b) \geq 0$ for every $x, y \in [0,1]^2$ such that $x \geq y$.

Let $\mathcal{C}$ denote the set of all copulas. Sklar's Theorem implies that for any joint distribution function F with continuous marginal distributions F_i for $i \in \{a, b\}$, there exists a unique copula C such that

$$F(v) = C(F_a(v_a), F_b(v_b)) \text{ for all } v \in V_a \times V_b.$$

A copula C can therefore be thought of as a sufficient statistic of the dependence structure between the two random variables. A commonly used copula exhibits no dependence between the random variables and is associated with the joint distribution $I(v) = F_a(v_a)F_b(v_b)$. Two other noteworthy copulas are known as the Frechét-Hoeffding (FH) bounds and are associated with the two joint distributions

$$\begin{aligned} K(v) &= \min\{F_a(v_a), F_b(v_b)\}; \\ L(v) &= \max\{F_a(v_a) + F_b(v_b) - 1, 0\}. \end{aligned}$$

Figure 8 depicts the support of these three copulas when marginals are uniform on $[0,1]$. In a classical contribution, Frechét and Hoeffding establish that the two FH copulas bound any other feasible copula C.

Proposition 3 *Any joint distribution F consistent with the two marginals, F_a and F_b, satisfies*

$$L(v) \leq F(v) \leq K(v) \quad \textit{for all} \quad v \in V_a \times V_b.$$

Random variables v_a and v_b are said to be *concordant* if $(v_i - v_i')(v_j - v_j') > 0$ for any two profiles v and v' in the support of the joint distribution. The random variables are instead

[1] We restrict attention to the environments in which the lowest values equal 0 to make results directly comparable to the one-variety no-gaps case where the Coase conjecture leads to 0 profits in the limit. The compact support assumption and the continuity assumption are convenient, but not essential.

said to be *discordant* if $(v_i - v_i')(v_j - v_j') < 0$ for any two profiles v and v' in the support. The FH bounds L and K characterize instances of extreme concordance and discordance, respectively, and they are thus associated with a Kendall's Tau of 1 and -1, respectively.[2] Informally, the FH upper bound identifies the most concordant copula, as large values of one random variable are associated with large values of the other, and small values of one with small values of the other. Conversely, the FH lower bound identifies the most discordant copula as large values of one random variable are associated with small values of the other, and vice versa. For instance, if the marginal distributions are either uniform or Gaussian, the FH bounds induce either perfect positive or perfect negative correlation between the two random variables.

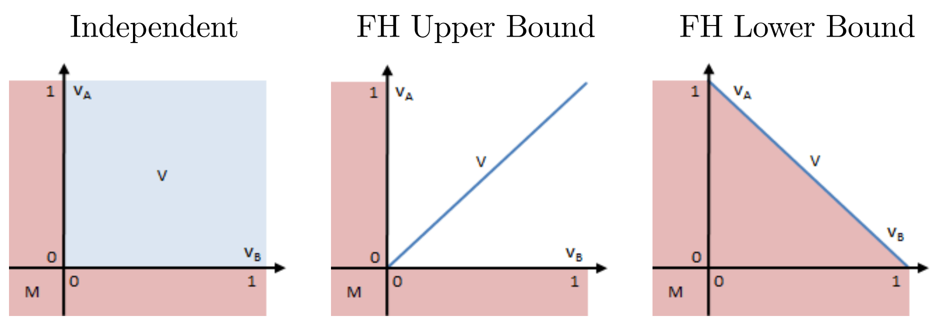

Figure 8: Three copulas for uniform marginals $F_a(x) = F_b(x) = x$.

The next result highlights a few well-known properties of the FH bounds that play an important role in our product design analysis. A subset V of $\mathbb{R}^2$ is *non-decreasing* if for any $v, v' \in V$, $v_a < v_a'$ implies $v_b \leq v_b'$. Similarly, a subset V of $\mathbb{R}^2$ is *non-increasing* if for any $v, v' \in V$, $v_a < v_a'$ implies $v_b \geq v_b'$.

Proposition 4 *The joint distribution F is equal to its:*
(1) FH upper bound K if and only if its support V is a non-decreasing subset of $V_a \times V_b$;
(2) FH lower bound L if and only if its support V is a non-increasing subset of $V_a \times V_b$.
Moreover, random variable v_a is almost surely a strictly monotonic function of v_b if and only if the copula coincides with one of the FH bounds.

The result implies that it is possible to construct monotone correspondences $v_a = k(v_b)$ and $v_a = l(v_b)$ that completely pin down the support of the FH bounds. The additional assumptions imposed on the marginal distributions further imply that these correspondences are functions such that $k'(x) \in (0, \infty)$ and $l'(x) \in (-\infty, 0)$ almost surely.

[2]For any two profiles v and v' in the support of the joint distribution, Kendall's Tau amounts to

$$\tau(F) = \Pr\left((v_i - v_i')(v_j - v_j') > 0\right) - \Pr\left((v_i - v_i')(v_j - v_j') < 0\right).$$

Product Design Results: Our product design exercises begin by analyzing how dependence (or association) between the random variables v_a and v_b affects the optimal market-clearing profits. By the analysis carried out in Section 4, our conclusions also identify how dependence affects the limiting stationary equilibrium profits of the dynamic pricing game. The first result fixes the marginal distributions F_i (postulating that any costs incurred in generating value depends only such marginals) and asks which joint distribution compatible with such marginals maximizes optimal market-clearing profits. Formally, let V^F denote the support of a joint distribution F, and let $\bar{\pi}^F$ denote the optimal market-clearing profits associated with the joint distribution F. Specifically, we solve the following problem:

$$\max_{C \in \mathcal{C}} \bar{\pi}^F \quad \text{s.t.} \quad F = C(F_a, F_b).$$

The next result establishes that optimal market-clearing profits are maximized at the FH lower bound. Thus, the seller always benefits from selling varieties that are most discordant. Intuitively, selling discordant varieties maximizes profits as segmenting the market guarantees that no value is wasted on varieties that are never purchased.

Proposition 5 *For any joint distribution F consistent with the two marginals, F_a and F_b,*

$$\bar{\pi}^F \leq \bar{\pi}^L.$$

The result implies that independent varieties are suboptimal since horizontal product differentiation allows for more effective market segmentation. As the monopolist loses its ability to intertemporally discriminate buyers (when the time between offers vanishes), statically screening consumers by selling different varieties is the only option available to extract any surplus. But such a task is most profitable when the market is segmented and values are discordant.

The other product design contribution looks at how variance in valuations affects the profitability of the monopolist. When a single product is for sale, the monopolist always prefers to minimize variance, as the durable good necessarily trades at the lowest valuation in the support. However, when multiple varieties can be sold, a trade-off emerges, since variance increases both buyers' information rents (which hurts profits) and total surplus (which benefits profits) as the maximal value grows. Thus, we ask whether the monopolist prefers a distribution F to the distribution $\hat{F}$ in which all buyers value products at the mean of F (that is, $\hat{F}$ is a degenerate distribution with unit measure at $E^F(v)$). The result establishes that the answer to this question depends on the details of the measure F when multiple varieties are for sale.

Remark 8 *Minimizing variance does not necessarily increase profits.*

The next examples prove this observation and show why variance is most beneficial when values are discordant and the support of the distribution is a concave map.

Product Differentiation Examples: Given that discordance increases profits by Proposition 5, one may conjecture that a lower bound on optimal market-clearing profits may then be pinned down by the other FH bound. This would be the case if separating high value consumers for the two varieties were most difficult with concordance. But, such a conclusion does not hold in general, as we show with the next example.

Consider two products with uniform marginal distributions on $V_a = [0,1]$ and $V_b = [0,x]$, respectively, for some $x \in (0,1]$. Three compatible designs are: (1) perfect concordance, when the joint distribution of consumers' preferences is K and the support is

$$V^K = \{v \in V_a \times V_b | v_b = xv_a\};$$

(2) independence, when the joint distribution of consumers' preferences is I and the support is $V(I) = V_a \times V_b$; (3) perfect discordance, when the joint distribution of consumers' preferences is L and the support is

$$V^L = \{v \in V_a \times V_b | v_b = x(1 - v_a)\}.$$

The first scenario corresponds to classical models of vertical price differentiation. The set of market-clearing prices, M^K, coincides with the set of price profiles in which at least one of the two prices is non-positive. When varieties are independent, the set of market-clearing prices does not change (that is, $M^I = M^K$). In both these scenarios, the optimal market-clearing profits are found by setting the price of the worse variety to zero (that is, $\bar{p}_b = 0$), while choosing the other price so as to maximize profits. When varieties are discordant, any price in the support of the distribution V^L clears the market, and optimal market-clearing prices belong to the support. The solutions for the three cases respectively satisfy:

Case	(1)	(2)	(2)	(3)
Subcase	$x \in [0,1]$	$x \in [0,2/3]$	$x \in [2/3,1]$	$x \in [0,1]$
$\bar{p}_a$	$(1-x)/2$	$(2-x)/4$	$1/3$	$1/2$
$\bar{p}_b$	0	0	0	$x/2$
$\bar{d}_a$	$1/2$	$(2-x)/4$	$2/(9x)$	$1/2$
$\bar{\pi}$	$(1-x)/4$	$(2-x)^2/16$	$2/(27x)$	$(1+x)/4$

where $\bar{d}_a$ denotes the demand for variety a at the optimal market-clearing prices for the proposed design. As argued in Proposition 5, the monopolist always prefers to choose a design that induces perfect discordance, as $\bar{\pi}^L \geq \max\{\bar{\pi}^I, \bar{\pi}^K\}$. Moreover, optimal market-clearing profits increase in x when varieties are discordant, but they decrease in x otherwise (see Figure 9). With vertically differentiated products, optimal market-clearing profits decrease to 0 when products become less differentiated (that is, when $x \to 1$), as expected from Proposition 1. Surprisingly however, perfectly concordant varieties raise more revenue than independently distributed ones when x is small. This follows because when x is small, the monopolist sells the expensive product to a larger measure of buyers with concordance than with independence at any market-clearing price.

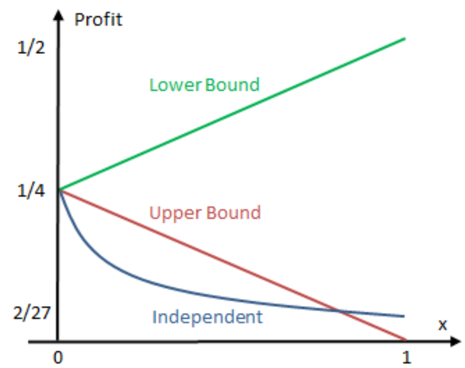

Figure 9: Optimal market-clearing profits for the three copulas as a function of x.

The second example illustrates why reducing variance in valuations can have ambiguous effects on optimal market-clearing profits. The insight provides a novel rationale for a monopolist to sell products with volatile valuations (thus establishing why low-volatility mass products are not necessarily a profit-maximizing design). First, consider the discordant distribution

$$L(v) = \max\{v_a^2 + v_b^2 - 1, 0\}.$$

The support of L is given by the decreasing set $V^L = \{v \mid v_a^2 + v_b^2 = 1\}$. The degenerate distribution $\hat{L} = [E^L(v)]$ has support $V^{\hat{L}} = \{(2/3, 2/3)\}$. Optimal market-clearing profits associated with L can be found by solving

$$\bar{\pi}^L = \arg\max_p p_a(1 - p_a^2) + p_b(1 - p_b^2) \text{ s.t. } p_a^2 + p_b^2 = 1.$$

The monopolist optimally clears the market by selling all units at a price of $1/\sqrt{2}$. In this scenario, the seller benefits from the variance, as

$$\bar{\pi}^L = 1/\sqrt{2} > 2/3 = \bar{\pi}^{\hat{L}}.$$

More volatile niche products are optimal. When the measure of high-value buyers is large, the minimal valuation for the preferred variety exceeds the average valuation, and thus niche products are preferred.

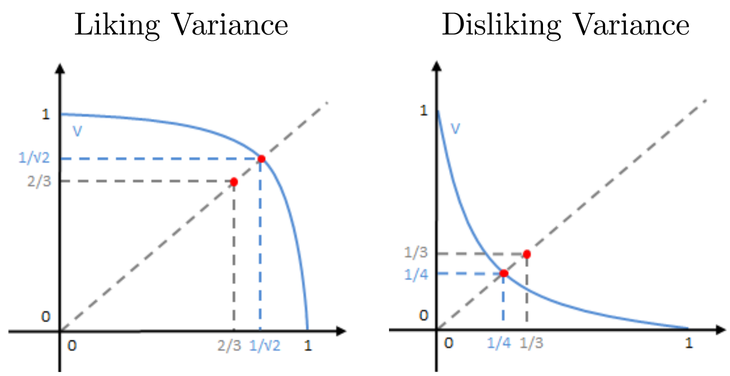

Figure 10: Effects of volatility on optimal market-clearing prices in two markets.

Next, consider the discordant distribution

$$L(v) = \max\{v_a^{1/2} + v_b^{1/2} - 1, 0\}.$$

The support of L is given by the decreasing set $V^L = \left\{v \mid v_a^{1/2} + v_b^{1/2} = 1\right\}$. The degenerate distribution $\hat{L} = \left[E^L(v)\right]$ has support $V^{\hat{L}} = \{(1/3, 1/3)\}$. The optimal market-clearing profits associated with L can be found by solving

$$\bar{\pi}^L = \arg\max_p p_a(1 - p_a^{1/2}) + p_b(1 - p_b^{1/2}) \;\text{ s.t. }\; p_a^{1/2} + p_b^{1/2} = 1.$$

The monopolist optimally clears the market by selling all units at a price of $1/4$. The monopolist is now hurt by the variance, as

$$\bar{\pi}^L = 1/4 < 1/3 = \bar{\pi}^{\hat{L}}.$$

Less volatile mass products are optimal. When the measure of low-value buyers is large, the average valuation exceeds the minimal valuation for the preferred variety, and thus mass products are preferred. Figure 10 depicts the measures considered and optimal market-clearing prices for the last two examples.

One may postulate that the monopolist prefers variance whenever varieties are perfectly discordant and the support is a concave map. However, this is not true in general. For instance, when $L(v) = \max\{v_a^2 + v_b - 1, 0\}$, the monopolist dislikes variance despite the support being concave.

2 Mixing Example

Consider a market in which $\delta = 3/4$, and in which the support of valuations is

$$V = (1,1) \cup \left\{ v \in [0,1]^2 \mid v_j = (1-v_i)/3 \text{ for any } v_i \in [1/4,1] \ \& \text{ any } i \in \{a,b\} \right\}.$$

The dark blue region in Figure 11 depicts this support. As regularity is mainly needed for existence, we consider an atomic example failing regularity for the sake of tractability. Of course, similar conclusions would hold for regular markets in which the support is the convex hull of V (the light blue shaded region in Figure 11) and in which almost all of the measure is near V.

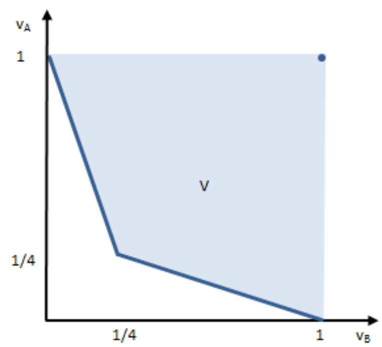

Figure 11: The support of the measure considered in the example and its convex hull.

Consider the following joint distribution on support V:

$$F(v) = \begin{cases} 1 & \text{if } \ v = (1,1) \\ (6v_j + 6v_i - 3)/10 & \text{if } \ v_i \geq 1/4 \ \& \ v_j \in [1/4, 1) \\ (18v_j + 6v_i - 6)/10 & \text{if } \ v_i \geq 1/4 \ \& \ v_j \in [1-3v_i, 1/4] \\ 0 & \text{if } \ \text{otherwise} \end{cases}.$$

Intuitively, such a distribution puts 1/10 of its measure on the atom at $(1,1)$, while 9/10 of the measure is uniformly distributed on the rest of V. If so, the marginal distribution for each variety amounts to

$$F_i(v_i) = \begin{cases} 9v_i/5 & \text{if } \ v_i \in [0, 1/4) \\ (6v_i + 3)/10 & \text{if } \ v_i \in [1/4, 1) \\ 1 & \text{if } \ v_i = 1 \end{cases}.$$

For convenience, define the following function which identifies the larger component of the support V:

$$g(x) = \max\{1/3 - x/3, 1 - 3x\} \ \text{ for } x \in [0,1].$$

Optimal Pricing: First consider the benchmark setting in which the monopolist can commit ex-ante to a constant price profile. If so, for any $p \notin M$ such that $p_i \geq p_j$, the seller's profits would simply amount to

$$\Pi(p) = p_i(9/10 - F_i(p_i)) + p_j(1 - F_j(p_j)),$$

and the associated optimal commitment prices and profits would be equal to

$$p_a^* = p_b^* = 13/24 \;\; \& \;\; \Pi^* = 169/480 = 0.352.$$

Optimal Clearing: Then consider the alternative benchmark in which the monopolist must set a price profile that clears the market instantaneously. Clearly, the seller would never set prices in the interior of the market-clearing price set. Thus, if it sold variety i at price $p_i \geq 1/4$, it would sell variety j at price $p_j = g(p_i)$, and for any such a price profile p, profits would amount to

$$\Pi(p_i, g(p_i)) = p_i(9/10 - F_i(p_i)) + g(p_i)(1/10 + F_i(p_i)).$$

The associated optimal market-clearing prices and profits would then be equal to

$$\bar{p}_i = 5/12, \;\; \bar{p}_j = 7/36 \;\; \& \;\; \bar{\Pi}^0 = 49/180 = 0.272.$$

Constructing Mixed Strategy Equilibria: We look for equilibria in which the market clears in two periods, both varieties are sold at a common price in the first period, and the seller randomizes with equal probability between two symmetric market-clearing price profiles in the second period. Throughout the example, we denote by i the variety that is sold at a higher price in the final period and by j the other variety.

We begin by constructing the equilibrium path of this symmetric mixed strategy equilibrium. To nest all cases, consider subgames in which a fraction $1 - \alpha$ of the measure at $(1, 1)$ has already purchased in the first period. Consider the equilibrium path active buyer set

$$A = \left[V \cap [0, x]^2\right] \cup (1, 1)$$

for some $x \in [1/4, 1)$. In subgame A, the payoff to the monopolist at any market-clearing price $p^0 \in M \backslash \bar{M}$ amounts to

$$\Pi(p^0|A) = p_i^0(F_i(x) - F_i(p_i^0)) + g(p_i^0)(\alpha/10 + F_i(p_i^0) - F_i(g(x))).$$

The subgame possesses two optimal market-clearing prices, which amount to

$$\bar{p}^0(x) = (\bar{p}_i^0(x), \bar{p}_j^0(x)) = ((9 - \alpha + 12x)/48, (39 + \alpha - 12x)/144) \quad \text{for} \quad i \in \{a, b\}.$$

We look for fixed points of the game in which: (1) the seller sells both varieties at a common price p in the initial period; (2) the atom at $(1,1)$ purchases in the initial period; and (3) the boundary of the active player set is determined by buyers who are indifferent between buying the same variety today and tomorrow. Fixed point conditions in general require that the boundary of the active player set be pinned down by the incentive constraint

$$\max\{v_a, v_b\} - p = \delta \max\left\{\max\{v_a, v_b\} - \frac{\bar{p}_a^0(x) + \bar{p}_b^0(x)}{2}, \frac{v_a + v_b}{2} - \min\{\bar{p}_a^0(x), \bar{p}_b^0(x)\}\right\}.$$

As the boundary of the active player set x is determined by buyers who are indifferent between buying a variety today and tomorrow,

$$x - p = \delta\left(x - \left(\bar{p}_a^0(x) + \bar{p}_b^0(x)\right)/2\right) \quad \Rightarrow \quad x(p) = (64p - 11)/20.$$

From this, the subgame prices can be pinned down as a function of the initial price quoted by the seller:

$$p^0(p) = \bar{p}^0(x(p)) = ((16p + 1)/20, (19 - 16p)/60).$$

For the atom to be willing to purchase in the initial period, it must be that $1 - p \geq \delta\left(1 - p_j^0(p)\right)$, which in turn requires that $p < 13/32$. Finally, to have such a fixed point, the marginal buyer must satisfy

$$x(p) - \left(p_i^0(p) + p_j^0(p)\right)/2 \geq (x(p) + g(x(p)))/2 - p_j^0(p),$$

which holds for $p \geq 1/4$. Therefore, such a fixed point can exist and be consistent with equilibrium behavior provided that $p < 13/32$.

The previous conjectures on the structure of the equilibrium imply that the objective function of the monopolist at the initial stage when setting the same price p for the two varieties simply amounts to

$$\Pi(p, p) = p(19/10 - 2F_i(x(p))) + \delta\Pi(\bar{p}^0(p)|x(p)).$$

Maximizing this objective function with respect to the initial price p identifies the optimal opening as $p^1 = 311/864$. Since $p^1 < 13/32$, the monopolist's behavior in the initial stage is consistent with buyers with value $(1,1)$ purchasing in the first period. At such a price, the present value of profits amounts to

$$\bar{\Pi}^1 = 2521/8640 = 0.29178.$$

This profit exceeds the optimal market-clearing profit. Thus, the seller does not benefit from clearing the market in the opening round. In the conjectured equilibrium, the monopolist initially sells both varieties at price $p^1 = 311/864$, then randomizes with equal probability between two market-clearing profiles $\{(5/12, 7/36), (7/36, 5/12)\}$ in period 2.

Asymmetric Deviations and Consistent Beliefs: Next we construct strategies and beliefs for asymmetric subgames in which the market is expected to clear in 2 periods. As before, to nest all cases, let α denote the fraction of the measure at $(1,1)$ purchasing in the second period. Consider asymmetric subgames in which the active buyer set satisfies

$$A = [V \cap [0, x_a] \times [0, x_b]] \cup (1,1),$$

for some vector $x \notin M$.[3] Without loss of generality, further suppose that $x_i \geq x_j$. In subgame A, the payoff to the monopolist at a market-clearing price $p^0 \in M \backslash \bar{M}$ amounts to

$$\Pi(p^0|A) = p_i^0(F_i(x_i) - F_i(p_i^0)) + g(p_i^0)(\alpha/10 + F_i(p_i^0) - F_i(g(x_j))).$$

In such a subgame, the optimal market-clearing prices satisfy

$$\bar{p}^0(x) = (\bar{p}_i^0(x), \bar{p}_j^0(x)) = \begin{cases} \left(\frac{9-\alpha+18x_i-6x_j}{48}, \frac{39+\alpha-18x_i+6x_j}{144}\right) & \text{if} \quad x_j \geq \frac{1}{4} \\ \left(\frac{8-\alpha+2x_i-18x_j}{16}, \frac{3\alpha-8-6x_i+54x_j}{16}\right) & \text{if} \quad x_j \leq \frac{1}{4} \quad \& \quad x_i + 7x_j \geq 2 + \frac{\alpha}{18} \\ (1-3x_j, x_j) & \text{if} \quad x_j \leq \frac{1}{4} \quad \& \quad x_i + 7x_j \leq 2 + \frac{\alpha}{18} \end{cases}.$$

Moreover, such prices are unique whenever $x_i > x_j$.

We look for fixed points of the game in which: (1) in the initial period, the seller posts a price profile p such that $p_i \geq p_j$; (2) the seller does not randomize in the final stage; (3) the atom at $(1,1)$ purchases in the initial period; and (4) the boundary of the active player set is determined by buyers who are indifferent between buying the same variety today and tomorrow.

Restriction (4) requires that the boundary of the active player set be pinned down by the two incentive constraints

$$x_a - p_a = \delta\left(x_a - \bar{p}_a^0(x)\right) \quad \text{and} \quad x_b - p_b = \delta\left(x_b - \bar{p}_b^0(x)\right),$$

[3] If $x \in M$, then A must have measure zero. Thus, we neglect this scenario.

and it further requires that

$$x_a - \bar{p}^0_a(x) \geq g(x_a) - \bar{p}^0_b(x) \text{ and } x_b - \bar{p}^0_b(x) \geq g(x_b) - \bar{p}^0_a(x).$$

As before, these conditions identify marginal buyers given the current prices, $x(p)$, and therefore they identify subgame prices as a function of current prices:

$$p^0(p) = \bar{p}^0(x(p)) = \left(\left(5 + 24p_i - 8p_j\right)/36, \left(31 + 8p_j - 24p_i\right)/108 \right).$$

As restrictions (3) and (1) require that the atom purchase in the first period and that $p_i \geq p_j$, equilibrium further implies that $1 - p_j \geq \delta\left(1 - \min\{p^0_a(p), p^0_b(p)\}\right)$.

Given the conjectures made about the fixed point, we look for equilibria in which $p_i \geq p_i$ and $x_i(p) \geq x_j(p)$. The objective function of the monopolist at the initial stage when setting prices for the two varieties amounts to

$$\Pi(p) = p_i(9/10 - F_i(x_i(p))) + p_j((10 - a)/10 - F_j(x_j(p))) + \delta\Pi(\bar{p}^0(p)|A(p)).$$

Solving this problem, we find that the opening prices set by the seller satisfy

$$p^1 = (p^1_i, p^1_j) = (171/416, 125/416) = (0.41106, 0.30048),$$

and consequently that marginal buyers satisfy

$$x(p^1) = \left(x_i(p^1), x_j(p^1)\right) = (63/104, 57/104) = (0.60577, 0.54808).$$

The monopolist's behavior in the initial stage is consistent with buyers with value $(1, 1)$ purchasing in the initial period, as

$$1 - p^1_j = 291/416 > 51/104 = \delta(1 - p^0_j).$$

Moreover, marginal buyers are not tempted by the other variety as

$$x_i - \bar{p}^0_i(x) = 27/104 \geq -9/104 = g(x_i) - \bar{p}^0_j(x),$$
$$x_j - \bar{p}^0_j(x) = 103/312 \geq -61/312 = g(x_j) - \bar{p}^0_i(x).$$

Thus, all fixed point conditions hold at such prices, and profits amounts to

$$\bar{\Pi}^1 = 479/1664 = 0.28786.$$

Profits exceed the optimal market-clearing profits, but they are smaller than the profits attainable when clearing the market symmetrically by randomizing in the final period.

Many Periods: If the seller were allowed a third period to clear the market, the most profitable strategy consistent with clearing the market in exactly three periods would require: selling both varieties at a common price in the first period, $p^2 = (0.38033, 0.38033)$; randomizing with equal probability between two price profiles in second period,

$$p^1 \in \{(0.32663, 0.25041), (0.25041, 0.32663)\};$$

setting one of two market-clearing price profiles in the third period contingent on the outcome of the randomization in the previous period:

$$p^0 = \begin{cases} (0.30097, 0.23300) & \text{if} \quad p^1 = (0.32663, 0.25041) \\ (0.23300, 0.30097) & \text{if} \quad p^1 = (0.25041, 0.32663) \end{cases}.$$

Mixing would take place in the second period, as the seller always prefers to clear the market asymmetrically once the atom has purchased,[4] and because the atom necessarily purchases in the first period in this example. Alternative strategies in which a measure of buyers purchases goods in any one of the three periods are necessarily less profitable.

When following such a strategy, the present value of profits would amount to

$$\bar{\Pi}^2 = 0.29057$$

and would be smaller than the profits the seller was making by clearing the market symmetrically in two periods. Intuitively, in this example, discounting losses would dominate price discrimination gains because buyers are sufficiently patient. If the seller had more than three periods in which to sell the product, a similar logic would hold, and profits would further decline under any strategy in which a positive measure of buyers purchases products in more than three periods.

MPE Beliefs: In any stationary equilibrium after any price history, the market clears in at most two periods. In the notation of the proof of Proposition 2, $X^t = \emptyset$ for any $t > 2$. The seller never benefits by clearing the market in more than two periods if buyers' beliefs are consistent and $\delta = 3/4$. Moreover, given these beliefs, clearing the market in more than two periods would be even less profitable since buyers would be more inclined to wait for a discount if the market was expected to clear sooner.

[4] In any subgame in which the atom has purchased, the measure of active buyers is discordant, and clearing the market asymmetrically maximizes the seller's profit.

In the desired equilibrium, the seller stochastically clears the market in two periods. In such an equilibrium, whenever $p_a = p_b = p$ and $p \in X^1$, buyers believe that the prices in the following period will coincide with the prices they would expect if the market had to clear symmetrically in exactly two periods. Thus, buyers expect one of the following two profiles with equal probability:

$$p^0(p) = (p_i^0(p), p_j^0(p)) = ((16p+1)/20, (19-16p)/60).$$

Whenever $p_i > p_j$ and $p \in X^1$, buyers believe that the prices in the following period will coincide with the prices they would expect if the market had to clear asymmetrically in exactly two periods:

$$p^0(p) = (p_i^0(p), p_j^0(p)) = ((5+24p_i-8p_j)/36, (31+8p_j-24p_i)/108).$$

For brevity's sake, we omit the derivation of beliefs associated with prices in X^2, while beliefs associated with prices in X^0 are trivial. By construction, these beliefs are consistent with the seller's strategy, and the seller's strategy maximizes profits given these beliefs.

3 Omitted Proofs

Proof Remark 1. (1) Consider the price $p^g = (\underline{v}_g, \underline{v}_g)$. Such a price belongs to M, as for all $v \in V$,

$$\max_{i\in\{a,b\}}\{v_i - p_i^g\} \geq \max_{i\in\{a,b\}} v_i - \underline{v}_g \geq 0.$$

By setting price p^g, the monopolist obviously achieves a profit of $\underline{v}_g$. Thus, when $\underline{v}_a < \underline{v}_b$, optimal market-clearing profits weakly exceed $\underline{v}_b$ and strictly exceed $\underline{v}_a$.

(2) If varieties are unranked, further consider the market-clearing price profile $p^\varepsilon = (\underline{v}_b + \varepsilon, \underline{v}_b)$ for some small number $\varepsilon > 0$. By definition of static demand, $\lim_{\varepsilon\to 0} d_i(p^\varepsilon) \geq \mathcal{F}(v_i > v_j)$ for any $i \in \{a,b\}$. As products are unranked, $\mathcal{F}(v_i > v_j) > 0$. But if so, optimal market-clearing profits must strictly exceed $\underline{v}_b$ since $d_a(p^\varepsilon) > 0$ for ε sufficiently small.

(3) If $\underline{v}_a = \underline{v}_b = \underline{v}$, again consider the market-clearing price profile $p^\varepsilon = (\underline{v}+\varepsilon, \underline{v})$. Again, it must be that $\lim_{\varepsilon\to 0} d_i(p^\varepsilon) \geq \mathcal{F}(v_i > v_j)$ for any $i \in \{a,b\}$. If at p^ε some consumers were to purchase variety a, optimal market-clearing profits would strictly exceed $\underline{v}$. If instead optimal market-clearing profits were equal to $\underline{v}$ for any $\varepsilon > 0$, then $\lim_{\varepsilon\to 0} d_a(p^\varepsilon) = \mathcal{F}(v_a > v_b) = 0$, and thus $v_a \leq v_b$ for any $v \in V$. A symmetric argument would then establish that $v_b \leq v_a$ for any $v \in V$. So, optimal market-clearing profits could be equal to $\underline{v}$ only if $v_a = v_b$ for any $v \in V$. Similarly, if varieties were identical, any market-clearing price would set one of the

two prices no higher than $\underline{v}$, as $(\underline{v}, \underline{v}) \in V$.

(4) But if so, optimal market-clearing profits would amount $\underline{v}$, as all players would purchase the cheapest variety. Thus, profits equal 0 if and only if varieties are identical and $\underline{v} = 0$.

(5) If $\underline{v}_a = \underline{v}_b = \underline{v}$ and $(\underline{v}, \underline{v}) \in V$, then $\underline{v}_g = \underline{v}$. If varieties are not identical, it must be that $\mathcal{F}(v_i > v_j) > 0$ for some $i \in \{a, b\}$. If so, the price profile $(p_i, p_j) = (\underline{v} + \varepsilon, \underline{v}) \in M$ would raise more profits than $\underline{v}_g$ for ε sufficiently small. ■

Proof Remark 2. To prove the first part note that by the proof of Remark 1 part (5), the optimal market-clearing price must equal $(p_i, p_j) = (\underline{v} + \varepsilon, \underline{v})$ for some $\varepsilon > 0$ and some $i \in \{a, b\}$. Since V is connected, $v \in V$, and $(\underline{v}, \underline{v}) \in V$, for any $\kappa \in (0, \varepsilon)$ there exists a $v^\kappa \in V$ such that

$$v_i^\kappa - v_j^\kappa = \kappa.$$

Thus, the price cannot be efficient if $\varepsilon > 0$, as any buyer v^κ purchases variety j despite preferring variety i.

To prove the second part, suppose $\underline{v}_a \leq \underline{v}_b$. When varieties are unranked and V is connected, efficiency of a price $(p_a, p_b) \in M$ requires $p_a = p_b$. Hence, the most profitable efficient market-clearing price must satisfy $p_a = p_b = \underline{v}_b$, because V is a Cartesian product. But, such a price cannot be optimal by the proof of Remark 1 part (2). ■

Proof Remark 3. Assume $\underline{v}_a^+ \geq c_a$. Consider setting price $p^\varepsilon = (c_a, c_b + \varepsilon)$ for some $\varepsilon > 0$. Such a price belongs to M, as for all $v \in V^+$,

$$\max_{i \in \{a,b\}} \{v_i - p_i\} \geq v_a - p_a = v_a - c_a \geq 0.$$

When varieties are unranked, $\mathcal{F}(v_a - c_a > v_b - c_b) > 0$. But if so,

$$\lim_{\varepsilon \to 0} d_b(p^\varepsilon) = \mathcal{F}(v_a - c_a > v_b - c_b) > 0.$$

Hence, optimal market-clearing profits are strictly positive for ε sufficiently small. ■

Proof Remark 4. The proof is essentially identical to that of Lemma 2 in the main text. Let $\bar{M}^+$ denote the "interior" of the market-clearing price set M^+. We show that, in any PBE, all consumers accept any price in $\bar{M}^+$ at any information set. Suppose this were not the case. Select any equilibrium, and let P denote the set of prices that are accepted by all buyers in any possible subgame. By contradiction, suppose that $\bar{M}^+$ is not contained in P. As before, $p \in P$ whenever $\min_{i \in \{a,b\}} p_i < -1$. To show the latter, observe that Lemma 1 is unaffected by costs, and that its proof, as before, implies that buyers' value functions at any buyer-history $\hat{h} \in \hat{H}$ are non-decreasing in v and have a modulus of continuity of less than

1, since for $v' \geq v$,

$$U(\hat{h}, v') - U(\hat{h}, v) \leq \max_{i\in\{a,b\}} \{v'_i - v_i\}.$$

Thus, in any PBE we have that $U(\hat{h}, v) \leq 1$ for all $v \in V$, and that all buyers strictly prefer to purchase a variety when $\min_{i\in\{a,b\}} p_i < -1$, as

$$\max_{i\in\{a,b\}} \{v_i - p_i\} > 1 > \delta U(\hat{h}, v) \text{ for all } \hat{h} \in \hat{H}. \tag{1}$$

If so, for any $\varepsilon > 0$ there is a price $\hat{p} \in \bar{M}^+ \backslash P \neq \emptyset$ such that

$$p_i \leq \hat{p}_i - \varepsilon \;\&\; p_j \leq \hat{p}_j \text{ for some } i \text{ implies } p \in P. \tag{2}$$

To find such a price $\hat{p}$, let $\tilde{p}_a = \inf_{q\in\bar{M}^+\backslash P} q_a$, and for some $\eta \in (0, \varepsilon)$ let

$$\tilde{p}_b = \inf_{q\in\bar{M}^+\backslash P} q_b \text{ s.t. } q_a \leq \tilde{p}_a + \eta,$$

where $\min_{i\in\{a,b\}} \tilde{p}_i \geq -1$ by (1). Then set $\hat{p}$ to be any price in $\bar{M}^+\backslash P$ such that $\hat{p} \leq \tilde{p}+(\eta,\eta)$. Such a price exists by definition of $\tilde{p}$ for any sufficiently small η. Moreover, (2) holds because: (a) $p \in P$ whenever $p_a \leq \hat{p}_a - \varepsilon < \tilde{p}_a$ by definition of $\tilde{p}_a$; and (b) $p \in P$ for any $p_a \leq \hat{p}_a$ when $p_b \leq \hat{p}_b - \varepsilon$ by definition of $\tilde{p}_b$.

But, when ε is sufficiently small,

$$\max_{i\in\{a,b\}} \{v_i - \hat{p}_i\} > \delta \max_{i\in\{a,b\}} \{v_i - \hat{p}_i + \varepsilon\} \quad \Leftrightarrow \quad \varepsilon < \frac{1-\delta}{\delta} \max_{i\in\{a,b\}} \{v_i - \hat{p}_i\}.$$

If so, all consumers would accept $\hat{p}$ at any buyer-history $(\hat{p}, h) \in \hat{H}$. If a type was to reject an offer, they could agree no sooner than tomorrow, and the most they could expect any one price to drop is ε as any further drop would lead to acceptance by all buyers. Thus, for all $v \in V$, the continuation value satisfies

$$\max_{i\in\{a,b\}} \{v_i - \hat{p}_i + \varepsilon\} \geq U((\hat{p}, h), v).$$

But this in turn implies that

$$\max_{i\in\{a,b\}} \{v_i - \hat{p}_i\} > \delta U((\hat{p}, h), v) \text{ for any } h \in H.$$

As $\hat{p} \notin P$, the latter contradicts the definition of P and consequently establishes the result. Since every consumer buys when prices belong to $\bar{M}^+$, the seller can secure a payoff arbitrarily close to the optimal market-clearing profits $\bar{\pi}(A) > 0$ (where $A = A(h)$ denotes the active player set associated with history h) by choosing a price in $\bar{M}^+$.

The proof of the second part of the remark essentially coincides with the proof of Proposition 3 and is omitted for sake of brevity. ■

Proof Remark 5. First, we introduce some notation. In this setting, all buyers are active in every period. However, the measure of values evolves over time. For any profile of values $v \in V$, denote the *residual value* of an owner of variety i by

$$v^i = (v_i^i, v_j^i) = (0, v_j - v_i).$$

Let $\mathcal{R}_i(h^t) \in \mathcal{P}^*([0,1]^2)$ denote the measure of residual values of consumers purchasing variety $i \in \{a, b\}$ at any history $h^t \in H$. Then, define the measure of values at history h^t as

$$\mathcal{F}(E|h^t) = \mathcal{F}(E|h^{t-1}) + \textstyle\sum_{i \in \{a,b\}} \left[\mathcal{R}_i(E|h^t) - \mathcal{D}_i(E|h^t)\right] \quad \text{for any} \quad E \in \Omega([0,1]^2).$$

Let $V(h^t)$ be the support of $\mathcal{F}(h^t)$. Define the set of buyers for whom gains from trade are positive at history h^t as

$$A^*(h^t) = \{v \in V(h^t) \mid \max_{i \in \{a,b\}}\{v_i - c_i\} > 0\}.$$

Refer to buyers in $A^*(h^t)$ as *essentially active*. The proof then follows along the same steps of Remark 3. We show that in any PBE, at any history $h \in H$, the set $A^*(p, h)$ must be empty if $p \in \bar{M}^*$, where $\bar{M}^*$ denotes the "interior" of the market-clearing price set M^*. Suppose this were not the case. Select any equilibrium, and let P denote those prices p such that $A^*(p, h) = \emptyset$ after any history h. By contradiction, suppose that $\bar{M}^*$ is not contained in P.

First, we show that $p \in P$ whenever $p \in \bar{M}^*$ and $\min_{i \in \{a,b\}} p_i < -1$. As before, buyers' value functions are non-decreasing in v and have a modulus of continuity of less than 1 at any buyer-history $h \in \hat{H}$, since an argument equivalent to Lemma 1 implies that for all $v' \geq v$,

$$U(h, v') - U(h, v) \leq \max_{i \in \{a,b\}} \{v'_i - v_i\}.$$

Thus, in any PBE we have that $U(h, v) \leq 1$ for all $v \in V$. But then, all buyers must purchase a variety when $\min_{i \in \{a,b\}} p_i < -1$, as

$$\max_{i \in \{a,b\}} \left\{v_i - p_i + \delta U(\hat{h}, v^i)\right\} > 1 > \delta U(\hat{h}, v) \quad \text{for all} \quad \hat{h} = (p, h) \in \hat{H}, \tag{3}$$

where $U(\hat{h}, v^i) \geq 0$ denotes the continuation value of buyer v once the value transitions to v^i. But if so, as $p \in \bar{M}^*$, for every player purchasing i we have that $c_j \geq v_j - v_i = v_j^i$, and similarly for every player purchasing j we have that $c_i \geq v_i - v_i = v_i^j$. Thus, $A^*(\hat{h}) = \emptyset$ and $p \in P$.

By the previous argument, for any $\varepsilon > 0$ there is a price $\hat{p} \in \bar{M}^+ \backslash P \neq \emptyset$ such that

$$p_i \leq \hat{p}_i - \varepsilon \ \& \ p_j \leq \hat{p}_j \text{ for some } i \text{ implies } p \in P. \tag{4}$$

To find such a price $\hat{p}$, let $\tilde{p}_a = \inf_{q \in \bar{M}^+ \backslash P} q_a$, and for some $\eta \in (0, \varepsilon)$ let

$$\tilde{p}_b = \inf\nolimits_{q \in \bar{M}^+ \backslash P} q_b \ \text{ s.t. } \ q_a \leq \tilde{p}_a + \eta,$$

where $\min_{i \in \{a,b\}} \tilde{p}_i \geq -1$ by (3). Then set $\hat{p}$ to be any price in $\bar{M}^+ \backslash P$ such that $\hat{p} \leq \tilde{p} + (\eta, \eta)$. Such a price exists by definition of $\tilde{p}$ for any sufficiently small η. Moreover, (4) holds because: (a) $\hat{p} \in \bar{M}^+$ given that $\tilde{p} \in \bar{M}^+$; (b) $p \in P$ whenever $p_a \leq \hat{p}_a - \varepsilon < \tilde{p}_a$ by definition of $\tilde{p}_a$; and (c) $p \in P$ for any $p_a \leq \hat{p}_a$ when $p_b \leq \hat{p}_b - \varepsilon$ by definition of $\tilde{p}_b$.

When ε is sufficiently small though,

$$\max\nolimits_{i \in \{a,b\}} \{v_i - \hat{p}_i\} > \delta \max\nolimits_{i \in \{a,b\}} \{v_i - \hat{p}_i + \varepsilon\} \quad \Leftrightarrow \quad \varepsilon < \frac{1-\delta}{\delta} \max\nolimits_{i \in \{a,b\}} \{v_i - \hat{p}_i\}.$$

But if ε is small, all consumers must purchase a variety at any buyer-history $(\hat{p}, h) = \hat{h} \in \hat{H}$. If a type was to reject an offer, they could agree no sooner than tomorrow, and the most they could expect any one price to drop is ε as any further drop would lead to acceptance by all buyers. But in that scenario, no more trade could be expected as $\hat{p} - (\varepsilon, \varepsilon) \in P$ and therefore no buyer would be essentially active. Thus, for all $v \in V(h)$ the continuation value must satisfy

$$\max\nolimits_{i \in \{a,b\}} \{v_i - \hat{p}_i + \varepsilon\} \geq U(\hat{h}, v).$$

In turn, this implies that for any $(\hat{p}, h) = \hat{h} \in \hat{H}$,

$$\max\nolimits_{i \in \{a,b\}} \left\{v_i - \hat{p}_i + \delta U(\hat{h}, v^i)\right\} > \delta \max\nolimits_{i \in \{a,b\}} \{v_i - \hat{p}_i + \varepsilon\} > \delta U(\hat{h}, v).$$

But then, all buyers must purchase a variety at $\hat{p}$. As $\hat{p} \in \bar{M}^*$, though, the latter implies that $A^*(\hat{h}) = \emptyset$, which contradicts $\hat{p} \notin P$. Thus, $\hat{p} \in P$ and $\bar{M}^* \subseteq P$. As every consumer buys when prices belong to $\bar{M}^*$ and no more trades take place after that, the seller can always secure a payoff arbitrarily close to the optimal market-clearing profits in M^* by charging a price in $\bar{M}^*$.

The proof of the second part of the remark again coincides with the proof of Proposition 3 and is again omitted for sake of brevity. ∎

Proof Remark 6. As the market is regular, $\mathcal{F}(A^\infty) = f\mathcal{L}(A^\infty)$ for some $f \in (\underline{f}, \overline{f})$ and hence $\mathcal{L}(A^\infty) = 0$. Thus, for any $v \in A^\infty$ and for any $v' \in V$ such that $v_a - v_b = v'_a - v'_b$, it must be that $v' \geq v$. If $v' < v$, then $v' \in A^\infty$ by part (3) of Lemma 1. But, this would lead to a contradiction when the market is regular since it would imply that $\mathcal{L}(A^\infty) > 0$, given that all values in a neighborhood of v' would also remain active as $v' \in V$.

Moreover, A^∞ must be an increasing set (that is, $v'_i > v_i \;\Rightarrow\; v'_j \geq v_j$ for any $v, v' \in A^\infty$). If this were not the case, buyer

$$v'' = (\max\{v_a, v'_a\}, \max\{v_b, v'_b\})$$

would also prefer not to buy any variety by part (1) of Lemma 1 (that is, $v'' \in V \;\Rightarrow\; v'' \in A^\infty$). If so, by part (3) of the Lemma 1, any buyer $\hat{v} \in V$ such that $\hat{v} < v''$ would strictly prefer not to buy (that is, $\hat{v} \in A^\infty$) provided that

$$\delta \max\nolimits_{i \in \{a,b\}} \{v''_i - \hat{v}_i\} < \min\nolimits_{i \in \{a,b\}} \{v''_i - \hat{v}_i\}. \tag{5}$$

When the market is regular, however, V is convex, and thus there exist buyers

$$v(\kappa) = \kappa v + (1 - \kappa) v' \in V \text{ for all } \kappa \in (0, 1).$$

If so, there exists a value $\hat{\kappa}$ such that $v''_a - v''_b = v_a(\hat{\kappa}) - v_b(\hat{\kappa})$ and consequently $v(\hat{\kappa}) \in A^\infty$. This, however, again leads to a contradiction, as $\mathcal{L}(A^\infty)$ would be strictly positive given that a positive measure of buyers would fulfill (5) if $v(\hat{\kappa}) \in V$.

At any history h, define the floor price of variety i at date t as $\underline{p}_i^t(h) = \min_{s \leq t} p_i^s$. As $\mathcal{F}(A^\infty) = 0$, it must be that for any equilibrium history h^∞ associated with A^∞, $\lim_{t\to\infty} \underline{p}^t(h^\infty) \in M$, or else the market could not possibly clear. Thus, $A^\infty \subseteq M \backslash \bar{M}$, as only buyers whose values are on the boundary of the market-clearing set M could remain active in the limit if $\lim_{t\to\infty} \underline{p}^t(h^\infty) \in M$, by Lemma 1 and because $\mathcal{F}(A^\infty) = 0$. However, $M \backslash \bar{M}$ is a decreasing set (that is, $p'_i > p_i \;\Rightarrow\; p'_j \leq p_j$ for any $p, p' \in M \backslash \bar{M}$), as $p > p'$ and $p \in M$ imply $p' \in \bar{M}$.

As A^∞ is both an increasing and a decreasing set, there exists an i such that $v, v' \in A^\infty$ implies $v_i = v'_i$. But if so, for all $\varepsilon > 0$, there must exist t sufficiently large such that $|v_i - v'_i| \leq \varepsilon$ for some i and for all $v, v' \in A^t$. ■

Proof Remark 7. All of the operators defined in the first part of proof of Proposition 1 are independent of s when z is sufficiently large. That is:

$$\Pi^s(A) = \Pi(A); \;\; B^s(A) = B(A); \;\; U^s(p, v) = U(p, v); \;\; \underline{A}^s(p, \rho | A) = \underline{A}(p, \rho | A).$$

Furthermore, the map $\Pi(A)$ is equicontinuous in A. To see this, consider any two active player sets $A' \subseteq A \subseteq V$. Let $p(A)$ denote any degenerate distribution in $B(A)$. It follows that

$$\Pi(A) - \Pi(A') \leq (\mathcal{F}(A) - \mathcal{F}(A')) \max\nolimits_i p_i(A) \leq \overline{f}(\mathcal{L}(A) - \mathcal{L}(A')),$$

where the first inequality follows because the seller could always set price $p(A)$ even when it believes that players in A' are active; and where the second inequality follows because the market is regular and $v \leq (1,1)$ for all $v \in V$. Also, the operator $\underline{A}(p, \rho|A)$ is equicontinuous in p and in ρ for all $A \in \mathcal{K}(V)$, since $U(\rho|v)$ is continuous in ρ by the proof of Lemma 1.

When $\underline{v}_g = 0$, consider a sequence of games where the measure of buyers $\{\mathcal{F}_n\}_{n=0}^{\infty}$ satisfies

$$f_n(v) = \begin{cases} f(v) & \text{if} \quad v \geq \underline{v}^n \\ 0 & \text{otherwise} \end{cases}$$

for a strictly positive decreasing sequence $\{\underline{v}^n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \underline{v}^n = \underline{v}$. As each of these games displays gaps ($\underline{v}_g^n \geq \max_i \underline{v}_i^n > 0$), there exists a stationary equilibrium $\{\sigma_n, \alpha_n\}$ in each of these games by the first part of the proof of Proposition 1.

Since the sequence of functions $\{\Pi_n, \underline{A}_n\}_{n=0}^{\infty}$ is equicontinuous, it has a uniformly convergent sub-sequence converging to continuous functions $\{\Pi, \underline{A}\}$. For notational ease, posit that the sequence actually converges to this limit. For any $A \in \mathcal{K}(V)$, label by $J(A, \Pi, \underline{A}, \sigma)$ the limit problem of the seller,

$$\max_{\rho \in \mathcal{P}\left([0,1]^2\right)} \int_{[0,1]^2} p d(p|A) + \delta \Pi(\underline{A}(p, \sigma(p)|A)) d\rho(p),$$

and by $J_n(A, \Pi_n, \underline{A}_n, \sigma_n)$ the same problem for the n^{th} element of the sequence of games. Since J_n is Lipschitz continuous in $(\Pi_n, \underline{A}_n, \sigma_n)$ in the uniform topology for all $A \in \mathcal{K}(V)$, J_n converges to J, and the limit points of σ_n must converge to an equilibrium of the limit game σ at any continuity point of σ by the theorem of the maximum. Thus, for all $p \in [0,1]^2$, we have that

$$\lim_{n\to\infty} \sigma_n(p) = \sigma(p) \quad \Rightarrow \quad \sigma(p) \in B(\underline{A}(p, \sigma(p)|A)).$$

Consequently, σ is an equilibrium of the limit game, and thus a WME exists even when $\underline{v}_g = 0$. ■

4 Additional Proofs

Proof Proposition 5. For clarity, let $d_i^F(p)$ denote the static demand for product i when prices are p and the measure of buyers is F. Recall that for a joint distribution F, the optimal market-clearing profits satisfy

$$\begin{aligned} \bar{\pi}^F &= d_a^F(\bar{p}^F)\bar{p}_a^F + (1 - d_a^F(\bar{p}^F))\bar{p}_b^F \\ &\geq d_a^F(p)p_a + (1 - d_a^F(p))p_b \ \text{ for any } p \in M(F), \end{aligned}$$

where $\bar{p}^F \in M^F$ denotes the optimal market-clearing price (which exists by regularity). We want to establish that $\bar{\pi}^F \leq \bar{\pi}^L$. To do so, it suffices to find prices $p \in M^L$ such that $p \geq \bar{p}^F$ and $d_a^F(\bar{p}^F) = d_a^L(p)$. For any market-clearing price $p \in M^F$ such that $p_a \geq p_b$, we have by regularity that

$$d_a^F(p) = \mathcal{F}^F \left(v_a - v_b \geq p_a - p_b\right).$$

Let V^L denote the support of the distribution L. Since V^L is a non-increasing set by Proposition 4, $v_a < p_a$ implies $v_b \geq p_b$ for any $v, p \in V^L$. Thus, $V^L \subseteq M^L$. Proposition 4 also implies that for any $p \in V^L$,

$$d_a^L(p) = \mathcal{F}^L \left(v_a - v_b \geq p_a - p_b\right) = \mathcal{F}^L \left(v_a - l(v_a) \geq p_a - l(p_a)\right) = \mathcal{F}^L \left(v_a \geq p_a\right) = 1 - F_a(p_a),$$

where the second equality holds since V^L is a non-increasing set, and the third holds since $l' \leq 0$. As marginal distributions are continuous by regularity, they admit an inverse. Thus, it is possible to find a price p_a satisfying

$$p_a = F_a^{-1}(1 - d_a^F(\bar{p}^F)).$$

To conclude, we establish that $p_a \geq \bar{p}_a^F$. By construction, it must be that

$$\mathcal{F}^F \left(v_a - v_b \geq \bar{p}_a^F - \bar{p}_b^F\right) = 1 - F_a(p_a).$$

However, since $\bar{p}^F \in M^F$, it is also the case that

$$\mathcal{F}^F \left(v_a - v_b \geq \bar{p}_a^F - \bar{p}_b^F\right) \leq 1 - F_a(\bar{p}_a^F),$$

as $v_a - \bar{p}_a^F \geq v_b - \bar{p}_b^F$ implies $v_a - \bar{p}_a^F \geq 0$ for any $\bar{p}^F \in M^F$. The last two observations together imply that $F_a(p_a) \geq F_a(\bar{p}_a^F)$, and thus $p_a \geq \bar{p}_a^F$. A similar argument applies for variety b. ■

References

[1] Sklar, A.,1959, "Fonctions de Répartition à n Dimensions et leurs Marges", *Publications Institut Statistique de l'Université de Paris*, 8, 229-231.