Learning from Others' Outcomes: Online Appendix
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8.1 Properties of F

Non-empty: If (x_0, x_1) satisfies (4) (resp., the opposite of (4)) then $p(k; x_0, x_1)$ is increasing (resp., decreasing) in k. In either case, $p(k; x_0, x_1)$ is monotone in k, so there exists $k^* \in \{0, \ldots, K\}$ such that either $p(k^* - 1; x_0, x_1) \leq p^* \leq p(k^*; x_0, x_1)$ or $p(k^*; x_0, x_1) \geq p^* \geq p(k^* + 1; x_0, x_1)$, where in both cases the first (last) inequality is vacuous if $k^* = 0$ ($k^* = K$). With this value of k^*, let $s = 1$ if $p(k^*; x_0, x_1) \geq p^*$ and let $s = 0$ if $p(k^*; x_0, x_1) < p^*$. Next, with these values of k^* and s, let (x_0', x_1') be computed as in (6) or (8).

   Compact-valued: Boundedness is trivial. For closedness, fix (x_0, x_1) and a sequence $(x_0'^n, x_1'^n) \to (x_0', x_1')$, with $(x_0'^n, x_1'^n) \in F(x_0, x_1)$, and let $(k^{*,n}, s^n)$ be arbitrarily chosen corresponding values of k^* and s. Taking a convergent subsequence $(k^{*,n}, s^n) \to (k^{*,\infty}, s^\infty)$, continuity of ϕ_θ implies that, with $k^* = k^{*,\infty}$ and $s = s^\infty$, (x_0', x_1') satisfies the conditions for inclusion in $F(x_0, x_1)$.

   Convex-valued: Recall that there is at most one value of $k^* \in \{0, \ldots, K\}$ such that $p(k^*; x_0, x_1) = p^*$. So, if there are distinct elements of $F(x_0, x_1)$, (x_0', x_1') and (x_0'', x_1''), it must be that (x_0', x_1') and (x_0'', x_1'') are computed as in (6) or (8) with distinct values $s', s'' \in [0, 1]$. But then, for all $\beta \in [0, 1]$, letting $s = \beta s' + (1 - \beta) s''$, it follows that $(\beta x_0' + (1 - \beta) x_0'', \beta x_1' + (1 - \beta) x_1'') \in F(x_0, x_1)$.

   Upper hemi-continuous: Fix sequences $(x_0^n, x_1^n) \to (x_0, x_1)$ and $(x_0'^n, x_1'^n) \to (x_0', x_1')$, with $(x_0'^n, x_1'^n) \in F(x_0^n, x_1^n)$, and let $(k^{*,n}, s^n)$ be arbitrarily chosen corresponding values of k^* and s. Taking a convergent subsequence $(k^{*,n}, s^n) \to (k^{*,\infty}, s^\infty)$, continuity of ϕ_θ implies that, with $k^* = k^{*,\infty}$ and $s = s^\infty$, (x_0', x_1') satisfies the conditions for inclusion in $F(x_0, x_1)$.
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8.2 Equilibrium Uniqueness when $K = 1$

Proposition 11 *When $K = 1$, there is a unique equilibrium, and it is aligned. In this equilibrium, players adopt with probability 1 after observing a success and adopt with probability less than 1 after observing a failure.*

Proof. Fix an equilibrium, and suppose players adopt with probability s_1 after observing a success and adopt with probability s_0 after observing a failure. Then, for $\theta = 0, 1$,

$$x_\theta = [\chi + x_\theta(\pi_\theta - \chi)] s_1 + [1 - \chi - x_\theta(\pi_\theta - \chi)] s_0,$$

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or

$$x_\theta = \frac{s_0 + \chi\,(s_1 - s_0)}{1 - (\pi_\theta - \chi)\,(s_1 - s_0)}. \tag{14}$$

Suppose toward a contradiction that $s_0 = 1$. As $s_0 = s_1 = 1$ would lead to $x_0 = x_1 = 1$, which is not an equilibrium by (2), this implies that $s_0 > s_1$. But $s_0 > s_1$ implies that $x_0 > x_1$, which contradicts Lemma 3. Hence, $s_0 < 1$.

Now, $s_0 < 1$ implies that $p\,(0; x_0, x_1) \le p^*$. As $p > p^*$ and p is a convex combination of $p\,(0; x_0, x_1)$ and $p\,(1; x_0, x_1)$ (by the law of total probability), this implies that $p\,(1; x_0, x_1) > p^*$. Hence, $s_1 = 1$.

Next, using (14) and $s_1 = 1$,

$$p\,(0; x_0, x_1) = \left[1 + \frac{1 - \pi_0}{1 - \pi_1} \frac{1 - (\pi_1 - \chi)\,(1 - s_0)}{1 - (\pi_0 - \chi)\,(1 - s_0)} \frac{1 - p}{p}\right]^{-1}.$$

Therefore, $(s_0 = 0, s_1 = 1)$ corresponds to an equilibrium if and only if

$$\frac{1 - \pi_0}{1 - \pi_1} \frac{1 - \pi_1 + \chi}{1 - \pi_0 + \chi} \ge \frac{p}{1 - p} \frac{1 - p^*}{p^*}. \tag{15}$$

On the other hand, $(s_0 = s, s_1 = 1)$ with $s > 0$ corresponds to an equilibrium if and only if

$$\frac{1 - \pi_0}{1 - \pi_1} \frac{1 - (\pi_1 - \chi)\,(1 - s)}{1 - (\pi_0 - \chi)\,(1 - s)} = \frac{p}{1 - p} \frac{1 - p^*}{p^*}. \tag{16}$$

The left-hand side of (16) is increasing in s, and by (2) it exceeds the right-hand side when $s = 1$. Hence, by the intermediate value theorem, either there is a unique equilibrium given by $(s_0 = 0, s_1 = 1)$ and (14), or there exists a unique value $s > 0$ such that the unique equilibrium is given by $(s_0 = s, s_1 = 1)$, (16), and (14). ∎

8.3 Examples of Misaligned Equilibria

Example 1: An Unstable Misaligned Equilibrium

Let $K = 2$, $\chi = 1$, $\pi_0 = 0$, $\pi_1 = \frac{1}{3}$, $p = \frac{1}{2}$, and $c = -\frac{8}{9}$. I claim that the misaligned point $\left(x_0 = 0, x_1 = \frac{3}{4}\right)$, together with the strategy of adopting if and only if at least one observation is a failure, is an equilibrium. (This point is misaligned because the success rate is 1 in state 0 and $1 - x_1\,(1 - \pi_1) = \frac{1}{2}$ in state 1.)

This follows because $p^* = \frac{\chi + c - \pi_0}{\pi_1 - \pi_0} = \frac{1}{3}$, while the posterior probability that $\theta = 1$ after observing at least one failure is $1 > p^*$, and the posterior probability that $\theta = 1$ after

observing zero failures is

$$\left[1 + \frac{1-p}{p}\frac{1}{\left(\frac{1}{2}\right)^2}\right]^{-1} = \frac{1}{5} < p^*.$$

The stated strategy is therefore optimal. In addition, (x_0, x_1) is an stationary point because the probability of observing at least one failure is 0 in state 0 and $1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$ in state 1.

The equilibrium is however unstable, as the probability of observing at least one failure in state 0 when fraction x_0 adopts equals $1 - (1 - x_0)^2$, which is greater than x_0 for all $x_0 \in (0, 1)$.

Example 2: A Stable Misaligned Equilibrium (and Two Stable Aligned Equilibria)

Let $K = 3$, $\chi = \frac{9}{10}$, $\pi_0 = 0$, $\pi_1 = \frac{1}{10}$, $p = \frac{1}{2}$, and $c = -\frac{1701}{2000}$.[27] Under the strategy of adopting if and only if at least two observations are failures, the equation for x_θ to be a stationary point is

$$x_\theta = (1 - \chi + x_\theta (\chi - \pi_\theta))^3 + 3 (1 - \chi + x_\theta (\chi - \pi_\theta))^2 (\chi - x_\theta (\chi - \pi_\theta)).$$

Consider the point (x_0, x_1) given by taking the smallest solution to this cubic equation for $\theta = 0$ and the largest solution for $\theta = 1$: $(x_0, x_1) \approx (.07407, .9419)$. This point is misaligned because the success rate is $\chi - x_1 (\chi - \pi_1) \approx 0.1465$ in state 1 and $\chi - x_0 (\chi - \pi_0) \approx 0.8333$ in state 0. It is straightforward to check that this point is stable: for $\theta = 0, 1$, the above cubic equation has three roots, of which the middle one is unstable. Finally, to see that the proposed strategy is optimal, note that $p^* = \frac{\chi + c - \pi_0}{\pi_1 - \pi_0} = 0.495$, while the posterior probability that $\theta = 1$ after observing two failures is

$$\left[1 + \frac{1-p}{p}\frac{(1 - \chi + x_0 (\chi - \pi_0))^2 (\chi - x_0 (\chi - \pi_0))}{(1 - \chi + x_1 (\chi - \pi_1))^2 (\chi - x_1 (\chi - \pi_1))}\right]^{-1} \approx 0.8217 > p^*,$$

while the posterior probability that $\theta = 1$ after observing one failure is

$$\left[1 + \frac{1-p}{p}\frac{(1 - \chi + x_0 (\chi - \pi_0)) (\chi - x_0 (\chi - \pi_0))^2}{(1 - \chi + x_1 (\chi - \pi_1)) (\chi - x_1 (\chi - \pi_1))^2}\right]^{-1} \approx 0.1366 < p^*.$$

In fact, it is not hard to see that a stable misaligned equilibrium cannot exist when $K = 2$, so $K = 3$ is the minimum sample size for which a stable misaligned equilibrium can exist.

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[27] The explanation for this oddly precise choice of c is that, if $c = -\frac{17}{20}$, the analysis of the example would be exactly the same except that one would have $p^* = p$, which violates (1). As the only role of c in the model is to determine p^*, it suffices to let $c = -\frac{17}{20} - \varepsilon$ for any sufficiently small $\varepsilon > 0$.

The reason is that, when $K = 2$, the fraction of players observing at least k failures in state θ is at most quadratic in x_θ, so there is a unique stable misaligned stationary point (x_0, x_1). But, since failure is more likely for a given fraction of adopters in state 0, this unique stable point always has $x_0 > x_1$, so by Lemma 3 it cannot be an equilibrium.

This same example also admits two stable aligned equilibria. Thus, there can be multiple stable aligned equilibria, and they can coexist with a stable misaligned equilibrium.

Specifically, I claim that a point $(x_0', x_1') \approx (0.4681, 0.5)$, together with the strategy of adopting if and only if at least two successes are observed, is an equilibrium; and that so is a point $(x_0'', x_1'') \approx (0.6625, 0.7061)$, together with the strategy of adopting if and only if at least one success is observed. The intuition for this multiplicity is that, when the "bar" for adopting is raised from one observed success to two, this reduces the steady-state adoption rate, which makes failure less likely in both states (as $\chi > \pi_0, \pi_1$), and thus makes failure more informative. This in turn justifies the greater number of observed successes required for adoption.

For the formal construction, note that, under the strategy of adopting if and only if at least two successes are observed, the equation for x_θ to be a stationary point is

$$x_\theta = (\chi - x_\theta (\chi - \pi_\theta))^3 + 3 (\chi - x_\theta (\chi - \pi_\theta))^2 (1 - \chi + x_\theta (\chi - \pi_\theta)).$$

Let (x_0', x_1') be the unique solutions to this equation for $\theta = 0, 1$, given by $(x_0', x_1') \approx (0.4681, 0.5)$. Then the posterior probability that $\theta = 1$ after observing two successes is

$$\left[1 + \frac{1-p}{p} \frac{(1 - \chi + x_0' (\chi - \pi_0)) (\chi - x_0' (\chi - \pi_0))^2}{(1 - \chi + x_1' (\chi - \pi_1)) (\chi - x_1' (\chi - \pi_1))^2} \right]^{-1} \approx 0.5113 > p^*,$$

while the posterior probability that $\theta = 1$ after observing one success is

$$\left[1 + \frac{1-p}{p} \frac{(1 - \chi + x_0' (\chi - \pi_0))^2 (\chi - x_0' (\chi - \pi_0))}{(1 - \chi + x_1' (\chi - \pi_1))^2 (\chi - x_1' (\chi - \pi_1))} \right]^{-1} \approx 0.4900 < p^*.$$

So this is an equilibrium. It is also easily seen to be stable, as the curve $(\chi - x_\theta (\chi - \pi_\theta))^3 + 3 (\chi - x_\theta (\chi - \pi_\theta))^2 (1 - \chi + x_\theta (\chi - \pi_\theta))$ crosses x_θ from above, for $\theta = 0, 1$.

Similarly, under the strategy of adopting if and only if at least one successes is observed, the equation for x_θ to be a stationary point is given by

$$x_\theta = 1 - (1 - \chi + x_\theta (\chi - \pi_\theta))^3.$$

Let (x_0'', x_1'') be the unique solutions, given by $(x_0'', x_1'') \approx (0.6625, 0.7061)$. Then the posterior

probability that $\theta = 1$ after observing one successes is

$$\left[1 + \frac{1-p}{p}\frac{(1-\chi + x_0''(\chi - \pi_0))^2(\chi - x_0''(\chi - \pi_0))}{(1-\chi + x_1''(\chi - \pi_1))^2(\chi - x_1''(\chi - \pi_1))}\right]^{-1} \approx 0.5015 > p^*,$$

while the posterior probability that $\theta = 1$ after observing zero successes is

$$\left[1 + \frac{1-p}{p}\frac{(1-\chi + x_0''(\chi - \pi_0))^3}{(1-\chi + x_1''(\chi - \pi_1))^3}\right]^{-1} \approx 0.4655 < p^*.$$

So this is also an equilibrium, and it is also easily seen to be stable.

Finally if one considers this example with $K = 2$ rather than $K = 3$, one finds that there is a unique stable aligned equilibrium $(x_0, x_1) \approx (0.5955, 0.6327)$ (corresponding to the strategy of adopting if and only if at least one success if observed), and welfare in this steady state lies in between that in the two stable aligned steady states that arise when $K = 3$. This shows that welfare does not always unambiguously increase when players observe larger samples, even within the class of stable aligned equilibria.

8.4 Proof of Proposition 3

1. If $\chi = 0$, then (15) is violated (as $\frac{p}{1-p}\frac{1-p^*}{p^*} > 1$), so the unique equilibrium is given by (16) and (14). Solving for x_0, x_1, and s gives

$$
\begin{aligned}
x_0 &= \frac{(p - p^*)(1 - \pi_1)}{p^*(1 - p)(\pi_1 - \pi_0)}, \\
x_1 &= \frac{(p - p^*)(1 - \pi_0)}{p(1 - p^*)(\pi_1 - \pi_0)}, \quad \text{and} \\
s &= \frac{(p - p^*)(1 - \pi_0)(1 - \pi_1)}{p^*(1 - p)(1 - \pi_0)\pi_1 - p(1 - p^*)\pi_0(1 - \pi_1)}.
\end{aligned}
$$

Noting that $p^* \to \hat{p}$ as $\pi_1 \to 1$, it follows that $x_0 \to 0$ and $x_1 \to \frac{p - \hat{p}}{p(1 - \hat{p})}$ as $\pi_1 \to 1$.

2. If $\chi > 0$, then (15) holds when π_1 is close enough to 1. In this case, (14) gives $x_\theta = \frac{\chi}{1 - \pi_\theta + \chi}$ for $\theta = 0, 1$. Hence, $x_0 = \frac{\chi}{1 - \pi_0 + \chi}$ and $x_1 \to 1$ as $\pi_1 \to 1$. ∎

8.5 Proof of Proposition 4

1. Fix a sequence of parameters $(\pi_0^n, \pi_1^n) \to (\pi_0, \pi_1) = (0, 1)$ and fix a corresponding sequence of equilibria $(x_0^n, x_1^n, k^{*,n}, s^n) \to (x_0, x_1, k^*, s)$. Note that (x_0, x_1, k^*, s) must be an equilibrium. Suppose toward a contradiction that $x_1 < 1$. By Lemma 3, $x_1 \geq \frac{p - p^*}{1 - p^*}$, so

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$x_1 \pi_1 \in (0, 1)$. On the other hand, $x_0 \pi_0 = 0$. Therefore, $p(k; x_0, x_1) > p^*$ for all $k \geq 1$, and hence $k^* = 0$. The steady state equation then implies that

$$x_1 = 1 - (1 - x_1)^K (1 - s) \geq 1 - (1 - x_1)^2 = x_1 (2 - x_1).$$

But this implies that $x_1 = 1$, a contradiction.

To show that $x_0 = 0$, let $\hat{s}$ be the probability with which players adopt after observing K failures in the equilibrium (x_0, x_1, k^*, s). (Thus, $\hat{s} = 0$ if $k^* > 0$, and $\hat{s} = s$ if $k^* = 0$.) Then the steady state equation implies that $x_0 = \hat{s}$. Next, note that $p(0; x_0, 1) = p(0; 1, 1) < p^*$ (by $\pi_0 = 0$ and (2)). Hence, $\hat{s} = 0$.

2. Fix a sequence of parameters $(\pi_0^n, \pi_1^n) \rightarrow (\pi_0, \pi_1) = (0, 1)$ and a corresponding sequence of aligned equilibria $(x_0^n, x_1^n, k^{*,n}, s^n) \rightarrow (x_0, x_1, k^*, s)$. Note that (x_0, x_1, k^*, s) must be an aligned equilibrium. I claim that $x_0 > 0$. To see this, note that, in any aligned equilibrium $p(K; x_0, x_1) > p > p^*$, and therefore players adopt with probability 1 after observing K successes. Thus, if $x_0 = 0$ and $\chi = 1$, then in state 0 players would observe all successes with probability 1; and therefore x_0 would equal 1, a contradiction.

Next, as $x_0 > 0$, $\pi_0 = 0$, and $\chi + x_1 (\pi_1 - \chi) = 1$, $p(k; x_0, x_1) = 0$ for all $k < K$, so players adopt with probability 0 after observing even a single failure. On the other hand, I have shown that players adopt with probability 1 after observing all successes, so

$$x_\theta = (1 - x_\theta (1 - \pi_\theta))^K \text{ for } \theta = 0, 1.$$

As $\pi_0 = 0$ and $\pi_1 = 1$, this implies that $x_0 = (1 - x_0)^K$ and $x_1 = 1$.

The last part of the proposition follows as the solution to the equation $x_0 = (1 - x_0)^K$ converges to 0 as $K \rightarrow \infty$. ∎

8.6 Proof of Proposition 5

Given adoption rates (x_0, x_1), the posterior belief that $\theta = 1$ after observing failure equals

$$\left[1 + \frac{1 - p}{p} \frac{1 - \chi - x_0 (\pi_0 - \chi)}{1 - \chi - x_1 (\pi_1 - \chi)} \right]^{-1}.$$

This posterior equals p^* if and only if

$$\frac{1 - \chi - x_0 (\pi_0 - \chi)}{1 - \chi - x_1 (\pi_1 - \chi)} = \frac{p}{1 - p} \frac{1 - p^*}{p^*}.$$

This equation defines a line $\hat{L}$ in (x_0, x_1) space. Let H be half-space where the posterior exceeds p^* and let H^c be the half-space where the posterior is less than p^*; thus $\hat{L}$ marks the boundary between H and H^c. Recall from the proof of Proposition 11 that there are two possible cases: either the equilibrium is $\left(x_0 = \frac{\chi}{1-\pi_0+\chi}, x_1 = \frac{\chi}{1-\pi_1+\chi}\right)$ and this point lies in the half-space H^c, or the equilibrium lies on the line $\hat{L}$.

At any point $(x_0, x_1) \in H$, it follows that $\dot{x}_\theta = 1 - x_\theta$ for $\theta = 0, 1$, so the vector $(\dot{x}_0, \dot{x}_1)$ points from (x_0, x_1) toward the point $(1, 1)$. By (2), the point $(1, 1)$ lies in the complementary half-space H^c. Hence, if the initial point $(x_0(0), x_1(0))$ lies in H, the distance between $(x_0(t), x_1(t))$ and the line $\hat{L}$ is decreasing in t and reaches 0 in finite time.

Similarly, if $(x_0, x_1) \in H^c$, then $\dot{x}_\theta = \chi - x_\theta(1 - \pi_\theta + \chi)$ for $\theta = 0, 1$. Hence, $(x_0(t), x_1(t))$ converges monotonically toward the point $\left(\frac{\chi}{1+\chi-\pi_0}, \frac{\chi}{1+\chi-\pi_1}\right)$, so long as $(x_0(t), x_1(t))$ remains in H^c. Thus, if the equilibrium is $\left(\frac{\chi}{1+\chi-\pi_0}, \frac{\chi}{1+\chi-\pi_1}\right)$ then the population dynamic converges monotonically to the equilibrium starting from any point in H^c, and otherwise the population dynamic converges monotonically toward the point $\left(\frac{\chi}{1+\chi-\pi_0}, \frac{\chi}{1+\chi-\pi_1}\right)$ until it hits the line $\hat{L}$ (which again occurs in finite time).

Next, if $(x_0(t), x_1(t)) \in \hat{L}$ then $\dot{x}_\theta \geq \chi - x_\theta(1 - \pi_\theta + \chi)$ for $\theta = 0, 1$. Hence, if the equilibrium is $\left(\frac{\chi}{1+\chi-\pi_0}, \frac{\chi}{1+\chi-\pi_1}\right)$, then the population dynamic converges toward this point from any point in $\hat{L}$. Combining the observations made so far, it follows that when the equilibrium is $\left(\frac{\chi}{1+\chi-\pi_0}, \frac{\chi}{1+\chi-\pi_1}\right)$, it is globally attracting.

Finally, if $(x_0(t), x_1(t)) \in \hat{L}$ and $(x_0^*, x_1^*) \in \hat{L}$, then the population dynamic remains in $\hat{L}$ forever: this follows because, as I have shown, the gradient $(\dot{x}_0, \dot{x}_1)$ points toward $\hat{L}$ whenever $(x_0, x_1) \notin \hat{L}$. Next, for any point $(x_0, x_1) \in \hat{L}$, there is a unique mixing probability conditional on observing failure, $s((x_0, x_1))$, such that the gradient $(\dot{x}_0, \dot{x}_1)$ is parallel to $\hat{L}$, and in addition the mixing probability $s((x_0, x_1))$ is itself continuous in (x_0, x_1).[28] As the vector $(\dot{x}_0, \dot{x}_1)$ is continuous in (x_0, x_1) and the mixing probability s, it may therefore also be viewed as a continuous function of (x_0, x_1). Furthermore, as any stationary point in $\hat{L}$ is an equilibrium, (x_0^*, x_1^*) is the unique point in $\hat{L}$ such that $(\dot{x}_0, \dot{x}_1) = (0, 0)$. Hence, as $(\dot{x}_0, \dot{x}_1)$ is continuous in (x_0, x_1), it must be that $(\dot{x}_0, \dot{x}_1)$ points toward the steady state, and in addition $(\dot{x}_0(t), \dot{x}_1(t))$ can converge to 0 only if $(x_0(t), x_1(t))$ converges to (x_0^*, x_1^*). Therefore, $(x_0(t), x_1(t))$ must converge to (x_0^*, x_1^*) starting from any initial point in $\hat{L}$. As

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[28] To see this, note that if $s = 0$, then $(\dot{x}_0, \dot{x}_1) = (\chi - x_0(1 - \pi_0 + \chi), \chi - x_1(1 - \pi_1 + \chi))$, which points into H (when $\left(\frac{\chi}{1+\chi-\pi_0}, \frac{\chi}{1+\chi-\pi_1}\right) \in H$, or equivalently when the equilibrium is in $\hat{L}$), and if $s = 1$ then $(\dot{x}_0, \dot{x}_1) = (1 - x_0, 1 - x_1)$, which points into H^c. Denote these vectors by $(\dot{x}_0^0, \dot{x}_1^0)$ and $(\dot{x}_0^1, \dot{x}_1^1)$, and let $\dot{x}_\theta^s = (1 - s)\dot{x}_\theta^0 + s\dot{x}_\theta^1$ for $\theta = 0, 1$. By the intermediate value theorem, there is a unique mixing probability $s((x_0, x_1))$ such that $\dot{x}^{s((x_0, x_1))}$ is parallel to $\hat{L}$, and $s((x_0, x_1))$ is continuous in (x_0, x_1) because $(\dot{x}_0^0, \dot{x}_1^0)$ and $(\dot{x}_0^1, \dot{x}_1^1)$ are continuous in (x_0, x_1).

I have shown that $(x_0(t), x_1(t))$ reaches $\hat{L}$ in finite time starting from any initial point in $[0,1]^2$, it follows that (x_0^*, x_1^*) is globally attracting. ■

8.7   Proof of Proposition 6

As $(x_0(0), x_1(0))$ is aligned, Theorem 1 implies that $(x_0(t), x_1(t))$ is aligned for all t. Hence, players adopt with probability 1 after observing a success. On the other hand, a player's posterior after observing a failure at time t is given by

$$p(0; x_0(t), x_1(t)) = \left[1 + \frac{1-p}{p}\frac{1}{1-x_1(t)\pi_1}\right]^{-1}.$$

This posterior is less than p^* at time 0 by (2), and it remains less than p^* until $x_1(t)$ reaches the value

$$x_1^* = \frac{1}{\pi_1}\left(1 - \frac{1-p}{p}\frac{p^*}{1-p^*}\right) < 1.$$

(Note that this equation defines the line $\hat{L}$ introduced in the proof of Proposition 5.) Letting T be the first time when $x_1(t)$ reaches x_1^*, it follows that $\dot{x}_\theta(t) = -x_\theta(t)(1-\pi_\theta)$ for all $t < T$ and $\theta = 0, 1$. Combined with the initial condition $(x_0(0), x_1(0)) = (1,1)$, this gives $x_\theta(t) = \exp(-(1-\pi_\theta)t)$ for $\theta = 0, 1$.

Next, as shown in the proof of Proposition 5, $(x_0(t), x_1(t))$ remains on the line $\hat{L}$ for all $t > T$: that is, $x_1(t) = x_1^*$ for all $t > T$. It follows that $s(t) = s$ for all $t > T$, where s is given by $x_1^* = x_1^*\pi_1 + (1 - x_1^*\pi_1)s$, or

$$s = \frac{1-\pi_1}{\pi_1}\left(\frac{p}{1-p}\frac{1-p^*}{p^*} - 1\right).$$

In addition, for $t > T$, $\dot{x}_0(t) = s - x_0(t)$, so $x_0(t)$ converges monotonically to its steady-state value of s.

Finally, the time T satisfies

$$T = \frac{1}{1-\pi_1}\left[\log\pi_1 - \log\left(1 - \frac{1-p}{p}\frac{p^*}{1-p^*}\right)\right].$$

Hence,

$$x_0(T) = \exp(-T) = \left(\frac{1}{\pi_1}\left(1 - \frac{1-p}{p}\frac{p^*}{1-p^*}\right)\right)^{\frac{1}{1-\pi_1}}.$$

In particular, $x_0(T) < s$ if and only if

$$1 - \frac{1-p}{p}\frac{p^*}{1-p^*} < \pi_1 \left(\frac{1-\pi_1}{\pi_1}\right)^{1-\pi_1} \left(\frac{p}{1-p}\frac{1-p^*}{p^*} - 1\right)^{1-\pi_1}.$$

The right-hand side of this inequality goes to 1 as $\pi_1 \to 1$, so $x_0(T) < s$ whenever π_1 is close enough to 1. ∎

8.8 Proof of Proposition 7

Fix $\varepsilon \in (0, (\chi - \pi_{\max \Theta^*})/(1 + \chi - \pi_{\max \Theta^*}))$. Suppose an asymptotically efficient path exists. Then there exists $\bar{K} > 0$ such that if $K > \bar{K}$ then $(X_0^K(0), \ldots, X_n^K(0)) = (1, \ldots, 1)$ and $\lim_{t \to 0} X_\theta^K(t) < \varepsilon$ (resp., $> 1 - \varepsilon$) for all $\theta \le \theta^*$ (resp., $> \theta^*$). For such a K, the success rate at $t = 0$ conditional on the event $\theta \le \theta^*$ equals $(1/a)\sum_{\theta=0}^{\theta^*} p_\theta \pi_\theta$ and the success rate at $t = 0$ conditional on the event $\theta \in \Theta^*$ equals $(1/b)\sum_{\theta \in \Theta^*} p_\theta \pi_\theta$, which is larger. On the other hand, as $t \to \infty$ the success rate conditional on the event $\theta \le \theta^*$ converges to a number greater than $(1-\varepsilon)\chi$, while the success rate conditional on the event $\theta \in \Theta^*$ converges to a number less than $\varepsilon + (1-\varepsilon)\pi_{\max \Theta^*}$, which is smaller. Hence, there must exist a time t^* such that (i) at $t = t^*$, the success rate conditional on the event $\theta \le \theta^*$ equals the success rate conditional on the event $\theta \in \Theta^*$, and (ii) for all $t > t^*$, the success rate conditional on the event $\theta \le \theta^*$ is larger than the success rate conditional on the event $\theta \in \Theta^*$.

Now, at $t = t^*$, after observing any sample a player's relative assessment of the probability of the events $\theta \le \theta^*$ and $\theta \in \Theta^*$ equals the prior probability $a/(a+b)$. Thus, (13) implies that (after observing any sample at $t = t^*$) action 1 is optimal conditional on the event $\theta \in \{1, \ldots, \theta^*\} \cup \Theta^*$. In addition, action 1 is optimal at any state $\theta \notin (\{1, \ldots, \theta^*\} \cup \Theta^*)$. Hence, $\dot{X}_\theta(t^*) = 1 - X_\theta(t^*)$ for all θ. Therefore,

$$
\begin{aligned}
\frac{1}{a}\sum_{\theta=0}^{\theta^*} p_\theta \dot{X}_\theta(t^*)(\pi_\theta - \chi) &= \left[\frac{1}{a}\sum_{\theta=0}^{\theta^*} p_\theta(\pi_\theta - \chi)\right] - \left[\frac{1}{a}\sum_{\theta=0}^{\theta^*} p_\theta X_\theta(t^*)(\pi_\theta - \chi)\right] \\
&= \left[\frac{1}{a}\sum_{\theta=0}^{\theta^*} p_\theta(\pi_\theta - \chi)\right] - \left[\frac{1}{b}\sum_{\theta \in \Theta^*} p_\theta X_\theta(t^*)(\pi_\theta - \chi)\right] \\
&< \left[\frac{1}{b}\sum_{\theta \in \Theta^*} p_\theta(\pi_\theta - \chi)\right] - \left[\frac{1}{b}\sum_{\theta \in \Theta^*} p_\theta X_\theta(t^*)(\pi_\theta - \chi)\right] \\
&= \frac{1}{b}\sum_{\theta \in \Theta^*} p_\theta \dot{X}_\theta(t^*)(\pi_\theta - \chi).
\end{aligned}
$$

But this implies that, just after time t^*, the success rate conditional on the event $\theta \le \theta^*$ is

smaller than the success rate conditional on the event $\theta \in \Theta^*$, a contradiction. ∎

8.9 Proof of Proposition 8

Suppose $\pi_\theta < \chi$ for some innovation-optimal state θ, and suppose a simple asymptotically efficient path exists. As in the proof of Proposition 7, for large enough K, at $t = 0$ the success rate in each innovation-optimal state is greater than the success rate in each status-quo optimal state, and the situation is reversed for large enough t. Hence, there must exist a time t^* at which the success rates in a status-quo optimal state and an innovation-optimal state cross for the first time: that is, a time t^* such that (i) $X_\theta (t^*) (\pi_\theta - \chi) \leq X_{\theta'} (t^*) (\pi_{\theta'} - \chi)$ for all $\theta \leq \theta^* < \theta'$, and (ii) there exists $\varepsilon > 0$ and $\theta \leq \theta^* < \theta'$ such that $X_\theta (t) (\pi_\theta - \chi) > X_{\theta'} (t) (\pi_{\theta'} - \chi)$ for all $t \in (t^*, t^* + \varepsilon)$.

The proof is completed by considering separately the case where $X_\theta (t^*) (\pi_\theta - \chi) = X_{\theta'} (t^*) (\pi_{\theta'} - \chi)$ for all θ, θ' and the case where $X_\theta (t^*) (\pi_\theta - \chi) < X_{\theta'} (t^*) (\pi_{\theta'} - \chi)$ for some θ, θ', and deriving a contradiction in each.

In the first case, the success rate is equal in all states at time t^*, and hence $\dot{X}_\theta (t^*) = 1 - X_\theta (t^*)$ for all θ. But, as in the proofs of Theorem 1 and Proposition 7, this implies that there cannot be a pair of states $\theta < \theta'$ with $X_\theta (t^*) (\pi_\theta - \chi) = X_{\theta'} (t^*) (\pi_{\theta'} - \chi)$ and $X_\theta (t) (\pi_\theta - \chi) > X_{\theta'} (t) (\pi_{\theta'} - \chi)$ for all $t \in (t^*, t^* + \varepsilon)$, a contradiction.

In the second case, there are three states with either (i) $\theta_0 < \theta \leq \theta^* < \theta'$ and

$$ X_{\theta_0} (t^*) (\pi_{\theta_0} - \chi) < X_\theta (t^*) (\pi_\theta - \chi) = X_{\theta'} (t^*) (\pi_{\theta'} - \chi) $$

or (ii) $\theta \leq \theta^* < \theta' < \theta_0$ and

$$ X_\theta (t^*) (\pi_\theta - \chi) = X_{\theta'} (t^*) (\pi_{\theta'} - \chi) < X_{\theta_0} (t^*) (\pi_{\theta_0} - \chi) . $$

Consider the first case (the second is symmetric). Then, as $X_\theta (t)$ is continuous for all θ, for sufficiently small $\varepsilon > 0$,

$$ X_{\theta_0} (t^* + \varepsilon) (\pi_{\theta_0} - \chi) < X_{\theta'} (t^* + \varepsilon) (\pi_{\theta'} - \chi) < X_\theta (t^* + \varepsilon) (\pi_\theta - \chi) . $$

But then the path is not simple. ∎

8.10 Proof of Proposition 9

By assumption, $X_0 (0) (\pi_0 - \chi) \leq X_1 (0) (\pi_1 - \chi)$. As X_0 and X_1 are continuous, if there exists a time t' with $X_0 (t') (\pi_0 - \chi) > X_1 (t') (\pi_1 - \chi)$, then there must exist another time

t where $X_0(t)(\pi_0 - \chi) = X_1(t)(\pi_1 - \chi)$ but it is not the case that $\dot{X}_0(t)(\pi_0 - \chi) < \dot{X}_1(t)(\pi_1 - \chi)$. By Theorem 2, a misaligned equilibrium cannot exist in the outcome-improving case, so $\pi_1 < \chi$. Hence, $X_0(t)(\pi_0 - \chi) = X_1(t)(\pi_1 - \chi)$ implies $X_0(t) < X_1(t)$. But, by definition of $\dot{X}_\theta(t)$, if $X_0(t)(\pi_0 - \chi) = X_1(t)(\pi_1 - \chi)$ and $X_0(t) < X_1(t)$, then $\dot{X}_0(t) > \dot{X}_1(t)$, and hence $\dot{X}_0(t)(\pi_0 - \chi) < \dot{X}_1(t)(\pi_1 - \chi)$. So there can be no such time t'. ∎

8.11   Proof of Proposition 10

It follows immediately from the definition of $\dot{X}_\theta(t)$ and stability from above that $\dot{X}_\theta(t)$ is bounded below 0 for all t such that $X_\theta(t)$ is bounded above x_θ^*. It is also straightforward to argue by contradiction that $X_\theta(t)$ can never cross x_θ^*, completing the proof. ∎