Why Has Urban Inequality Increased?

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Online Mathematical Appendix

1 Setup for Cost Minimization

Here, v denotes the cost of capital, and w_j^u and w_j^s denote the unskilled and skilled workers' wages, respectively. Factors of production are capital K_j, skilled labor S_j and unskilled labor U_j. j indexes location. We also have factor augmenting technical change that shows up as multipliers on factors; these multipliers do not differ by location. The underlying problem is:

$$\min_{K_j,S_j,U_j} \quad vK_j + w_j^s S_j + w_j^u U_j$$

$$\text{subject to} \quad Y_j = A_j \left[cA_s^\sigma D_j^{\sigma\mu_s} S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right)^{\frac{\sigma}{\rho}}\right]^{\frac{1}{\sigma}}$$

The parameters that may change over time are $A_s, A_k, A_u, \mu_s, \mu_k$ and μ_u.

1.1 Equation 2

The first order conditions from cost minimization are given by the following equations, where θ_j denotes the Lagrange multiplier:

$$\frac{\partial L}{\partial K_j} = v - \theta_j \frac{\partial Y_j}{\partial K_j} = 0 \tag{1}$$

$$\frac{\partial L}{\partial U_j} = w_j^u - \theta_j \frac{\partial Y_j}{\partial U_j} = 0 \tag{2}$$

$$\frac{\partial L}{\partial S_j} = w_j^s - \theta_j \frac{\partial Y_j}{\partial S_j} = 0 \tag{3}$$

Following are the marginal products of the three factors:

$$\begin{aligned}
\frac{\partial Y_j}{\partial K_j} &= A_j\left[cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma+(1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho+(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{\frac{\sigma}{\rho}}\right]^{\frac{1-\sigma}{\sigma}} \\
&\times(1-c)\left[\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho+(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}}\times\lambda A_k^\rho D_j^{\rho\mu_k}K_j^{\rho-1} \\
\frac{\partial Y_j}{\partial U_j} &= A_j\left[cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma+(1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho+(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{\frac{\sigma}{\rho}}\right]^{\frac{1-\sigma}{\sigma}} \\
&\times(1-c)\left[\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho+(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}}\times(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^{\rho-1} \\
\frac{\partial Y_j}{\partial S_j} &= A_j\left[cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma+(1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho+(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{\frac{\sigma}{\rho}}\right]^{\frac{1-\sigma}{\sigma}}\times cA_s^\sigma D_j^{\sigma\mu_s}S_j^{\sigma-1}
\end{aligned}$$

Dividing the first order condition for S_j given by equation 3 by the one for U_j in equation 2:

$$\frac{w_j^s}{w_j^u}=\frac{cD_j^{\sigma\mu_s}A_s^\sigma S_j^{\sigma-1}}{(1-c)\left[\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho+(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}}(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^{\rho-1}}$$

This equation gives a relationship between the relative wages of skilled and unskilled workers and the inputs and agglomeration parameters of the model. Multiplying and dividing the right hand side by $\left[A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}}$:

$$\begin{aligned}
\frac{w_j^s}{w_j^u} &= \frac{cA_s^\sigma D_j^{\sigma\mu_s}S_j^{\sigma-1}}{(1-c)\left[\left(A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{-1}\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho+(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)\right]^{\frac{\sigma-\rho}{\rho}}\left[A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}}(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^{\rho-1}} \\
&\quad \text{calculating the product in the first square bracket, grouping terms and simplifying:} \\
&= \frac{c}{1-c}\frac{1}{1-\lambda}\left[\lambda\left(\frac{A_k}{A_u}\right)^\rho D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^\rho+(1-\lambda)\right]^{\frac{-(\sigma-\rho)}{\rho}}\left(\frac{A_s}{A_u}\right)^\sigma D_j^{\sigma(\mu_s-\mu_u)}\left(\frac{S_j}{U_j}\right)^{\sigma-1}
\end{aligned}$$

Taking logarithms on both sides of the previous expression:

$$\begin{aligned}
\ln\left(\frac{w_j^s}{w_j^u}\right) &= \ln\left(\frac{c}{1-c}\right)-\ln(1-\lambda)-\frac{\sigma-\rho}{\rho}\ln\left(\lambda\left(\frac{A_k}{A_u}\right)^\rho D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^\rho+(1-\lambda)\right) \\
&+\sigma\ln\left(\frac{A_s}{A_u}\right)+\sigma(\mu_s-\mu_u)\ln D_j+(\sigma-1)\ln\left(\frac{S_j}{U_j}\right) \qquad (4)
\end{aligned}$$

In order to now take the total derivative of equation 4, consider each component:

$$\frac{\partial\ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial\ln S_j}d\ln S_j = (\sigma-1)d\ln S_j$$

$$\frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \ln U_j} d\ln U_j = \frac{\sigma-\rho}{\rho}\frac{\rho\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho} d\ln U_j}{\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho}+(1-\lambda)} - (\sigma-1)d\ln U_j$$

expanding and multiplying and dividing the first term by $A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}$

$$= (\sigma-\rho)\frac{A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\lambda A_k^{\rho} A_u^{-\rho} D_j^{\rho\mu_k} D_j^{-\rho\mu_u} K_j^{\rho} U_j^{-\rho}}{A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\left(\lambda A_k^{\rho} A_u^{-\rho} D_j^{\rho\mu_k} D_j^{-\rho\mu_u} K_j^{\rho} U_j^{-\rho}+(1-\lambda)\right)} d\ln U_j - (\sigma-1)d\ln U_j$$

$$= (\sigma-\rho)\frac{\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho}}{\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho}+(1-\lambda)A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}} d\ln U_j - (\sigma-1)d\ln U_j$$

$$= (\sigma-\rho)\omega_j^c d\ln U_j - (\sigma-1)d\ln U_j$$

$$\frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \ln K_j} d\ln K_j = -\left(\frac{\sigma-\rho}{\rho}\right)\frac{\rho\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho} d\ln K_j}{\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho}+(1-\lambda)}$$

multiplying and dividing by $A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}$

$$= -(\sigma-\rho)\omega_j^c d\ln K_j$$

$$\frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \mu_s} d\mu_s = \sigma d\mu_s \ln D_j$$

$$\frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \mu_u} d\mu_u = -\left(\frac{\sigma-\rho}{\rho}\right)\frac{\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho}(-\rho)d\mu_u \ln D_j}{\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho}+(1-\lambda)} - \sigma d\mu_u \ln D_j$$

multiplying and dividing by $A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}$

$$= (\sigma-\rho)\omega_j^c d\mu_u \ln D_j - \sigma d\mu_u \ln D_j$$

$$\frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \mu_k} d\mu_k = -\left(\frac{\sigma-\rho}{\rho}\right)\frac{\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho}\rho d\mu_k \ln D_j}{\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho}+(1-\lambda)}$$

multiplying and dividing by $A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}$

$$= -(\sigma-\rho)\omega_j^c d\mu_k \ln D_j$$

$$\frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \ln A_s} d\ln A_s = \sigma d\ln A_s$$

$$\frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \ln A_u} d\ln A_u = \frac{\sigma-\rho}{\rho}\frac{\rho\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho} d\ln A_u}{\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho}+(1-\lambda)} - \sigma d\ln A_u$$

multiplying and dividing the first term by $A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}$

$$= (\sigma-\rho)\omega_j^c d\ln A_u - \sigma d\ln A_u$$

$$\frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \ln A_k} d\ln A_k = -\left(\frac{\sigma-\rho}{\rho}\right)\frac{\rho\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho} d\ln A_k}{\lambda\left(\frac{A_k}{A_u}\right)^{\rho} D_j^{\rho(\mu_k-\mu_u)}\left(\frac{K_j}{U_j}\right)^{\rho}+(1-\lambda)}$$

multiplying and dividing by $A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}$

$$= -(\sigma-\rho)\omega_j^c d\ln A_k$$

where $\omega_j^c = \frac{\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho}}{\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho}+(1-\lambda)A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}}$ (analogously defined to that one in the previous section). And hence the total derivative of equation 4 is given by:

$$\begin{aligned}
d\ln\left(\frac{w_j^s}{w_j^u}\right) &= \frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial U_j} dU_j + \frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial S_j} dS_j + \frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial K_j} dK_j + \frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \mu_u} d\mu_u + \frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \mu_s} d\mu_s \\
&+ \frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \mu_k} d\mu_k + \frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \ln A_u} d\ln A_u + \frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \ln A_s} d\ln A_s + \frac{\partial \ln\left(\frac{w_j^s}{w_j^u}\right)}{\partial \ln A_k} d\ln A_k \\
&= (\sigma-\rho)\omega_j^c d\ln U_j - (\sigma-1)d\ln U_j + (\sigma-1)d\ln S_j - (\sigma-\rho)\omega_j^c d\ln K_j - \sigma d\mu_u \ln D_j \\
&+ (\sigma-\rho)\omega_j^c d\mu_u \ln D_j + \sigma d\mu_s \ln D_j - (\sigma-\rho)\omega_j^c d\mu_k \ln D_j - \sigma d\ln A_u \\
&+ (\sigma-\rho)\omega_j^c d\ln A_u + \sigma d\ln A_s - (\sigma-\rho)\omega_j^c d\ln A_k \\
&\text{grouping similar terms} \\
&= (\sigma-1)(d\ln S_j - d\ln U_j) + \sigma \ln D_j (d\mu_s - d\mu_u) - (\sigma-\rho)\omega_j^c (d\ln K_j - d\ln U_j) \\
&- (\sigma-\rho)\omega_j^c \ln D_j (d\mu_k - d\mu_u) + \sigma(d\ln A_s - d\ln A_u) - (\sigma-\rho)\omega_j^c (d\ln A_k - d\ln A_u) \\
&= \sigma d(\mu_s-\mu_u)\ln D_j + (\sigma-1)d\ln\left(\frac{S_j}{U_j}\right) - (\sigma-\rho)\omega_j^c d\ln\left(\frac{K_j}{U_j}\right) - (\sigma-\rho)\omega_j^c d(\mu_k-\mu_u)\ln D_j \\
&+ \sigma d\ln\left(\frac{A_s}{A_u}\right) - (\sigma-\rho)\omega_j^c d\ln\left(\frac{A_k}{A_u}\right)
\end{aligned}$$

which is Equation 2 in the text.

1.2 Equation 3

Starting with the first order condition from profit maximization for capital close to equation 1, take the log on both sides:

$$\begin{aligned}\ln v &= \ln A_j + \frac{1-\sigma}{\sigma}\ln\left[cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{\frac{\sigma}{\rho}}\right] + \ln(1-c)\\ &+\frac{\sigma-\rho}{\rho}\ln\left[\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right] + \ln\lambda + \rho\ln A_k + \rho\mu_k\ln D_j + (\rho-1)\ln K_j \qquad (5)\end{aligned}$$

Consider the following equations that will help with totally differentiating equation 5:

$$\frac{\partial \ln v}{\partial \ln A_j}d\ln A_j = d\ln A_j$$

$$\begin{aligned}\frac{\partial \ln v}{\partial \ln U_j}d\ln U_j &= \left(\frac{1-\sigma}{\sigma}\right)\frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}}\rho(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho d\ln U_j}{cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{\frac{\sigma}{\rho}}}\\ &+\left(\frac{\sigma-\rho}{\rho}\right)\frac{\rho(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho d\ln U_j}{\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho}\\ &\text{canceling terms and rewriting}\\ &= (1-\sigma)\omega_j^{cu}(1-\omega_j^c)d\ln U_j + (\sigma-\rho)(1-\omega_j^c)d\ln U_j\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln v}{\partial \ln S_j}d\ln S_j &= \left(\frac{1-\sigma}{\sigma}\right)\frac{\sigma cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma d\ln S_j}{cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{\frac{\sigma}{\rho}}}\\ &\text{canceling } \sigma\text{, rewrite as}\\ &= (1-\sigma)(1-\omega_j^{cu})d\ln S_j\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln v}{\partial \ln K_j}d\ln K_j &= \left(\frac{1-\sigma}{\sigma}\right)\frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}}\rho\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho d\ln K_j}{cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{\frac{\sigma}{\rho}}}\\ &+\left(\frac{\sigma-\rho}{\rho}\right)\frac{\rho\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho d\ln K_j}{\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho} + (\rho-1)d\ln K_j\\ &\text{canceling terms and rewriting}\\ &= (1-\sigma)\omega_j^{cu}\omega_j^c d\ln K_j + (\sigma-\rho)\omega_j^c d\ln K_j + (\rho-1)d\ln K_j\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \ln v}{\partial \mu_u} d\mu_u &= \left(\frac{1-\sigma}{\sigma}\right) \frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}} \rho(1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho d\mu_u \ln D_j}{c A_s^\sigma D_j^{\sigma\mu_s} S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right)^{\frac{\sigma}{\rho}}} \\
&\quad + \left(\frac{\sigma-\rho}{\rho}\right) \frac{\rho(1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho d\mu_u \ln D_j}{\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho} \\
&\quad \text{canceling terms and rewriting} \\
&= (1-\sigma)\omega_j^{cu}(1-\omega_j^c) d\mu_u \ln D_j + (\sigma-\rho)(1-\omega_j^c) d\mu_u \ln D_j
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln v}{\partial \mu_s} d\mu_s &= \left(\frac{1-\sigma}{\sigma}\right) \frac{\sigma c A_s^\sigma D_j^{\sigma\mu_s} S_j^\sigma d\mu_s \ln D_j}{c A_s^\sigma D_j^{\sigma\mu_s} S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right)^{\frac{\sigma}{\rho}}} \\
&\quad \text{canceling terms and rewriting} \\
&= (1-\sigma)(1-\omega_j^{cu}) d\mu_s \ln D_j
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln v}{\partial \mu_k} d\mu_k &= \left(\frac{1-\sigma}{\sigma}\right) \frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}} \rho\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho d\mu_k \ln D_j}{\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho} \\
&\quad + \left(\frac{\sigma-\rho}{\rho}\right) \frac{\rho\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho d\mu_k \ln D_j}{\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho} + \rho d\mu_k \ln D_j \\
&\quad \text{canceling terms and rewriting} \\
&= (1-\sigma)\omega_j^{cu}\omega_j^c d\mu_k \ln D_j + (\sigma-\rho)\omega_j^c d\mu_k \ln D_j + \rho d\mu_k \ln D_j
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln v}{\partial \ln A_u} d\ln A_u &= \left(\frac{1-\sigma}{\sigma}\right) \frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}} \rho(1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho d\ln A_u}{c A_s^\sigma D_j^{\sigma\mu_s} S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right)^{\frac{\sigma}{\rho}}} \\
&\quad + \left(\frac{\sigma-\rho}{\rho}\right) \frac{\rho(1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho d\ln A_u}{\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho} \\
&\quad \text{canceling terms and rewriting} \\
&= (1-\sigma)\omega_j^{cu}(1-\omega_j^c) d\ln A_u + (\sigma-\rho)(1-\omega_j^c) d\ln A_u
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln v}{\partial \ln A_s} d\ln A_s &= \left(\frac{1-\sigma}{\sigma}\right) \frac{\sigma c A_s^\sigma D_j^{\sigma\mu_s} S_j^\sigma d\ln A_s}{c A_s^\sigma D_j^{\sigma\mu_s} S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right)^{\frac{\sigma}{\rho}}} \\
&\quad \text{canceling } \sigma\text{, rewrite as} \\
&= (1-\sigma)(1-\omega_j^{cu}) d\ln A_s
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial \ln v}{\partial \ln A_k} d\ln A_k &= \left(\frac{1-\sigma}{\sigma}\right) \frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}} \rho\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho d\ln A_k}{c A_s^\sigma D_j^{\sigma\mu_s} S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right)^{\frac{\sigma}{\rho}}} \\
&\quad + \left(\frac{\sigma-\rho}{\rho}\right) \frac{\rho\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho d\ln A_k}{\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho} + \rho d\ln A_k \\
&\quad \text{canceling terms and rewriting} \\
&= (1-\sigma)\omega_j^{cu}\omega_j^c d\ln A_k + (\sigma-\rho)\omega_j^c d\ln A_k + \rho d\ln A_k
\end{aligned}$$

where ω_j^{cu} is analogously defined as $\omega_j^{cu} = \frac{(1-c)\left[\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right]^{\frac{\sigma}{\rho}}}{c A_s^\sigma D_j^{\sigma\mu_s} S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right)^{\frac{\sigma}{\rho}}}$.

Note that the preceding equations made use of the following expressions as well:

$$\begin{aligned}
1-\omega_j^{cu} &= \frac{c A_s^\sigma D_j^{\sigma\mu_s} S_j^\sigma}{c A_s^\sigma D_j^{\sigma\mu_s} S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho\right)^{\frac{\sigma}{\rho}}} \\
1-\omega_j^{c} &= \frac{(1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho}{\lambda A_k^\rho D_j^{\rho\mu_k} K_j^\rho + (1-\lambda) A_u^\rho D_j^{\rho\mu_u} U_j^\rho}
\end{aligned}$$

Hence, the total derivative of equation 5 is given by:

$$\begin{aligned}
d\ln v &= d\ln A_j + (1-\sigma)\omega_j^{cu}(1-\omega_j^c) d\ln U_j + (\sigma-\rho)(1-\omega_j^c) d\ln U_j + (1-\sigma)(1-\omega_j^{cu}) d\ln S_j \\
&\quad +(1-\sigma)\omega_j^{cu}\omega_j^c d\ln K_j + (\sigma-\rho)\omega_j^c d\ln K_j + (\rho-1) d\ln K_j + (1-\sigma)\omega_j^{cu}(1-\omega_j^c) d\mu_u \ln D_j \\
&\quad +(\sigma-\rho)(1-\omega_j^c) d\mu_u \ln D_j + (1-\sigma)(1-\omega_j^{cu}) d\mu_s \ln D_j + (1-\sigma)\omega_j^{cu}\omega_j^c d\mu_k \ln D_j \\
&\quad +(\sigma-\rho)\omega_j^c d\mu_k \ln D_j + \rho d\mu_k \ln D_j + (1-\sigma)\omega_j^{cu}(1-\omega_j^c) d\ln A_u + (\sigma-\rho)(1-\omega_j^c) d\ln A_u \\
&\quad +(1-\sigma)(1-\omega_j^{cu}) d\ln A_s + (1-\sigma)\omega_j^{cu}\omega_j^c d\ln A_k + (\sigma-\rho)\omega_j^c d\ln A_k + \rho d\ln A_k \\
&\quad \text{grouping terms} \\
&= d\ln A_j + \left[(1-\sigma)\omega_j^{cu}(1-\omega_j^c) + (\sigma-\rho)(1-\omega_j^c)\right] d\ln U_j + (1-\sigma)(1-\omega_j^{cu}) d\ln S_j \\
&\quad + \left[(1-\sigma)\omega_j^{cu}\omega_j^c + (\sigma-\rho)\omega_j^c + \rho - 1\right] d\ln K_j + \left[(1-\sigma)\omega_j^{cu}(1-\omega_j^c) + (\sigma-\rho)(1-\omega_j^c)\right] d\mu_u \ln D_j \\
&\quad +(1-\sigma)(1-\omega_j^{cu}) d\mu_s \ln D_j + \left[(1-\sigma)\omega_j^{cu}\omega_j^c + (\sigma-\rho)\omega_j^c + \rho\right] d\mu_k \ln D_j \\
&\quad +[(1-\sigma)\omega_j^{cu}(1-\omega_j^c) + (\sigma-\rho)(1-\omega_j^c)] d\ln A_u + (1-\sigma)(1-\omega_j^{cu}) d\ln A_s \\
&\quad +[(1-\sigma)\omega_j^{cu}\omega_j^c + (\sigma-\rho)\omega_j^c + \rho] d\ln A_k \qquad (6)
\end{aligned}$$

Now rearrange the coefficients of some of the terms. For $d\ln U_j$ and $d\ln A_u$, start by expanding:

$$\begin{aligned}
(1-\sigma)\omega_j^{cu}(1-\omega_j^c) &+ (\sigma-\rho)(1-\omega_j^c) = \\
&= \omega_j^{cu} - \sigma\omega_j^{cu} - \omega_j^{cu}\omega_j^c + \sigma\omega_j^{cu}\omega_j^c + \sigma - \sigma\omega_j^c - \rho + \rho\omega_j^c \\
&\quad \text{adding and subtracting } 1+\rho\omega_j^{cu}\omega_j^c\text{, and grouping terms} \\
&= (1-\omega_j^{cu}\omega_j^c) - \rho(1-\omega_j^{cu}\omega_j^c) - (1-\omega_j^{cu}) + \sigma(1-\omega_j^{cu}) - \sigma\omega_j^c(1-\omega_j^{cu}) + \rho\omega_j^c(1-\omega_j^{cu}) \\
&= -(\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) + (1-\rho)(1-\omega_j^{cu}\omega_j^c) - (1-\sigma)(1-\omega_j^{cu})
\end{aligned}$$

Expanding the coefficient for $d\ln K_j$:

$$\begin{aligned}
(1-\sigma)\omega_j^{cu}\omega_j^c &+ (\sigma-\rho)\omega_j^c + \rho - 1 = \\
&= \omega_j^{cu}\omega_j^c - \sigma\omega_j^{cu}\omega_j^c + (\sigma-\rho)\omega_j^c + \rho - 1 \\
&\quad \text{adding and subtracting } \rho\omega_j^{cu}\omega_j^c\text{, and grouping terms} \\
&= (\sigma-\rho)\omega_j^c - \sigma\omega_j^{cu}\omega_j^c + \rho\omega_j^{cu}\omega_j^c - (1-\omega_j^{cu}\omega_j^c) + \rho(1-\omega_j^{cu}\omega_j^c) \\
&= (\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c)
\end{aligned}$$

Similarly, the coefficient for $d\ln A_k$ is:

$$\begin{aligned}
(1-\sigma)\omega_j^{cu}\omega_j^c &+ (\sigma-\rho)\omega_j^c + \rho = \\
&= (\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c) + 1
\end{aligned}$$

And lastly, expanding the coefficient for $d\mu_u \ln D_j$:

$$\begin{aligned}
(1-\sigma)\omega_j^{cu}(1-\omega_j^c) &+ (\sigma-\rho)(1-\omega_j^c) = \\
&= \omega_j^{cu} - \omega_j^{cu}\omega_j^c - \sigma\omega_j^{cu} + \sigma\omega_j^{cu}\omega_j^c + \sigma - \sigma\omega_j^c - \rho + \rho\omega_j^c \\
&\quad \text{adding and subtracting 1, and grouping terms} \\
&= (\sigma-1)(1-\omega_j^{cu}) + (\sigma-1)\omega_j^{cu}\omega_j^c - (\sigma-\rho)\omega_j^c - \rho + 1 \\
&= -(1-\sigma)(1-\omega_j^{cu}) - (1-\sigma)\omega_j^{cu}\omega_j^c - (\sigma-\rho)\omega_j^c - \rho + 1
\end{aligned}$$

Substituting the arranged coefficients in the total derivative given by equation 6:

$$\begin{aligned}
d\ln v &= d\ln A_j + (1-\sigma)(1-\omega_j^{cu})d\ln S_j + \left[-(\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) + (1-\rho)(1-\omega_j^{cu}\omega_j^c) - (1-\sigma)(1-\omega_j^{cu})\right] d\ln U_j \\
&+ \left[(\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c)\right] d\ln K_j + (1-\sigma)(1-\omega_j^{cu})d\mu_s \ln D_j \\
&+ \left[-(1-\sigma)(1-\omega_j^{cu}) - (1-\sigma)\omega_j^{cu}\omega_j^c - (\sigma-\rho)\omega_j^c - \rho + 1\right] d\mu_u \ln D_j \\
&+ \left[(1-\sigma)\omega_j^{cu}\omega_j^c + (\sigma-\rho)\omega_j^c + \rho\right] d\mu_k \ln D_j + (1-\sigma)(1-\omega_j^{cu})d\ln A_s \\
&-[(\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c) + (1-\sigma)(1-\omega_j^{cu})]d\ln A_u \\
&+[(\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c) + 1]d\ln A_k
\end{aligned}$$

For now, let $Z \equiv (\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c)$. Rearranging terms:

$$\begin{aligned}
Zd\ln K_j - Zd\ln U_j &= d\ln v - d\ln A_j + (1-\sigma)(1-\omega_j^{cu})d\ln U_j - (1-\sigma)(1-\omega_j^{cu})d\ln S_j \\
&+(1-\sigma)(1-\omega_j^{cu})d\mu_u \ln D_j - (1-\sigma)(1-\omega_j^{cu})d\mu_s \ln D_j \\
&+ \left[(1-\sigma)\omega_j^{cu}\omega_j^c + (\sigma-\rho)\omega_j^c + \rho\right] d\mu_u \ln D_j \\
&- \left[(1-\sigma)\omega_j^{cu}\omega_j^c + (\sigma-\rho)\omega_j^c + \rho\right] d\mu_k \ln D_j - d\mu_u \ln D_j \\
&-(1-\sigma)(1-\omega_j^{cu})d\ln A_s + [Z + (1-\sigma)(1-\omega_j^{cu})]d\ln A_u - (Z+1)d\ln A_k \\
&\text{grouping terms} \\
Zd\ln\left(\frac{K_j}{U_j}\right) &= d\ln v - d\ln A_j - (1-\sigma)(1-\omega_j^{cu})d\ln\left(\frac{S_j}{U_j}\right) + (1-\sigma)(1-\omega_j^{cu})d(\mu_u-\mu_s)\ln D_j \\
&+ \left[(1-\sigma)\omega_j^{cu}\omega_j^c + (\sigma-\rho)\omega_j^c + \rho\right] d(\mu_u-\mu_k)\ln D_j - d\mu_u \ln D_j \\
&-(1-\sigma)(1-\omega_j^{cu})d\ln\left(\frac{A_s}{A_u}\right) - (Z+1)d\ln\left(\frac{A_k}{A_u}\right) - d\ln A_u \\
&\text{dividing by and substituting for } Z
\end{aligned}$$

$$\begin{aligned}
d\ln\left(\frac{K_j}{U_j}\right) &= \frac{d\ln v - d\ln A_j}{(\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c)} \\
&- \frac{(1-\sigma)(1-\omega_j^{cu})}{(\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c)} d\ln\left(\frac{S_j}{U_j}\right) \\
&+ \frac{(1-\sigma)(1-\omega_j^{cu})d(\mu_u-\mu_s) + \left[(1-\sigma)\omega_j^{cu}\omega_j^c + (\sigma-\rho)\omega_j^c + \rho\right] d(\mu_u-\mu_k) - d\mu_u}{(\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c)} \ln D_j \\
&- \frac{(1-\sigma)(1-\omega_j^{cu})}{(\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c)} d\ln\left(\frac{A_s}{A_u}\right)
\end{aligned}$$

$$
-\frac{(1-\sigma)\omega_j^{cu}\omega_j^c+(\sigma-\rho)\omega_j^c+\rho}{(\sigma-\rho)\omega_j^c(1-\omega_j^{cu})-(1-\rho)(1-\omega_j^{cu}\omega_j^c)}d\ln\left(\frac{A_k}{A_u}\right)
$$
$$
-\frac{d\ln A_u}{(\sigma-\rho)\omega_j^c(1-\omega_j^{cu})-(1-\rho)(1-\omega_j^{cu}\omega_j^c)}
$$

which can be rearranged into equation 3 in the text.

1.3 Equations 4 and 5

Start by dividing the first order conditions given by equation 2 and equation 1:

$$
\begin{aligned}
\frac{w_j^u}{v} &= \frac{(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^{\rho-1}}{\lambda A_k^\rho D_j^{\rho\mu_k}K_j^{\rho-1}} \\
&= \frac{1-\lambda}{\lambda}\left(\frac{A_k}{A_u}\right)^{-\rho} D_j^{\rho(\mu_u-\mu_k)}\left(\frac{K_j}{U_j}\right)^{1-\rho}
\end{aligned}
$$

Taking logs on both sides yields:

$$
\begin{aligned}
\ln\left(\frac{w_j^u}{v}\right) &= \ln\left(\frac{1-\lambda}{\lambda}\right)-\rho\ln\left(\frac{A_k}{A_u}\right)+\rho(\mu_u-\mu_k)\ln D_j+(1-\rho)\ln\left(\frac{K_j}{U_j}\right) \\
&\quad \text{rearranging terms} \\
\ln\left(\frac{K_j}{U_j}\right) &= \frac{1}{1-\rho}\ln\left(\frac{w_j^u}{v}\right)+\frac{\rho}{1-\rho}(\mu_k-\mu_u)\ln D_j+\frac{\rho}{1-\rho}\ln\left(\frac{A_k}{A_u}\right)-\frac{1}{1-\rho}\ln\left(\frac{1-\lambda}{\lambda}\right)
\end{aligned}
$$

Totally differentiating the previous expression yields the analogous to the equation 4 in the text:

$$
d\ln\left(\frac{K_j}{U_j}\right)=\frac{1}{1-\rho}d\ln\left(\frac{w_j^u}{v}\right)+\frac{\rho}{1-\rho}d(\mu_k-\mu_u)\ln D_j+\frac{\rho}{1-\rho}d\ln\left(\frac{A_k}{A_u}\right)
$$

To obtain equation 5 in the text, substitute equation 3 into equation 2 and rearrange terms. This yields

$$
\begin{aligned}
d\ln\left(\frac{w_j^s}{w_j^u}\right) &= \psi_1^{SU}\left(d\mu_s,d\mu_k,d\mu_u,\sigma,\rho,\omega_j^c,\omega_j^{cu}\right)\ln D_j \\
&+ \psi_2^{SU}\left(\sigma,\rho,\omega_j^c,\omega_j^{cu}\right)\left[d\ln\left(\frac{S_j}{U_j}\right)+d\ln\left(\frac{A_s}{A_u}\right)\right]+d\ln\left(\frac{A_s}{A_u}\right) \\
&+ \psi_3^{SU}\left(\sigma,\rho,\omega_j^c,\omega_j^{cu}\right)\left[d\ln v-d\ln A_j-d\ln A_k\right]
\end{aligned}
$$

where, using equation 3 as derived above,

$$
\begin{aligned}
\psi_1^{SU} &= (\rho-\sigma)\omega_j^c\left[d(\mu_k-\mu_u)+\frac{(1-\sigma)(1-\omega_j^{cu})d(\mu_u-\mu_s)+[(1-\sigma)\omega_j^{cu}\omega_j^c+(\sigma-\rho)\omega_j^c+\rho]d(\mu_u-\mu_k)-d\mu_u}{(\sigma-\rho)\,\omega_j^c(1-\omega_j^{cu})-(1-\rho)(1-\omega_j^c\omega_j^{cu})}\right] \\
&\quad +\sigma d(\mu_s-\mu_u) \\
\psi_2^{SU} &= (\sigma-1)-\left[\frac{(\rho-\sigma)\omega_j^c(1-\sigma)(1-\omega_j^{cu})}{(\sigma-\rho)\omega_j^c(1-\omega_j^{cu})-(1-\rho)(1-\omega_j^{cu}\omega_j^c)}\right] \\
\psi_3^{SU} &= \frac{(\rho-\sigma)\omega_j^c}{(\sigma-\rho)\omega_j^c(1-\omega_j^{cu})-(1-\rho)(1-\omega_j^{cu}\omega_j^c)}
\end{aligned}
$$

1.4 Equation 6

Start by taking the log of the production function:

$$
\ln Y_j = \ln A_j + \frac{1}{\sigma}\ln\left[cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{\frac{\sigma}{\rho}}\right] \tag{7}
$$

Consider the following equations that will be useful for getting the total derivative of the previous equation:

$$
\frac{\partial \ln Y_j}{\partial \ln A_j}d\ln A_j = d\ln A_j
$$

$$
\begin{aligned}
\frac{\partial \ln Y_j}{\partial \ln U_j}d\ln U_j &= \left(\frac{1}{\sigma}\right)\frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}}\rho(1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho d\ln U_j}{cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{\frac{\sigma}{\rho}}} \\
&\quad \text{canceling terms and rewriting as above} \\
&= \omega_j^{cu}(1-\omega_j^c)d\ln U_j
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln Y_j}{\partial \ln S_j}d\ln S_j &= \left(\frac{1}{\sigma}\right)\frac{\sigma cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma d\ln S_j}{cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{\frac{\sigma}{\rho}}} \\
&\quad \text{canceling } \sigma \text{ and rewriting} \\
&= (1-\omega_j^{cu})d\ln S_j
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln Y_j}{\partial \ln K_j}d\ln K_j &= \left(\frac{1}{\sigma}\right)\frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right]^{\frac{\sigma-\rho}{\rho}}\rho\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho d\ln K_j}{cA_s^\sigma D_j^{\sigma\mu_s}S_j^\sigma + (1-c)\left(\lambda A_k^\rho D_j^{\rho\mu_k}K_j^\rho + (1-\lambda)A_u^\rho D_j^{\rho\mu_u}U_j^\rho\right)^{\frac{\sigma}{\rho}}} \\
&\quad \text{canceling terms and rewriting as above} \\
&= \omega_j^{cu}\omega_j^c d\ln K_j
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln Y_j}{\partial \mu_u} d\mu_u &= \left(\frac{1}{\sigma}\right) \frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} + (1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\right]^{\frac{\sigma-\rho}{\rho}} \rho(1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho} d\mu_u \ln D_j}{c A_s^{\sigma} D_j^{\sigma\mu_s} S_j^{\sigma} + (1-c)\left(\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} + (1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\right)^{\frac{\sigma}{\rho}}} \\
&\quad \text{canceling terms and rewriting as above} \\
&= \omega_j^{cu}(1-\omega_j^{c}) d\mu_u \ln D_j
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln Y_j}{\partial \mu_s} d\mu_s &= \left(\frac{1}{\sigma}\right) \frac{\sigma c A_s^{\sigma} D_j^{\sigma\mu_s} S_j^{\sigma} d\mu_s \ln D_j}{c A_s^{\sigma} D_j^{\sigma\mu_s} S_j^{\sigma} + (1-c)\left(\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} + (1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\right)^{\frac{\sigma}{\rho}}} \\
&\quad \text{canceling terms and rewriting as above} \\
&= (1-\omega_j^{cu}) d\mu_s \ln D_j
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln Y_j}{\partial \mu_k} d\mu_k &= \left(\frac{1}{\sigma}\right) \frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} + (1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\right]^{\frac{\sigma-\rho}{\rho}} \rho\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} d\mu_k \ln D_j}{c A_s^{\sigma} D_j^{\sigma\mu_s} S_j^{\sigma} + (1-c)\left(\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} + (1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\right)^{\frac{\sigma}{\rho}}} \\
&\quad \text{canceling terms and rewriting as above} \\
&= \omega_j^{cu}\omega_j^{c} d\mu_k \ln D_j
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln Y_j}{\partial \ln A_u} d\ln A_u &= \left(\frac{1}{\sigma}\right) \frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} + (1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\right]^{\frac{\sigma-\rho}{\rho}} \rho(1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho} d\ln A_u}{c A_s^{\sigma} D_j^{\sigma\mu_s} S_j^{\sigma} + (1-c)\left(\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} + (1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\right)^{\frac{\sigma}{\rho}}} \\
&\quad \text{canceling terms and rewriting as above} \\
&= \omega_j^{cu}(1-\omega_j^{c}) d\ln A_u
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln Y_j}{\partial \ln A_s} d\ln A_s &= \left(\frac{1}{\sigma}\right) \frac{\sigma c A_s^{\sigma} D_j^{\sigma\mu_s} S_j^{\sigma} d\ln A_s}{c A_s^{\sigma} D_j^{\sigma\mu_s} S_j^{\sigma} + (1-c)\left(\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} + (1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\right)^{\frac{\sigma}{\rho}}} \\
&\quad \text{canceling } \sigma \text{ and rewriting} \\
&= (1-\omega_j^{cu}) d\ln A_s
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln Y_j}{\partial \ln A_k} d\ln A_k &= \left(\frac{1}{\sigma}\right) \frac{(1-c)\frac{\sigma}{\rho}\left[\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} + (1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\right]^{\frac{\sigma-\rho}{\rho}} \rho\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} d\ln A_k}{c A_s^{\sigma} D_j^{\sigma\mu_s} S_j^{\sigma} + (1-c)\left(\lambda A_k^{\rho} D_j^{\rho\mu_k} K_j^{\rho} + (1-\lambda) A_u^{\rho} D_j^{\rho\mu_u} U_j^{\rho}\right)^{\frac{\sigma}{\rho}}} \\
&\quad \text{canceling terms and rewriting as above} \\
&= \omega_j^{cu}\omega_j^{c} d\ln A_k
\end{aligned}
$$

Where ω_j^{cu} and ω_j^c are as defined in sections 2.2 and 2.3. Hence, the total derivative of equation 7 is:

$$\begin{aligned} d\ln Y_j &= d\ln A_j + (1-\omega_j^{cu})d\ln S_j + \omega_j^{cu}(1-\omega_j^c)d\ln U_j + \omega_j^{cu}\omega_j^c d\ln K_j \\ &+(1-\omega_j^{cu})d\mu_s \ln D_j + \omega_j^{cu}(1-\omega_j^c)d\mu_u \ln D_j + \omega_j^{cu}\omega_j^c d\mu_k \ln D_j \\ &+(1-\omega_j^{cu})d\ln A_s + \omega_j^{cu}(1-\omega_j^c)d\ln A_u + \omega_j^{cu}\omega_j^c d\ln A_k \end{aligned} \tag{8}$$

Now consider equation 3 in the text, which can be rewritten in terms of Z as follows:

$$\begin{aligned} d\ln\left(\frac{K_j}{U_j}\right) &= \frac{d\ln v - d\ln A_j}{Z} - \frac{(1-\sigma)(1-\omega_j^{cu})}{Z} d\ln\left(\frac{S_j}{U_j}\right) \\ &+ \frac{(1-\sigma)(1-\omega_j^{cu})d(\mu_u-\mu_s) + (Z+1)d(\mu_u-\mu_k) - d\mu_u}{Z}\ln D_j \\ &- \frac{(1-\sigma)(1-\omega_j^{cu})}{Z} d\ln\left(\frac{A_s}{A_u}\right) - \frac{Z+1}{Z} d\ln\left(\frac{A_k}{A_u}\right) - \frac{1}{Z} d\ln A_u \\ &\text{taking } d\ln U_j \text{ to the right hand side} \end{aligned}$$

$$\begin{aligned} d\ln K_j &= \frac{d\ln v - d\ln A_j}{Z} - \frac{(1-\sigma)(1-\omega_j^{cu})}{Z} d\ln\left(\frac{S_j}{U_j}\right) + d\ln U_j \\ &+ \frac{(1-\sigma)(1-\omega_j^{cu})d(\mu_u-\mu_s) + (Z+1)d(\mu_u-\mu_k) - d\mu_u}{Z}\ln D_j \\ &- \frac{(1-\sigma)(1-\omega_j^{cu})}{Z} d\ln\left(\frac{A_s}{A_u}\right) - \frac{Z+1}{Z} d\ln\left(\frac{A_k}{A_u}\right) - \frac{1}{Z} d\ln A_u \\ &\text{subtracting } d\ln Y_j \text{ on both sides using equation 8} \end{aligned}$$

$$\begin{aligned} d\ln K_j - d\ln Y_j &= \frac{d\ln v - d\ln A_j}{Z} - \frac{(1-\sigma)(1-\omega_j^{cu})}{Z} d\ln\left(\frac{S_j}{U_j}\right) + d\ln U_j \\ &+ \frac{(1-\sigma)(1-\omega_j^{cu})d(\mu_u-\mu_s) + (Z+1)d(\mu_u-\mu_k) - d\mu_u}{Z}\ln D_j \\ &- \frac{(1-\sigma)(1-\omega_j^{cu})}{Z} d\ln\left(\frac{A_s}{A_u}\right) - \frac{Z+1}{Z} d\ln\left(\frac{A_k}{A_u}\right) - \frac{1}{Z} d\ln A_u \\ &- d\ln A_j - (1-\omega_j^{cu})d\ln S_j - \omega_j^{cu}(1-\omega_j^c)d\ln U_j - \omega_j^{cu}\omega_j^c d\ln K_j \\ &-(1-\omega_j^{cu})d\mu_s \ln D_j - \omega_j^{cu}(1-\omega_j^c)d\mu_u \ln D_j - \omega_j^{cu}\omega_j^c d\mu_k \ln D_j \\ &-(1-\omega_j^{cu})d\ln A_s - \omega_j^{cu}(1-\omega_j^c)d\ln A_u - \omega_j^{cu}\omega_j^c d\ln A_k \\ &\text{grouping some terms} \end{aligned}$$

$$= \frac{d\ln v - d\ln A_j}{Z} - \frac{(1-\sigma)(1-\omega_j^{cu})}{Z} d\ln\left(\frac{S_j}{U_j}\right)$$

$$
\begin{aligned}
&-\omega_j^{cu}\omega_j^c d\ln\left(\frac{K_j}{U_j}\right) - (1-\omega_j^{cu})d\ln\left(\frac{S_j}{U_j}\right) - d\ln A_j \\
&+\frac{(1-\sigma)(1-\omega_j^{cu})d(\mu_u-\mu_s)+(Z+1)d(\mu_u-\mu_k)-d\mu_u}{Z}\ln D_j \\
&-(1-\omega_j^{cu})d\mu_s\ln D_j - \omega_j^{cu}(1-\omega_j^c)d\mu_u\ln D_j - \omega_j^{cu}\omega_j^c d\mu_k\ln D_j \\
&-\frac{(1-\sigma)(1-\omega_j^{cu})}{Z}d\ln\left(\frac{A_s}{A_u}\right) - (1-\omega_j^{cu})d\ln\left(\frac{A_s}{A_u}\right) \\
&-\omega_j^{cu}\omega_j^c d\ln\left(\frac{A_k}{A_u}\right) - \frac{Z+1}{Z}d\ln A_k
\end{aligned}
$$

substituting for $d\ln\left(\frac{K_j}{U_j}\right)$ from equation 3 again and rearranging

$$
\begin{aligned}
= \quad & \frac{(1-\omega_j^{cu}\omega_j^c)(d\ln v - d\ln A_j)}{Z} - d\ln A_j \\
&-\left[\frac{(1-\sigma)(1-\omega_j^{cu})(1-\omega_j^{cu}\omega_j^c)}{Z} + (1-\omega_j^{cu})\right]d\ln\left(\frac{S_j}{U_j}\right) \\
&+\left[\frac{(1-\sigma)(1-\omega_j^{cu})(1-\omega_j^{cu}\omega_j^c)}{Z} + (1-\omega_j^{cu})\right]d\mu_u\ln D_j \\
&-\left[\frac{(1-\sigma)(1-\omega_j^{cu})(1-\omega_j^{cu}\omega_j^c)}{Z} + (1-\omega_j^{cu})\right]d\mu_s\ln D_j \\
&-\left[\frac{(Z+1)(1-\omega_j^{cu}\omega_j^c)}{Z} + \omega_j^{cu}\omega_j^c\right]d\mu_k\ln D_j \\
&-\left[\frac{(1-\sigma)(1-\omega_j^{cu})(1-\omega_j^{cu}\omega_j^c)}{Z} + (1-\omega_j^{cu})\right]d\ln\left(\frac{A_s}{A_u}\right) \\
&-\left[\frac{(Z+1)(1-\omega_j^{cu}\omega_j^c)}{Z} + \omega_j^{cu}\omega_j^c\right]d\ln A_k
\end{aligned}
$$

The next step involves rearranging the coefficient for the last four terms of the previous equation. Notice that the coefficient for $d\ln\left(\frac{S_j}{U_j}\right)$ is the same as the ones for $d\mu_s\ln D_j$, $d\ln\left(\frac{A_s}{A_u}\right)$, and for $d\mu_u\ln D_j$ (this last one with an opposite sign). Multiplying this coefficient by Z for a common denominator and expanding:

$$
\begin{aligned}
\frac{(1-\sigma)(1-\omega_j^{cu})(1-\omega_j^{cu}\omega_j^c)}{Z} \quad + \quad & (1-\omega_j^{cu}) = \\
= \quad & \frac{(1-\omega_j^{cu})\left[(1-\sigma)(1-\omega_j^{cu}\omega_j^c)+\sigma\omega_j^c-\sigma\omega_j^{cu}\omega_j^c-\rho\omega_j^c-1+\omega_j^{cu}\omega_j^c+\rho\right]}{Z} \\
= \quad & \frac{(1-\omega_j^{cu})\left[(1-\sigma)(1-\omega_j^{cu}\omega_j^c)+\sigma\omega_j^c-\sigma\omega_j^{cu}\omega_j^c-\rho\omega_j^c-1+\omega_j^{cu}\omega_j^c+\rho\right]}{Z} \\
= \quad & \frac{(1-\omega_j^{cu})\left[1-\omega_j^{cu}\omega_j^c-\sigma+\sigma\omega_j^{cu}\omega_j^c+\sigma\omega_j^c-\sigma\omega_j^{cu}\omega_j^c-\rho\omega_j^c-1+\omega_j^{cu}\omega_j^c+\rho\right]}{Z}
\end{aligned}
$$

canceling terms

$$
\begin{aligned}
&= \frac{(1-\omega_j^{cu})\left[-\sigma(1-\omega_j^c)+\rho(1-\omega_j^c)\right]}{Z} \\
&= \frac{(\rho-\sigma)(1-\omega_j^{cu})(1-\omega_j^c)}{Z}
\end{aligned}
$$

Finally, expanding the coefficient for $d\mu_k \ln D_j$:

$$
\begin{aligned}
-\frac{(Z+1)(1-\omega_j^{cu}\omega_j^c)}{Z} - \omega_j^{cu}\omega_j^c &= \frac{-(Z+1)(1-\omega_j^{cu}\omega_j^c) - \omega_j^{cu}\omega_j^c Z}{Z} \\
&\quad \text{expanding and canceling terms} \\
&= \frac{-(Z+1)+\omega_j^{cu}\omega_j^c}{Z} \\
&\quad \text{using the expression above for } Z+1 \text{ and canceling terms} \\
&= \frac{\sigma\omega_j^{cu}\omega_j^c + \rho\omega_j^c - \sigma\omega_j^c - \rho}{Z} \\
&= \frac{\sigma\omega_j^{cu}\omega_j^c + (\rho-\sigma)\omega_j^c - \rho}{Z}
\end{aligned}
$$

Also, note that this coefficient can also be written as:

$$
-\frac{(Z+1)(1-\omega_j^{cu}\omega_j^c)}{Z} - \omega_j^{cu}\omega_j^c = -\left[\frac{(1-\omega_j^{cu}\omega_j^c)}{Z}+1\right]
$$

Substituting these four coefficients for the previous ones:

$$
\begin{aligned}
d\ln\left(\frac{K_j}{Y_j}\right) &= \frac{(1-\omega_j^{cu}\omega_j^c)(d\ln v - d\ln A_j)}{Z} - d\ln A_j \\
&\quad -\frac{(\rho-\sigma)(1-\omega_j^{cu})(1-\omega_j^c)}{Z} d\ln\left(\frac{S_j}{U_j}\right) \\
&\quad +\frac{(\rho-\sigma)(1-\omega_j^{cu})(1-\omega_j^c)}{Z} d\mu_u \ln D_j \\
&\quad -\frac{(\rho-\sigma)(1-\omega_j^{cu})(1-\omega_j^c)}{Z} d\mu_s \ln D_j \\
&\quad +\frac{\sigma\omega_j^{cu}\omega_j^c + (\rho-\sigma)\omega_j^c - \rho}{Z} d\mu_k \ln D_j \\
&\quad -\frac{(\rho-\sigma)(1-\omega_j^{cu})(1-\omega_j^c)}{Z} d\ln\left(\frac{A_s}{A_u}\right) \\
&\quad -\left[\frac{(1-\omega_j^{cu}\omega_j^c)}{Z}+1\right] d\ln A_k \\
&\quad \text{grouping terms}
\end{aligned}
$$

$$\begin{aligned} &= \frac{(1-\omega_j^{cu}\omega_j^c)(d\ln v - d\ln A_j - d\ln A_k)}{Z} - d\ln A_j - d\ln A_k \\ &+ \frac{(\rho-\sigma)(1-\omega_j^{cu})(1-\omega_j^c)d(\mu_u-\mu_s) + \left[\sigma\omega_j^{cu}\omega_j^c + (\rho-\sigma)\omega_j^c - \rho\right] d\mu_k}{Z} \ln D_j \\ &- \frac{(\rho-\sigma)(1-\omega_j^{cu})(1-\omega_j^c)}{Z}\left[d\ln\left(\frac{S_j}{U_j}\right) + d\ln\left(\frac{A_s}{A_u}\right)\right] \end{aligned}$$

Substituting for Z, we get the analogous expression to equation 6 in the text:

$$\begin{aligned} d\ln\left(\frac{K_j}{Y_j}\right) &= \frac{(1-\omega_j^{cu}\omega_j^c)(d\ln v - d\ln A_j - d\ln A_k)}{(\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c)} - d\ln A_j - d\ln A_k \\ &+ \frac{(\rho-\sigma)(1-\omega_j^{cu})(1-\omega_j^c)d(\mu_u-\mu_s) + \left[\sigma\omega_j^{cu}\omega_j^c + (\rho-\sigma)\omega_j^c - \rho\right] d\mu_k}{(1-\sigma)\omega_j^{cu}\omega_j^c + (\sigma-\rho)\omega_j^c + (\rho-1)} \ln D_j \\ &- \frac{(\rho-\sigma)(1-\omega_j^{cu})(1-\omega_j^c)}{(\sigma-\rho)\omega_j^c(1-\omega_j^{cu}) - (1-\rho)(1-\omega_j^{cu}\omega_j^c)}\left[d\ln\left(\frac{S_j}{U_j}\right) + d\ln\left(\frac{A_s}{A_u}\right)\right] \end{aligned}$$

This is equation 6 in the text.

1.5 Equations 7 and 8

Start by dividing the first order condition for unskilled labor by the one for capital:

$$\begin{aligned} \frac{w_j^u}{v} &= \frac{1-\lambda}{\lambda}\left(\frac{A_k}{A_u}\right)^{-\rho} D_j^{\rho(\mu_u-\mu_k)}\left(\frac{K_j}{U_j}\right)^{1-\rho} \\ & \quad \text{taking logs on both sides} \\ \ln w_j^u - \ln v &= \ln\left(\frac{1-\lambda}{\lambda}\right) - \rho\ln\left(\frac{A_k}{A_u}\right) + \rho(\mu_u-\mu_k)\ln D_j + (1-\rho)\ln\left(\frac{K_j}{U_j}\right) \end{aligned}$$

Taking $\ln v$ to the right hand side and totally differentiating the expression results in the analogous to equation 7 in the text:

$$d\ln w_j^u = d\ln v - \rho d\ln\left(\frac{A_k}{A_u}\right) + \rho d(\mu_u-\mu_k)\ln D_j + (1-\rho)d\ln\left(\frac{K_j}{U_j}\right)$$

Substituting in for $d\ln\left(\frac{K_j}{U_j}\right)$ yields equation 8 in the text.