# Online Appendix
# Fiscal Unions

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## A Online Appendix: Proofs and Details for Static Model in Sections 2-3

### A.1 Proof of Proposition 1

We have already proved that the conditions in the proposition are necessary for an allocation together with prices to form part of an equilibrium with complete markets. We now need to establish these conditions are sufficient. The proof is constructive. Start with an allocation together with prices that satisfy these conditions. We choose wages $W^i(s)$ to satisfy the labor-leisure condition (5) for each $i \in I$ and $s \in S$. Given some set of state prices $Q(s)$, we pick portfolio taxes $\tau^i_D(s)$ to satisfy the risk sharing condition (3) for each $i \in I$ and $s \in S$. Note a first dimension of indeterminacy here: we can always multiply state prices $Q(s)$ and portfolio taxes $1 + \tau^i_D(s)$ by some arbitrary common function $\lambda(s)$ of $s$. We then pick labor taxes $\tau^i_L$ to satisfy the price setting equation (6). Finally, for a given set of ex-post fiscal transfers $\hat{T}^i(s)$ that satisfy the country budget constraint $\int Q(s)\hat{T}^i(s)\pi(s)ds = \int Q(s)\left[P_T(s)(C^i_T(s) - E^i_T(s))\right]\pi(s)$ and the condition that aggregate net international transfers are zero in every state (8), we compute transfers to households $T^i(s)$ using the government budget constraint (7). We can then compute the required portfolio positions $D^i(s)$ using the ex-post household budget constraint (2). These choices guarantee that the ex-ante household budget constraint (1) is verified. Note a second dimension of indeterminacy, as we have some degree of freedom in choosing ex-post fiscal transfers $\hat{T}^i(s)$.

### A.2 Proof of Proposition 2

We have already proved that the conditions in the proposition are necessary for an allocation together with prices to form part of an equilibrium with complete markets. We now need to establish these conditions are sufficient. The proof is constructive. Start with an allocation together with prices that satisfy these conditions. We choose wages $W^i(s)$ to satisfy the labor-leisure condition (5) for each $i \in I$ and $s \in S$. We then pick labor taxes $\tau^i_L$ to satisfy the prices setting equation (12). We choose transfers $T^i(s)$ to satisfy the household budget constraint (11). We then choose ex-post fiscal transfers $\hat{T}^i(s)$ to satisfy the government budget constraint (13). We can verify that these choices satisfy (8).

## A.3 Price Setting with Constant Elasticity of Substitution

We have

$$1-\frac{\int \tau^i(s)U^i_{C_{NT}}(s)C^i_{NT}(s)\pi(s)ds}{\int U^i_{C_{NT}}(s)C^i_{NT}(s)\pi(s)ds}=\frac{1}{1+\tau^i_L}\frac{\varepsilon-1}{\varepsilon}.$$

We can rewrite the first-order condition for $P^i_{NT}$ as

$$\int \frac{\alpha^i_p(s)}{\alpha^i(s)}p^i(s)\,\alpha^i(s)C^i_T(s)\,\frac{1}{p^i(s)}U^i_{C_T}(s)\,\tau^i(s)\,\pi(s)\,ds=0.$$

If $\frac{\alpha^i_p(s)}{\alpha^i(s)}p^i(s)$ is constant then this implies that

$$\int C^i_{NT}(s)\,U^i_{C_{NT}}(s)\,\tau^i(s)\,\pi(s)\,ds=0.$$

Thus in this case $\frac{1}{1+\tau^i_L}\frac{\varepsilon-1}{\varepsilon}=1$ or $\tau^i_L=-1/\varepsilon$.

## A.4 Proof of Proposition 7

Consider an equilibrium such that $\tau^i(s)\neq 0$ for some $i\in I, s\in S$. Assume, towards a contradiction, that the allocation is constrained Pareto efficient.

We consider two cases in turn. First, suppose that $V^i_{C_T}(s)=U^i_{C_T}(s)(1+\frac{\alpha^i(s)}{p^i(s)}\tau^i(s))<0$ for some set $\Omega\subset I\times S$ of positive measure of countries and states. Define the sections $\Omega(s)=\{i:(i,s)\in\Omega\}$. Then there exists a perturbation that for each $s\in S$ : (a) lowers $C^i_T(s)$ for $i\in\Omega(s)$ and improves welfare $V^i(s)$; (b) increases $C^i_T(s)$ for $i\notin\Omega(s)$ and improves welfare $V^i(s)$; and (c) satisfies the resource constraint $\int C^i_T(s)di=\int E^i_T(s)di$. This perturbation is feasible and creates a Pareto improvement, a contradiction.

Next, consider the case where $1+\frac{\alpha^i(s)}{p^i(s)}\tau^i(s)\geq 0$ for all $i\in I, s\in S$. For each state $s$ consider ranking countries by their weighted labor wedge $\frac{\alpha^i(s)}{p^i(s)}\tau^i(s)$. By Proposition 6 it must be that

$$\frac{1+\frac{\alpha^i(s)}{p^i(s)}\tau^i(s)}{1+\frac{\alpha^{i'}(s)}{p^{i'}(s)}\tau^{i'}(s)}=\frac{1+\frac{\alpha^i(s)}{p^i(s)}\tau^i(s')}{1+\frac{\alpha^{i'}(s)}{p^{i'}(s)}\tau^{i'}(s')}$$

for all $i$, $i'$, $s$ and $s'$. This implies that the ranking must be the same in all states $s$. It follows that there is a country $i^*$ that is at top of the ranking for all states $s$, i.e. $i^*\in\cap_{s\in S}\arg\max_{i\in I}\frac{\alpha^i(s)}{p^i(s)}\tau^i(s)$. Proposition 5 then implies that this country has a positive labor wedge: $\tau^{i^*}(s)\geq 0$ for all $s$. Proposition 4 then implies that $\tau^{i^*}(s)=0$ for all $s$. Therefore we have that $\tau^i(s)\leq 0$ for all $i\in I, s\in S$. Proposition 5 then implies that actually $\tau^i(s)=0$ for all $i\in I, s\in S$.

# B Online Appendix: Extensions for Static Model in Sections 2-3

## B.1 Sticky Wages

In order to have a well defined wage setting problem we assume that labor services are produced by combining a variety of differentiated labor inputs according to the constant returns CES technology

$$N^i(s) = \left(\int_0^1 N^{i,h}(s)^{1-\frac{1}{\varepsilon_w}}dh\right)^{\frac{1}{1-\frac{1}{\varepsilon_w}}}.$$

The rest of the technology is as before. We assume that in each country there is a continuum of workers $h \in [0,1]$, each supplying a particular variety $h \in [0,1]$ with preferences

$$\int U^i(C^{i,h}_{NT}(s), C^{i,h}_T(s), N^{i,h}(s); s)\pi(s)ds.$$

The budget constraints are the same as before

$$\int D^{i,h}(s)Q(s)\pi(s)ds \le 0,$$

$$\begin{aligned} P^i_{NT}(s)C^{i,h}_{NT}(s) + P_T(s)C^{i,h}_T(s) \le (1-\tau^i_L)W^{i,h}N^{i,h}(s) \\ + P_T(s)E^i_T(s) + \Pi^i(s) + T^i(s) + (1+\tau^i_D(s))D^{i,h}(s), \end{aligned}$$

except that the wage $W^{i,h}$ is now specific to each worker $h$ but independent of $s$ because wages are set in advance of the realization of the state $s$. Note that prices of non-traded goods are now state-contingent. For convenience, we now assume that the worker pays for the labor tax; firms are untaxed.

Workers set their own wages $W^{i,h}$ taking into account that in each state of the world $s$ labor demand is given by $N^i(s)(W^{i,h}/W^i)^{-\varepsilon_w}$ where $W^i = (\int(W^{i,h})^{1-\varepsilon_w}dh)^{1/(1-\varepsilon_w)}$ is the wage index for labor services. In a symmetric equilibrium, all workers set the same wage $W^{i,h} = W^i$, and consume and work the same so that $C^{i,h}_{NT}(s) = C^i_{NT}(s)$, $C^{i,h}_T(s) = C^i_T(s)$ and $N^{i,h}(s) = N^i(s)$. The wage $W^i$ is given by

$$W^i = \frac{1}{1-\tau^i_L}\frac{\epsilon_w}{\epsilon_w - 1}\frac{\int -N^i(s)U^i_N(s)\pi(s)ds}{\int \frac{U^i_{C_{NT}}(s)}{P^i_{NT}(s)}N^i(s)\pi(s)ds}.$$

All varieties sell at the same price so that $P^{i,j}_{NT}(s) = P^i_{NT}(s)$. This price is given by

$$P^i_{NT}(s) = \frac{\epsilon}{\epsilon - 1}\frac{W^i}{A^i(s)}.$$

All the results that we derived in the version of the model with sticky prices carry through with no modification to this specification with sticky wages. In particular, Propositions 1–12 are still valid. However, the corresponding allocations can be different than under sticky prices if there are productivity shocks.

## B.2 Limited Commitment

Explicit or implicit insurance (risk sharing) arrangements inevitably raise concerns of incentives. We have abstracted from these considerations, not because we believe them to be unimportant, but in order to isolate the effects that our aggregate demand externality has on optimal risk sharing. Modeling limits to insurance due to incentive problems requires making specific choices about the underlying shocks, the asymmetry of information, the available monitoring technologies, or the type of commitment problem, etc. Although the possibilities are vast and exploring them all is beyond the scope of this paper, we believe the main insights of our analysis would carry over.[43]

In the online appendix B.3, we analyze an example with moral hazard. Here instead, we develop an example with limited commitment. Consider the implementation with incomplete markets and international transfers, where all international risk sharing occurs through international transfers. Ex post, in every state of the world $s$, some countries $i$ are net contributors to the union with $\hat{T}^i(s) \leq 0$, and some countries $i$ are net beneficiaries with $\hat{T}^i(s) \geq 0$. This poses no particular problem to the extent that there exists a strong enough union-wide institutional enforcement mechanism. But with imperfect enforcement and limited commitment, the concern arises that governments of ex-post net contributor countries do no in fact contribute the transfers that were agreed upon ex ante behind the veil of ignorance before the realization of the shock.

To make things stark, consider the extreme case where there is no institutional enforcement mechanism. Governments can default on their promised transfers $\hat{T}^i(s)$, and have no ability to commit. We assume that default leads to a utility loss for which we adopt a flexible parametrization $K^i(s)$. The planning problem can now be written as

$$\max_{P_T, P^i_{NT}, C^i_T(s)} \int \int \lambda^i V^i \left( C^i_T(s), \frac{P_T}{P^i_{NT}}; s \right) \pi(s) di ds \tag{28}$$

subject to

$$\int C^i_T(s) di = \int E^i_T(s) di \tag{29}$$

and

$$V^i \left( C^i_T(s), \frac{P_T}{P^i_{NT}}; s \right) \geq V^i \left( E^i_T(s), \frac{P_T}{P^i_{NT}}; s \right) - K^i(s). \tag{30}$$

The only difference introduced by the limited commitment problem is the presence of the incen-

[43] In practice of course, institutional mechanisms exist to mitigate these agency problems. For example, most fiscal unions such as the US channel a large part of their transfers through more or less ex-ante-rules-based automatic stabilizers (through the unemployment insurance program, federal income and social security taxes, bailout funds), probably for reasons of political acceptability and transparency, but also to mitigate the difficulties associated with collective and distributing discretionary ex-post transfers in a world with limited commitment. Another example is state debt-limit in the US, or collective budget procedures and enforcement mechanisms that already exist in Europe.

tive compatibility constraint (30), requiring each country to be better off sticking to the fiscal union arrangement than defaulting and reverting to autarky while experiencing the endowment loss associated with default.

Let $\mu > 0$ be the multiplier on (29) and $\nu^i(s) \geq 0$ be the multiplier on (30). The condition for constrained efficient risk sharing becomes

$$U^i_{C_T}(s)[1 + \frac{\alpha^i(s)}{p^i(s)}\tau^i(s)][1 + \nu^i(s)] = \mu. \tag{31}$$

By contrast, the corresponding corresponding condition for a country outside the currency union is

$$U^i_{C_T}(s)[1 + \nu^i(s)] = \mu. \tag{32}$$

Condition (32) shows that even with flexible exchange rates, limited commitment endogenously limits insurance (risk sharing) possibilities. A high value of the multiplier $\nu^i(s)$ indicates that it is relatively tempting for country $i$ to default in a state $s$. The optimal contract then adjusts the transfer $\hat{T}^i(s)$ and the traded goods consumption $C^i_T(s)$ so that default is prevented. Condition (31) shows how the optimal provision of insurance (risk sharing) and incentives must be modified when the country is in a currency union. The provision of incentives requires the private consumption of traded goods to vary with the realization of government consumption. Because prices are sticky, this generates a non-zero pattern of labor wedges $\tau^i(s)$. This in turn opens up a wedge between the social and private marginal utility of income, which creates another force agains the perfect equalization of consumption of traded goods across states for each country.

This example shows that the optimal risk sharing arrangements are different for countries that belong to a currency union than that for countries who have a flexible exchange rate. This is true with or without enforcement frictions. In both cases, the optimal arrangement involves a key sufficient statistic, the social marginal utility of transfers given by $U^i_{C_T}(s)[1 + \frac{\alpha^i(s)}{p^i(s)}\tau^i(s)]$ for a country in a currency union and given by $U^i_{C_T}(s)$ for a country outside a currency union.

### B.3 Moral Hazard

Suppose that the government can exert effort $e$ ex ante to affect the distribution of the endowment of the traded good ex post, but that effort $e$ is not observable, creating a moral hazard problem. We focus on a single country $i \in [0, 1]$. We assume that the shock $s$ is purely idiosyncratic and only affects the value of the endowment $E^i_T(s)$ in country $i$. Naturally, monetary policy at the union level should not react to to the idiosyncratic shocks $s$ of an infinitesimal country, so that $P_T(s) = P_T$ is constant. These assumptions simplify the exposition. The principal-agent problem is then

$$\max_{P^i_{NT}, C^i_T(s), e} \int V^i \left( C^i_T(s), \frac{P_T}{P^i_{NT}} \right) \pi(s|e) ds - h(e) \tag{33}$$

subject to

$$\int (C^i_T(s) - E^i_T(s))\pi(s|e)ds \leq 0 \tag{34}$$

and

$$\int V^i\left(C^i_T(s), \frac{P_T}{P^i_{NT}}\right)\pi(s|e)ds - h(e) \geq \int V^i\left(C^i_T(s), \frac{P_T}{P^i_{NT}}\right)\pi(s|e')ds - h(e') \qquad \text{for all } e'. \tag{35}$$

The first constraint (34) simply conditions the average level of expected transfers; this reflects the fact that insurance is priced fairly i.e. ($Q(s) = 1$), since the shock is experienced by a single country and does not affect aggregate resources at the union level. The last constraint (35) is the incentive compatibility condition, requiring the country's effort to be optimal, taking the schedule $C^i_T(s)$ as given.

In the absence of nominal rigidities or for a country with flexible exchange rates and independent monetary policy, we would solve the same problem but using $V^{i*}(C^i_T(s)) = \max_{p^i} V^i\left(C^i_T(s), p^i\right)$ in place of $V^i\left(C^i_T(s), \frac{P_T}{P^i_{NT}}\right)$. Note that $V^i\left(C^i_T, p^i\right) \leq V^{i*}(C^i_T)$ with equality at a single value of $C^i_T$, so that $V^{i*}$ is an upper envelope of $V^i$. When prices are rigid it is *as if* the country were more risk averse, in the sense described earlier. In the presence of moral hazard, higher risk aversion affects the optimal insurance (risk sharing) contract $C^i_T(\cdot)$.

Consider the planning problem (33). Let $\mu$ be the multiplier on (34) and $d\nu(e')$ be the measure multiplier on (35). The corresponding for constrained efficient risk sharing becomes

$$U^i_{C_T}(s)[1 + \frac{\alpha^i(s)}{p^i(s)}\tau^i(s)][1 + \int \frac{\pi(s|e) - \pi(s|e')}{\pi(s|e)}d\nu(e')] = \mu. \tag{36}$$

By contrast, the corresponding corresponding condition for a country outside the currency union is

$$U^i_{C_T}(s)[1 + \int \frac{\pi(s|e) - \pi(s|e')}{\pi(s|e)}d\nu(e')] = \mu. \tag{37}$$

Condition (37) shows that even with flexible exchange rates, moral hazard endogenously limits insurance (risk sharing) possibilities. There is a meaningful tradeoff between insurance (risk sharing) and incentives and providing incentives for the country's government to exert the adequate level effort requires the government to have "skin in the game". The private consumption of traded goods must vary with the realization of government spending on traded goods. It must be high whenever the particular realization of government spending is more likely (as measured by the likelihood ration $\frac{\pi(s|e)-\pi(s|e')}{\pi(s|e)}$) under the desired effort level than under alternative effort levels that the government is tempted to exert (as measured by the measure multiplier $d\nu(e')$ on the corresponding incentive compatibility constraint). This is accomplished by reducing the level of transfers to the country when government spending on traded goods is high.

Condition (36) shows how the optimal provision of insurance (risk sharing) and incentives must be modified when the country is in a currency union. The provision of incentives requires the private consumption of traded goods to vary with the realization of government consumption. Because prices are sticky, this generates a non-zero pattern of labor wedges $\tau^i(s)$. This in turn opens up a wedge between the social and private marginal utility of income, which creates another force agains the perfect equalization of consumption of traded goods across states. As a result, the optimal insurance insurance (risk sharing) arrangements are different for countries that belong to a currency union than that for countries who have a flexible exchange rate.

## B.4 Government Spending

We introduce government spending in the model. We characterize the joint optimal use of international transfers and government spending. Our analysis underscores that both instruments should be used in conjunction. Moreover, we show that our characterization of fiscal unions is robust to the availability of government spending as an additional instrument. We also compare their relative performance depending on a number of deep economic parameters by studying a few limit cases.

**Introducing government spending.** Following the literature, we focus on the case where government spending is concentrated on non-traded goods, which we view as the most practically relevant case.[44] In each state $s$ and country $i$, the government spends $P^i_{NT} G^i_{NT}(s)$ to finance government consumption of $G^i_{NT}(s)$ of non-traded goods. As is standard, we capture agents' preferences of government consumption by including it in the utility function and write

$$U^i(G^i_{NT}(s), C^i_{NT}(s), C^i_T(s), N^i(s); s)$$

for the state-$s$ utility function of country $i$ agents. We assume that that preferences are weakly separable over government consumption on the one hand, and private consumption and labor on the other hand. In addition, we continue to assume that preferences over consumption goods are weakly separable from labor, and that the preference over consumption goods are homothetic.

Apart from that, there are only minor differences with the setup of the main model. These differences involve the government budget constraint, the resource constraint for non-traded goods, and the price setting conditions.[45] Our implementability results in Propositions 1 and 2 can be extended in a straightforward way.

[44]For example Beetsma and Jensen (2005) and Gali and Monacelli (2008) introduce government spending on domestic goods in models where all goods are traded, with or without home bias in consumption. The natural equivalent in our setup is to study government spending on non-traded goods. We have also analyzed government spending on traded goods. The analysis is available upon request.

[45]The government budget constraint is now $T^i(s) + P^i_{NT} G^i_{NT}(s) = \tau^i_L W^i(s) N^i(s) - \tau^i_D(s) D^i(s) + \hat{T}^i(s)$. The resource constraint for non-traded goods is now $C^i_{NT}(s) + G^i_{NT}(s) = A^i(s) N^i(s)$. The price setting constraint is now $P^i_{NT} = (1+\tau^i_L)\frac{\epsilon}{\epsilon-1}\frac{\int \frac{Q(s)}{1+\tau^i_D(s)}\frac{W^i(s)}{A^i(s)}\left[C^i_{NT}(s)+G^i_{NT}(s)\right]\pi(s)ds}{\int \frac{Q(s)}{1+\tau^i_D(s)}\left[C^i_{NT}(s)+G^i_{NT}(s)\right]\pi(s)ds}$.

**Second-Best Planning problem.** In order to write down the second-best Ramsey planning problem jointly characterizing international transfers and government spending, we modify the indirect utility function. We define

$$\tilde{V}^i(G^i_{NT}(s), C_T, p; s) = U^i\left(G^i_{NT}(s), \alpha^i(p;s)C_T, C_T, \frac{\alpha^i(p;s)C_T + G^i_{NT}(s)}{A^i(s)}; s\right).$$

In an equilibrium with $G^i_{NT}(s)$, $C^i_T(s)$ and $p^i(s)$, ex post welfare in state $s$ in country $i$ is then given by

$$\tilde{V}^i(G^i_{NT}(s), C^i_T(s), p^i(s); s).$$

The second-best planning problem is

$$\max_{G^i_{NT}(s), P_T(s), P^i_{NT}, C^i_T(s)} \int\int \tilde{V}^i\left(G^i_{NT}(s), C^i_T(s), \frac{P_T(s)}{P^i_{NT}}; s\right)\lambda^i \pi(s)\, di\, ds$$

subject to

$$\int C^i_T(s)di = \int E^i_T(s)di.$$

We can solve this planning problem recursively by defining

$$V^i(C_T, p; s) = \max_{G^i_{NT}(s)} U^i\left(G^i_{NT}(s), \alpha^i(p;s)C_T, C_T, \frac{\alpha^i(p;s)C_T + G^i_{NT}(s)}{A^i(s)}; s\right), \tag{38}$$

and then solving

$$\max_{P_T(s), P^i_{NT}, C^i_T(s)} \int\int V^i\left(C^i_T(s), \frac{P_T(s)}{P^i_{NT}}; s\right)\lambda^i \pi(s)\, di\, ds$$

subject to

$$\int C^i_T(s)di = \int E^i_T(s)di.$$

**Constrained Pareto efficient allocations.** With these notations, the analysis is identical to that of the model without government spending. Indeed, the derivatives of the indirect utility function $V^i(C_T, p; s)$ are given by exactly the same formula as in Proposition 3, and as a result, Propositions 4-10 as well as Proposition 12 carry through without any modification.[46] Hence our analysis of fiscal unions is robust to the availability of government spending as an additional instrument.

Of course, this does not mean that the resulting allocation is unchanged. Away from this case, optimal government spending can reduce the deviations of the labor wedge $\tau^i(s)$ from zero, but it does not eliminate them.[47] There are two informative ways to write the optimality condition for government spending, both of which follow directly from the definition of $V^i(C_T, p; s)$ in equation

[46]The exact conditions in Proposition 11 for the constrained efficiency of the complete markets equilibrium without portfolio taxes are different in the presence of government spending.

[47]Formally, this is true except in the knife-edge cases where the optimal allocation with flexible prices can be implemented with a fixed exchange rate.

(38):

$$U^i_{G_{NT}}(s) = -\frac{1}{A^i(s)} U^i_N(s), \tag{39}$$

or

$$U^i_{G_{NT}}(s) = (1 - \tau^i(s)) U^i_{C_{NT}}(s). \tag{40}$$

To understand these formulas, it is best to analyze first the case when prices or exchange rates are flexible. Optimal government spending is then characterized by equation (39) or equation (40) with $\tau^i(s) = 0$. Both equations equalize the marginal benefit $U^i_{G_{NT}}(s)$ of government consumption with its marginal cost, but express the marginal cost in two different (but equivalent) ways. Equation (39) expresses the marginal cost $-\frac{1}{A^i(s)} U^i_N(s)$ in terms of the marginal increase labor that would be required to service the marginal increase in government consumption, while equation (40) with $\tau^i(s) = 0$ expresses the marginal cost $U^i_{C_{NT}}(s)$ in terms of the marginal reduction in private consumption that would be required to service the marginal increase in government consumption. These are two equivalent ways of stating the Samuelson rule (see Samuelson 1954) for the optimal provision of public goods.

Depending on which of these formulations one prefers to focus on, rigid prices and fixed exchange rates either require no deviation from the Samuelson rule (equation (39)) or a deviation from the Samuelson rule (equation (40)). The reason is that the social marginal cost of government spending is still given by $-\frac{1}{A^i(s)} U^i_N(s)$ but not by $U^i_{C_{NT}}(s)$ and instead by $(1 - \tau^i(s)) U^i_{C_{NT}}(s)$. This is because the price of non-traded goods does not reflect the marginal cost of producing them. The discrepancy is precisely given by the labor wedge. The government internalizes this wedge when it decides its consumption of non-traded goods, but private agents do not. As a result, in recessions when $\tau^i(s) > 0$, it is optimal to tilt the mix of government and private consumption of non-traded goods in the direction of the former, and the opposite holds true in booms when $\tau^i(s) < 0$.[48]

Having characterized the jointly optimal use of international transfers and government spending and shown the robustness of our characterization of optimal international transfers to the availability of government spending as an additional instrument, we now compare the relative performance of government spending and international transfers in a few enlightening limit cases. We first treat the case of the closed-economy limit. We show that international transfers achieve perfect macroeconomic stabilization, with no residual role for government spending. By contrast, in the perfectly open economy limit, international transfers are not used for macroeconomic stabilization, but government spending is. We then treat the cases where the disutility of labor is linear or government spending is purely wasteful. In both cases, we show that even though the optimum is away from the first best, government spending is not useful for macroeconomic stabilization.

[48] This analysis assumes that prices are entirely rigid. If there is some adjustment in prices, then increases in government spending stimulate inflation. Given a fixed exchange rate, and other things equal, this leads to an appreciation of the real exchange rate which depresses private spending on non-traded goods and counteracts the direct effect of government spending on total spending on non-traded goods (see e.g. Farhi-Werning 2012). This lessens the macroeconomic stabilization role of government spending. The same holds true for international transfers, as we emphasize in Section 5.

**Closed-economy limit.** Consider first the closed-economy limit. This limit can be understood as follows. Suppose for simplicity that preferences are given by $v(G^i_{NT}(s)) + \frac{C^i(s)^{1-\gamma}}{1-\gamma} - \phi(N^i(s))$ where $C^i(s) = \left[(1-\alpha)^{\frac{1}{\eta}} C^i_{NT}(s)^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} C^i_T(s)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$. Then for any $p$, $\alpha^i(p;s)$ is decreasing in $\alpha$. The closed-economy limit is obtained in the limit by also scaling where $\alpha$ goes to zero and $\alpha^i(p;s)$ goes to infinity as long as we also scale $E^i_T(s)$ by $\frac{\alpha}{(1-\alpha)}$ so as to keep $\alpha^i(p;s)E^i_T(s)$ constant. In this limit, the first-best level of welfare is achieved. This is because international transfers are extremely powerful in relatively closed economies. Indeed, we have already emphasized that the "dollar-for-dollar" output multiplier of transfers is precisely given by the relative expenditure share of non-traded to traded goods. And this multiplier goes to infinity in the closed-economy limit. As we approach the closed-economy limit, vanishingly small departures (as a fraction of each country's nominal income) from the international transfers that support the first-best allocation are enough to perfectly stabilize the economy and deliver $\tau^i(s) = 0$ for all $i$ and $s$. There is no residual macroeconomic stabilization role for government spending, which then simply follows the first-best Samuelson rule.

**Perfectly-open economy limit.** The relative usefulness of international transfers and government spending is reversed in the limit where countries are perfectly open, which we capture by letting $\alpha$ go to one. In this limit, international transfers are not used for macroeconomic stabilization, in the sense that constrained efficient and privately optimal risk sharing coincide, so that optimal international transfers are only needed when markets are incomplete, in order to replicate the complete markets allocation with privately optimal risk sharing. By contrast, government spending is used for macroeconomic stabilization as characterized by the same optimality conditions (39) and (40).

**Linear disutility of labor.** Suppose now that the disutility from labor is linear. We maintain the same parametrization of preferences and assume in addition that $\phi(N^i(s)) = \phi N^i(s)$ for some constant $\phi > 0$. In this case, the first-order condition for optimal government spending (39) becomes $v'(G^i_{NT}(s)) = \frac{\phi}{A^i(s)}$. This formula, which pins down $G^i_{NT}(s)$ as a function of $A^i(s)$, holds both under rigid prices and fixed exchange rates, and under flexible prices or flexible exchange rates. Hence there is a sense in which government spending is not used for macroeconomic stabilization, despite the fact that macroeconomic stabilization is imperfect. The same is not true of international transfers.

**Purely wasteful government spending.** Another enlightening case is the case in which government spending is purely wasteful, so that it does not enter preferences.[49] In that case, formulas (39) and (40) indicate that it is optimal not to use government spending, both with rigid prices and fixed exchange rates and with flexible prices or flexible exchange rates. Hence, once again, there is a sense in which government spending is not used for macroeconomic stabilization, despite the fact that macroeconomic stabilization is imperfect. The same is not true of international transfers.

[49] Formally, this means that $U^i(G^i_{NT}(s), C^i_{NT}(s), C^i_T(s), N^i(s); s)$ is independent of $G^i_{NT}(s)$.

# C Online Appendix: Proofs and Details for Dynamic Model in Sections 4-5

## C.1 Nonlinear Calvo Price Setting Equations

The equilibrium conditions for the Calvo price setting model can be expressed as follows

$$\frac{1-\delta\Pi_{H,t}^{\epsilon-1}}{1-\delta} = \left(\frac{F_t}{K_t}\right)^{\epsilon-1},$$

$$K_t = \frac{\varepsilon}{\varepsilon-1}\frac{1+\tau^L}{A_{H,t}}Y_t N_t^{\phi} + \delta\beta\Pi_{H,t+1}^{\epsilon}K_{t+1},$$
$$F_t = Y_t C_t^{-\sigma}S_t^{-1}\mathcal{Q}_t + \delta\beta\Pi_{H,t+1}^{\epsilon-1}F_{t+1},$$

together with an equation determining the evolution of price dispersion

$$\Delta_t = h(\Delta_{t-1}, \Pi_{H,t}),$$

where $h(\Delta,\Pi) = \delta\Delta\Pi^{\epsilon} + (1-\delta)\left(\frac{1-\delta\Pi^{\epsilon-1}}{1-\delta}\right)^{\frac{\epsilon}{\epsilon-1}}$.

## C.2 Decomposing the Planning Problem (22)

We can break down the planning problem into two parts. First, there is an aggregate planning problem determining the average output gap and inflation $\hat{y}_t^*$ and $\pi_t^*$

$$\min \frac{1}{2}\int_0^{\infty} e^{-\rho t}\left[\alpha_\pi(\pi_t^*)^2 + (\hat{y}_t^*)^2\right]dt \tag{41}$$

subject to (27).

Second, there is a disaggregated planning problem determining deviations from the aggregates for output gap, home inflation and consumption smoothing, $\hat{\tilde{y}}_t^i$, $\hat{\tilde{\pi}}_{H,t}^i$ and $\hat{\tilde{\theta}}_t^i$

$$\min \frac{1}{2}\int_0^{\infty}\int_0^1 e^{-\rho t}\left[\alpha_\pi(\hat{\tilde{\pi}}_{H,t}^i)^2 + (\hat{\tilde{y}}_t^i)^2 + \alpha_\theta(\hat{\tilde{\theta}}^i)^2\right]didt \tag{42}$$

subject to (23), (24), (25), (26). Note that because the forcing variables in this linear quadratic problem satisfy $\int_0^1 \tilde{s}_t^i di = 0$, the aggregation constraint (26) is not binding. We can therefore drop it from the planning problem. The resulting relaxed planning problem can be broken down into separate component planning problems for each country $i \in [0,1]$

$$\min \frac{1}{2}\int_0^{\infty} e^{-\rho t}\left[\alpha_\pi(\hat{\tilde{\pi}}_{H,t}^i)^2 + (\hat{\tilde{y}}_t^i)^2 + \alpha_\theta(\hat{\tilde{\theta}}^i)^2\right]dt \tag{43}$$

subject to (23), (24) and (25).

### C.3 Incomplete Markets and No Transfers in a Currency Union

Here we analyze the solution with incomplete markets and no transfers. This solution imposes $\hat{\hat{\theta}}^i = 0$ and coincides with the solution with complete markets and no interventions in financial markets, a well-known property of the Cole-Obstfeld case, where the lack of complete markets is not a constraint on private risk sharing.

Using the fact that $\int_0^1 \hat{\hat{y}}_t^i di = \int_0^1 \hat{\hat{\pi}}_{H,t}^i di = 0$, we are led to the following planning problem:

$$\min \frac{1}{2} \int_0^\infty \int_0^1 e^{-\rho t} \Big[ \alpha_\pi (\hat{\hat{\pi}}_{H,t}^i)^2 + (\hat{\hat{y}}_t^i)^2 + \alpha_\pi (\pi_t^*)^2 + (\hat{y}_t^*)^2 \Big] di\, dt$$

subject to

$$\dot{\hat{\hat{\pi}}}_{H,t}^i = \rho \hat{\hat{\pi}}_{H,t}^i - \kappa_y \hat{\hat{y}}_t^i,$$
$$\dot{\hat{\hat{y}}}_t^i = -\hat{\hat{\pi}}_{H,t}^i - \dot{\tilde{s}}_t^i,$$

$$\hat{\hat{y}}_0^i = -\tilde{s}_0^i,$$

$$\dot{\pi}_t^* = \rho \pi_t^* - \kappa_y \hat{y}_t^*,$$

where the minimization is over the variables $\hat{\hat{\pi}}_{H,t}^i$, $\pi_t^*$, $\hat{\hat{y}}_t^i$, $\hat{y}_t^*$. Note that since $\hat{\hat{\theta}}^i = 0$, the two aggregation constraints $\int_0^1 \hat{\hat{y}}_t^i di = 0$ and $\int_0^1 \hat{\hat{\pi}}_{H,t}^i di = 0$ are automatically verified.

The solution of the planning problem is then simply $\hat{y}_t^* = \pi_t^* = 0$ for the aggregates. This result is a restatement of the result in Benigno (2004) and Gali and Monacelli (2008) that optimal monetary policy in a currency union ensures that the union average output gap and inflation are zero in every period. Monetary policy can be chosen at the union level so that monetary conditions are adapted to the average country. The disaggregated variables $\hat{\hat{\pi}}_{H,t}^i$ and $\hat{\hat{y}}_t^i$ solve the following system of differential equations,

$$\dot{\hat{\hat{\pi}}}_{H,t}^i = \rho \hat{\hat{\pi}}_{H,t}^i - \kappa_y \hat{\hat{y}}_t^i,$$
$$\dot{\hat{\hat{y}}}_t^i = -\hat{\hat{\pi}}_{H,t}^i - \dot{\tilde{s}}_t^i,$$

with initial condition

$$\hat{\hat{y}}_0^i = -\tilde{s}_0^i.$$

**Proposition 18.** *The solution with incomplete markets and no interventions in financial markets ($N\hat{\hat{F}}A_0^i = \hat{\hat{\theta}}^i = 0$) coincides with the solution with complete markets and no interventions in financial markets. In both cases, union-wide aggregates are zero*

$$\hat{y}_t^* = \pi_t^* = 0.$$

## C.4 Transfer Multipliers in a Currency Union

Before solving the normative problem it is useful to review the positive effects of transfers. The next proposition characterizes the response of the economy to a marginal increase in transfers.

**Proposition 19** (Transfer Multipliers)**.** *Let* $\nu = \frac{\rho - \sqrt{\rho^2 + 4\kappa_y}}{2}$*. Transfer multipliers are given by*

$$\frac{\partial \hat{\tilde{y}}_t^i}{\partial N\hat{\tilde{F}}A_0^i} = e^{\nu t}\rho\frac{1-\alpha}{\alpha} - (1 - e^{\nu t})\rho\frac{1}{1+\phi},$$

$$\frac{\partial \hat{\tilde{\pi}}_{H,t}^i}{\partial N\hat{\tilde{F}}A_0^i} = -\nu e^{\nu t}\left[\rho\frac{1-\alpha}{\alpha} + \rho\frac{1}{1+\phi}\right],$$

$$\frac{\partial \hat{\tilde{s}}_t^i}{\partial N\hat{\tilde{F}}A_0^i} = -[1 - e^{\nu t}]\left[\rho\frac{1-\alpha}{\alpha} + \rho\frac{1}{1+\phi}\right].$$

The presence of the discount factor $\rho$ in all these expressions is natural because what matters is the annuity value $\rho N\hat{\tilde{F}}A_0^i$ of the transfer. Note that the terms of trade gap equals accumulated inflation: $\hat{\tilde{s}}_t = -\int_0^t \hat{\tilde{\pi}}_{H,s}^i ds$.

Transfers have opposite effects on output in the short and long run. In the short run, when prices are rigid, there is a Keynesian effect due to the fact that transfers stimulate the demand for home goods: $\frac{\partial \hat{\tilde{y}}_0^i}{\partial N\hat{\tilde{F}}A_0^i} = \rho\frac{1-\alpha}{\alpha}$. In the long run, when prices adjust, the neoclassical wealth effect on labor supply lowers output: $\lim_{t\to\infty} \frac{\partial \hat{\tilde{y}}_t^i}{\partial N\hat{\tilde{F}}A_0^i} = -\rho\frac{1}{1+\phi}$. In the medium run, the speed of adjustment, from the Keynesian short-run response to the neoclassical long-run response, is controlled by the degree of price flexibility $\kappa_y$, which affects $\nu$.[50]

Note that the determinants of the Keynesian and neoclassical wealth effects are very different. The strength of the Keynesian effect hinges on the relative expenditure share of home goods $\frac{1-\alpha}{\alpha}$: the more closed the economy, the larger the Keynesian effect. The strength of the neoclassical wealth effect depends on the elasticity of labor supply $\phi$: the more elastic labor supply, the larger the neoclassical wealth effect.

Positive transfers also increase home inflation. The long-run cumulated response in the price of home produced goods equals $\rho\frac{1-\alpha}{\alpha} + \rho\frac{1}{1+\phi}$. The first term $\rho\frac{1-\alpha}{\alpha}$ comes from the fact that transfers increase the demand for home goods, due to home bias. The second term $\rho\frac{1}{1+\phi}$ is due to a neoclassical wealth effect that reduces labor supply, raising the wage. How fast this increase in the price of home goods occurs depends positively on the flexibility of prices through its effect on $\nu$.[51]

The effects echo the celebrated Transfer Problem controversy of Keynes (1929) and Ohlin (1929). With home bias, a transfer generates a boom when prices are sticky, and a real appreciation of the terms of trade when prices are flexible. The neoclassical wealth effect associated with a transfer comes into play when prices are flexible, and generates an output contraction and a further real

[50]Note that $\nu$ is decreasing in $\kappa_y$, with $\nu = 0$ when prices are rigid ($\kappa_y = 0$), and $\nu = -\infty$ when prices are flexible ($\kappa_y = \infty$).

[51]Recall that $\nu$ is decreasing in the degree of price flexibility $\kappa_y$.

appreciation.

## C.5 Proof of Proposition 14

In this case, $\kappa_y = 0$ and the constraint set boils down to $\hat{y}^i_t = (1-\alpha)\hat{\hat{\theta}}^i - \tilde{s}^i_t$, and we are therefore left with the following component planning problem

$$\min \frac{1}{2}\int_0^\infty e^{-\rho t}\left[((1-\alpha)\hat{\hat{\theta}}^i - \tilde{s}^i_t)^2 + \alpha_\theta(\hat{\hat{\theta}}^i)^2\right]dt.$$

The result follows.

## C.6 Proof of Proposition 19

We use the decomposition of the planning problem given in Appendix C.2. We focus on the component planning problem for a country $i$. We first solve the behavior of an economy for a given transfer $\hat{\hat{\theta}}^i$. Then in Appendix C.7, we solve for the optimal $\hat{\hat{\theta}}^i$.

$$\begin{aligned}
\dot{\hat{\pi}}^i_{H,t} &= \rho\hat{\pi}^i_{H,t} - \kappa_y\hat{y}^i_t - \lambda\alpha\hat{\hat{\theta}}^i, \\
\dot{\hat{y}}^i_t &= -\hat{\pi}^i_{H,t} - \dot{\tilde{s}}^i_t,
\end{aligned}$$

$$\hat{y}^i_0 = (1-\alpha)\hat{\hat{\theta}}^i - \tilde{s}^i_0.$$

Define $E_1 = [1,0]'$ and $E_2 = [0,1]'$. Let $X^i_t = [\hat{\pi}^i_{H,t}, \hat{y}^i_t]'$, $B^i_t = [-\lambda\alpha\hat{\hat{\theta}}^i, -\dot{\tilde{s}}^i_t]' = -\lambda\alpha\hat{\hat{\theta}}^i E_1 - \dot{\tilde{s}}^i_t E_2$. Define $A = \begin{bmatrix} \rho & -\kappa_y \\ -1 & 0 \end{bmatrix}$. Let $\nu = \frac{\rho - \sqrt{\rho^2 + 4\kappa_y}}{2} < 0$ be the (only) negative eigenvalue of $A$, and $X_\nu = [-\nu, 1]'$ and be an eigenvector associated with the negative eigenvalue of $A$. The solution is given by

$$X^i_t = e^{\nu t}\alpha^i_\nu X_\nu - \int_t^\infty e^{A(t-s)}B^i_s ds = e^{\nu t}\alpha^i_\nu X_\nu + \lambda\alpha\hat{\hat{\theta}}^i A^{-1}E_1 + \int_t^\infty \dot{\tilde{s}}^i_u e^{A(t-u)}E_2 du,$$

where

$$\begin{aligned}
X^i_0 + \int_0^\infty e^{-As}B^i_s ds &= \alpha^i_\nu X_\nu, \\
E_2' X^i_0 &= (1-\alpha)\hat{\hat{\theta}}^i - \tilde{s}^i_0.
\end{aligned}$$

We find

$$\alpha^i_\nu = \left[(1-\alpha) - \lambda\alpha E_2' A^{-1}E_1\right]\hat{\hat{\theta}}^i - \tilde{s}^i_0 - \int_0^\infty \dot{\tilde{s}}^i_t E_2' e^{-At}E_2 dt.$$

Using $E_2' A^{-1} E_1 = -\kappa_y^{-1}$, and $E_1' A^{-1} E_1 = 0$, we can infer the path for output $\hat{y}^i_t = E_2' X^i_t$ and inflation $\hat{\pi}^i_{H,t} = E_1' X^i_t$ as follows:

$$\hat{y}^i_t = e^{\nu t}\alpha^i_\nu - \frac{\lambda}{\kappa_y}\alpha\hat{\hat{\theta}}^i + \int_t^\infty \dot{\tilde{s}}^i_u E_2' e^{A(t-u)}E_2 du,$$

$$\hat{\tilde{\pi}}^i_{H,t} = -\nu e^{\nu t}\alpha^i_\nu + \int_t^\infty \dot{\tilde{s}}^i_u E_1' e^{A(t-u)} E_2 du.$$

The results in Proposition 19 follow by specializing these expressions to the case $\tilde{s}_t = 0$.

## C.7 Derivation of the Optimum in Section 6.2

In Appendix C.6, we solved for the behavior of the disaggregated variables $X^i_t = [\hat{\tilde{\pi}}^i_{H,t}, \hat{\tilde{y}}^i_t]'$ for a given $\hat{\tilde{\theta}}^i$. We now solve for the optimal $\hat{\tilde{\theta}}^i$. We apply the results of Appendix C.6 in the particular case $\tilde{s}^i_t = \tilde{s}^i_0 e^{-\psi t}$. We get

$$X^i_t = e^{\nu t}\alpha^i_\nu X_\nu + \lambda\alpha\hat{\tilde{\theta}}^i A^{-1}E_1 - \psi e^{-\psi t}\tilde{s}^i_0(A+\psi I)^{-1}E_2,$$

where

$$\alpha^i_\nu = \left[(1-\alpha) - \lambda\alpha E_2' A^{-1} E_1\right]\hat{\tilde{\theta}}^i - \tilde{s}^i_0 + \psi\tilde{s}^i_0 E_2'(A+\psi I)^{-1}E_2,$$

$E_1 = [1,0]'$, $E_2 = [0,1]'$, $A = \begin{bmatrix} \rho & -\kappa_y \\ -1 & 0 \end{bmatrix}$, $\nu = \frac{\rho - \sqrt{\rho^2+4\kappa_y}}{2} < 0$ is the negative eigenvalue of $A$, and $X_\nu = [-\nu, 1]'$ is an eigenvector associated with the negative eigenvalue of $A$. This yields

$$\hat{\tilde{\pi}}^i_{H,t} = -\alpha_\nu \nu e^{\nu t} - \frac{\psi\kappa_y\tilde{s}^i_0}{(\rho+\psi)\psi - \kappa_y}e^{-\psi t},$$
$$\hat{\tilde{y}}^i_t = \alpha_\nu e^{\nu t} - \frac{\alpha\lambda}{\kappa_y}\hat{\tilde{\theta}}^i - \frac{(\rho+\psi)\psi\tilde{s}^i_0}{(\rho+\psi)\psi - \kappa_y}e^{-\psi t}.$$

We need to solve
$$\min_{\hat{\tilde{\theta}}^i} \frac{1}{2}\frac{\alpha_\theta}{\rho}(\hat{\tilde{\theta}}^i)^2 + \frac{1}{2}\int_0^\infty e^{-\rho t}\left[\alpha_\pi(\hat{\tilde{\pi}}^i_{H,t})^2 + (\hat{\tilde{y}}^i_t)^2\right]dt.$$

This can be rewritten as

$$\begin{aligned}\min_{\hat{\tilde{\theta}}^i} \frac{1}{2}\frac{\alpha_\theta}{\rho}(\hat{\tilde{\theta}}^i)^2 + \frac{1}{2}\int_0^\infty e^{-\rho t}\Bigg[\alpha_\pi\Bigg(\alpha_\nu^2\nu^2 e^{2\nu t} + \Big(\frac{\psi\kappa_y\tilde{s}^i_0}{(\rho+\psi)\psi-\kappa_y}\Big)^2 e^{-2\psi t} + \frac{2\alpha_\nu\nu\psi\kappa_y\tilde{s}^i_0}{(\rho+\psi)\psi-\kappa_y}e^{-(\psi-\nu)t}\Bigg) \\ + \Bigg(\alpha_\nu^2 e^{2\nu t} + \Big(\frac{\alpha\lambda}{\kappa_y}\Big)^2(\hat{\tilde{\theta}}^i)^2 + \Big(\frac{(\rho+\psi)\psi\tilde{s}^i_0}{(\rho+\psi)\psi-\kappa_y}\Big)^2 e^{-2\psi t} - 2\alpha_\nu\frac{\alpha\lambda}{\kappa_y}\hat{\tilde{\theta}}^i e^{\nu t} \\ - \frac{2\alpha_\nu(\rho+\psi)\psi\tilde{s}^i_0}{(\rho+\psi)\psi-\kappa_y}e^{-(\psi-\nu)t} + \frac{2\alpha\lambda(\rho+\psi)\psi\tilde{s}^i_0}{\kappa_y[(\rho+\psi)\psi-\kappa_y]}\hat{\tilde{\theta}}^i e^{-\psi t}\Bigg)\Bigg]dt.\end{aligned}$$

Solving the integrals, we arrive at

$$\begin{aligned}\min_{\hat{\tilde{\theta}}^i}\frac{\alpha_\theta}{2\rho}(\hat{\tilde{\theta}}^i)^2 + \frac{(\alpha\lambda)^2}{2\rho\kappa^2}(\hat{\tilde{\theta}}^i)^2 + \frac{\alpha_\pi\nu^2+1}{2(\rho-2\nu)}(\alpha^i_\nu)^2 + \frac{(\alpha_\pi\kappa^2 + (\rho+\psi)^2)(\psi\tilde{s}^i_0)^2}{2(\rho+2\psi)[(\rho+\psi)\psi-\kappa_y]^2} \\ + \frac{(\alpha_\pi\nu\kappa_y - \rho - \psi)\psi\tilde{s}^i_0}{(\rho+\psi-\nu)[(\rho+\psi)\psi-\kappa_y]}\alpha^i_\nu + \frac{\alpha\lambda\tilde{s}^i_0\psi}{(\rho+\psi)\psi\kappa_y - \kappa_y^2}\hat{\tilde{\theta}}^i - \frac{\alpha\lambda}{(\rho-\nu)\kappa_y}\alpha^i_\nu\hat{\tilde{\theta}}^i.\end{aligned}$$

The solution is

$$\hat{\hat{\theta}}^i = \frac{\left[\frac{\alpha\lambda}{(\rho-\nu)\kappa_y} - \frac{\alpha_\pi \nu^2+1}{\rho-2\nu}(1-\alpha+\frac{\alpha\lambda}{\kappa_y})\right]\frac{\kappa_y \tilde{s}_0^i}{(\rho+\psi)\psi-\kappa_y} - \frac{\alpha\lambda\psi\tilde{s}_0^i}{(\rho+\psi)\psi\kappa_y-\kappa_y^2} - \frac{(\alpha_\pi\nu\kappa_y-\rho-\psi)\psi\tilde{s}_0^i(1-\alpha+\frac{\alpha\lambda}{\kappa_y})}{(\rho+\psi-\nu)((\rho+\psi)\psi-\kappa_y)}}{\frac{\alpha_\theta}{\rho}+\frac{(\alpha\lambda)^2}{\rho\kappa_y^2}+\frac{\alpha_\pi\nu^2+1}{\rho-2\nu}(1-\alpha+\frac{\alpha\lambda}{\kappa_y})^2-\frac{2\alpha\lambda}{(\rho-\nu)\kappa_y}(1-\alpha+\frac{\alpha\lambda}{\kappa_y})}.$$

### C.8 Proof of Proposition 15

In the closed-economy limit, as $\alpha \to 0$, we have $\alpha_\theta = 0$, and we see directly from Section C.7 that

$$\hat{\hat{\theta}}^i = -\frac{\kappa_y \tilde{s}_0^i}{(\rho+\psi)\psi-\kappa_y} - \frac{\rho-2\nu}{\alpha_\pi\nu^2+1}\frac{(\alpha_\pi\nu\kappa_y-\rho-\psi)\psi\tilde{s}_0^i}{(\rho+\psi-\nu)[(\rho+\psi)\psi-\kappa_y]}.$$

## D Online Appendix: Extensions of Dynamic Model in Sections 4-5 for Section 6

In this appendix, we detail two extensions of the dynamic model by introducing hand-to-mouth consumers and alternative macroeconomic instruments. This appendix is organized as follows. Section D.1 outlines the extended model. Section D.2 derives the allocations given exogenous policies. Section D.3 derives the loss function. Sections D.4-D.9 set up the planning problem as well as the solution method for alternative macroeconomic instruments: international transfers, capital controls, government spending, redistribution, deficits, and then all domestic fiscal policy instruments combined. Section D.10 specializes the previous sections to the case of no hand-to-mouth consumers. We first setup up the model allowing only for international transfers, government spending, and capital controls in Sections D.4-D.6. We then generalize it to allow for redistribution and deficits in Sections D.7-D.9.

### D.1 Model

This is a model of fiscal unions with two types of agents: hand-to-mouth (HtM) agents of measure $\chi \in [0,1]$ and optimizers of measure $1-\chi$. We assume throughout that countries are in a currency union. From the get go, we study idiosyncratic productivity shocks in a given country $i$, which will correspond to Home in the exposition of Sections 4-5.[52] We abstract from idiosyncratic shocks in other countries and from aggregate shocks. As a result, union-wide aggregates and policies are constant at their steady state where outputs in all countries are equal to one.

[52]There is no rest of the world (RoW) beyond the members of the fiscal union.

### D.1.1 Households

The two types of households in country $i$ share the same preferences, given by

$$\int_0^\infty e^{-\rho t}\left((1-v)\ln(C_t^{i,s}) - \frac{1}{1+\phi}(N_t^{i,s})^{1+\phi} + v\ln(G_t^i)\right)dt, \tag{44}$$

where $s \in \{o, r\}$ designates the agent type; $o$ denotes optimizers and $r$ HtM agents. As in the main paper, $N_t^{i,s}$ denotes labor supplied by agent $s$ in country $i$, and the consumption basket is given by

$$C_t^{i,s} = \left[(1-\alpha)^{\frac{1}{\eta}}(C_{H,t}^{i,s})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}(C_{F,t}^{i,s})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}},$$

where

$$C_{H,t}^{i,s} = \left(\int_0^1 (C_{H,t}^{i,s}(j))^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}}$$

is the basket of home goods consumed by agent $s$ in country $i$, with $j$ denoting an individual good variety. Similarly,

$$C_{F,t}^{i,s} = \left(\int_0^1 (C_{k,t}^{i,s})^{\frac{\gamma-1}{\gamma}} dk\right)^{\frac{\gamma}{\gamma-1}}$$

denotes the consumption index of imported goods consumed by agent $s$. We also have

$$C_{k,t}^{i,s} = \left(\int_0^1 (C_{k,t}^{i,s}(j))^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}}.$$

Throughout the paper, we focus exclusively on the Cole-Obstfeld calibration and subsequently set $\eta = \gamma = 1$. As is standard, the parameter $\alpha$ indexes the degree of home bias in goods consumption. As $\alpha \to 0$, the share of imported goods in domestic consumption vanishes. As $\alpha \to 1$, the share of home goods vanishes. The two agent types differ with respect to their per-period budget constraints, which we now describe.

### D.1.2 Budget Constraints, Transfers, and NFA

In our analysis of transfers, country $i$ will receive a transfer in every period from the remaining members of the fiscal union after the realization of a productivity shock. We impose that each optimizer and each HtM agent receive an equal share of this transfer in every period. To that end, the government will tax optimizers and rebate the proceeds to the HtM agents accordingly.

For optimizers, only the net present value of this transfer matters. We encode the net present value of this transfer in the initial net foreign asset position of country $i$, $NFA_0^i$, which we will later derive from the optimizers' budget constraint explicitly.

We start our discussion of transfers by considering the optimizers' budget constraint. Each optimizer faces the per-period constraint of

$$\int_0^1 P^i_{H,t}(j)C^{i,o}_{H,t}(j)dj + \int_0^1 \int_0^1 P^k_t(j)C^i_{k,t}(j)dkdj + D^i_{i,t+1} + \int_0^1 D^i_{k,t+1}dk \tag{45}$$

$$\leq W^i_t N^{i,o}_t + (1+i^i_{t-1})D^i_{i,t} + \int_0^1 (1+\tau^{cap,i}_{t-1})(1+i^*)D^i_{k,t}dk + \frac{1}{1-\chi}P^i_t Z^i_{H,t} + P^i_t T^{cap,i,o}_t + P^i_t T^{i,o}_t + P^i_t \mathcal{T}^i_t$$

In this expression, $D^i_{k,t+1}$ is country $i$'s holdings of country $k$ bonds, $D^i_{i,t+1}$ is country $i$'s holdings of country $i$ bonds, $P^i_t$ is the domestic consumer price index, $W^i_t$ is the domestic wage, $P^i_{H,t}(j)$ is the price of variety $j$ of the home good in country $i$, $P^k_t(j)$ is the price of variety $j$ of the good of country $k$. Besides labor income and income on bond holdings, the budget constraint of optimizers features three additional sources of transfers. We denote by $P^i_t Z^i_{H,t} = P^i_{H,t}Y^i_t - W^i_t N^i_t$ nominal domestic profits, where $P^i_{H,t}$ is the price of the home good in country $i$. Note that optimizers receive all the profits in the economy.

We now turn to the policy instruments that show up in the budget constraint. $T^{i,o}_t$ is the real transfer each optimizer in country $i$ receives from the domestic government. $\mathcal{T}^i_t$ denotes the transfer that each optimizer receives from the fiscal union. Note that for convenience, we adopt the convention that international transfers are paid to optimizers and then passed through to HtM consumers via government taxes and transfers. We could equivalently have specified that international transfers are paid to the government and then distributed to optimizers and HtM consumers. Finally $i^i_{t-1}$ is the interest rate in country $i$ and $i^*$ in the rest of the currency union with $\beta(1+i^*) = 1$. These can be different because of capital controls in the form of taxes on capital inflows $\tau^{cap,i}_{t-1}$ in country $i$. We rebate the proceeds from the capital controls lump sum to optimizers using $T^{cap,i,o}_t = -\int_0^1 \tau^{cap,i}_{t-1}(1+i^*)D^i_{k,t}dk$, and from now on, we omit these flows from the government budget constraint.

The HtM budget constraint given by

$$P^i_t C^{i,r}_t = W^i_t N^{i,r}_t + P^i_t T^{i,r}_t. \tag{46}$$

In our baseline model (we relax this later), the government must balance its budget in every period $t$, yielding the following government budget constraint

$$P^i_{H,t}G^i_t + \chi P^i_t T^{i,r}_t + (1-\chi)P^i_t T^{i,o}_t = 0,$$

where $G^i_t$ are real government purchases of domestic goods.

We impose that the lump sum transfers to optimizers $T^{i,o}_t$ and to hand-to-mouth agents $T^{i,r}_t$ are such that after these transfers: (i) each agent receives and equal share of domestic profits $P^i_t Z^i_{H,t}$ in every period; (ii) each agent receives and equal share of the international transfer $P^i_t \mathcal{T}^i_t$ in every period; (iii) each agent contributes equally towards the financing of domestic government spending $P^i_{H,t}G^i_t$ in every period.

For convenience, we introduce a separate notation, $\tau^{i,o}_t$ and $\tau^{i,r}_t$, for the part of these transfers that ensures equal distribution of profits across agents as described in point (i) above:

$$T_t^{i,r} - \tau_t^{i,r} = Z_{H,t}^i, \tag{47}$$

$$T_t^{i,o} - \tau_t^{i,o} = -\frac{\chi}{1-\chi} Z_{H,t}^i. \tag{48}$$

The government budget constraint can then equivalently be restated as

$$P_{H,t}^i G_t^i + \chi P_t^i \tau_t^{i,r} + (1-\chi) P_t^i \tau_t^{i,o} = 0. \tag{49}$$

We denote the net present value of international transfers $P_t^i \mathcal{T}_t^i$ discounted as the union interest rate $i^* = \beta^{-1} - 1$ in units of the initial domestic price index, $P_0^i$, which is the same thing as the price index for imports in country $i$, $P_F^i$, in every period:[53]

$$NFA_0^i = \sum_{t=0}^{\infty} \beta^t \frac{P_t^i \mathcal{T}_t^i}{P_0^i} = \sum_{t=0}^{\infty} \beta^t \frac{P_t^i \mathcal{T}_t^i}{P_F^i} = \sum_{t=0}^{\infty} \beta^t (S_t^i)^{-(1-\alpha)} \mathcal{T}_t^i.$$

Points (ii) and (iii) above imply that

$$NFA_0^i = \sum_{t=0}^{\infty} \beta^t [\tau_t^{i,r} + (S_t^i)^{-\alpha} G_t^i]. \tag{50}$$

In equilibrium, $NFA_0$ is equal to the net present value of trade balances, as required by the current account identity. Net exports are given by

$$NX_t^i = P_{H,t}^i Y_t^i - P_t^i C_t^i - P_{H,t}^i G_t^i. \tag{51}$$

Therefore, we have

$$NFA_0^i = -\sum_{t=0}^{\infty} \beta^t \frac{NX_t^i}{P_F^i} = -\sum_{t=0}^{\infty} \beta^t \Big( (S_t^i)^{-1} Y_t^i - (S_t^i)^{-(1-\alpha)} C_t^i - (S_t^i)^{-1} G_t^i \Big). \tag{52}$$

#### D.1.3 Firms

Our descriptions of firms here is identical to that in the body of the paper. Firms produce differentiated goods with a linear technology given by

$$Y_t^i(j) = A_t^i N_t^i(j), \tag{53}$$

where $A_t^i$ is the productivity in the home country. To offset the monopoly distortion, we introduce a constant employment tax $1 + \tau^L$ so that the real marginal cost deflated by the home producer price index is given by

$$MC_t^i = \frac{1+\tau^L}{A_t^i} \frac{W_t^i}{P_{H,t}^i}. \tag{54}$$

[53]In the absence of transfers we simply have $NFA_0^i = \mathcal{T}_t^i = 0$ and $\tau_t^{i,r} = \tau_t^{i,o} = -(S_t^i)^{-\alpha} G_t^i$.

This employment tax is set cooperatively at a symmetric steady state with flexible prices. As we show later, this implies that $\tau^L = -\frac{1}{\epsilon}$ or $\frac{\epsilon}{\epsilon-1}(1+\tau^L) = 1$.

As in the main paper, we assume that the Law of One Price holds at all times and maintain Producer Currency Pricing. Our price setting assumptions here are also analogous to those in the paper, except that we now use continuous time. As a result, we can write the supply side equilibrium conditions of the model as

$$\begin{aligned}
\frac{1-\delta(\Pi^i_{H,t})^{\epsilon-1}}{1-\delta} &= \left(\frac{F^i_t}{K^i_t}\right)^{\epsilon-1} \\
K^i_t &= \frac{\epsilon}{\epsilon-1}\frac{1+\tau^L}{A^i_t}Y^i_t(N^i_t)^{\phi} + \beta\delta(\Pi^i_{H,t+1})^{\epsilon}K^i_{t+1} \\
F^i_t &= \frac{Y^i_t Q^i_t}{C^i_t S^i_t} + \beta\delta(\Pi^i_{H,t+1})^{\epsilon}F^i_{t+1} \\
\Pi^i_{H,t} &= \frac{P^i_{H,t}}{P^i_{H,t-1}}.
\end{aligned}$$

### D.1.4 Terms of trade, Exchange rates, and UIP

We briefly restate here our conventions about price indices and exchange rates from the body of the paper.

We can write the consumer price index (CPI) as

$$P^i_t = \left[(1-\alpha)(P^i_{H,t})^{1-\eta} + \alpha(P^i_F)^{1-\eta}\right]^{\frac{1}{1-\eta}}, \tag{55}$$

and the home producer price index (PPI) as

$$P^i_{H,t} = \left(\int_0^1 P^i_{H,t}(j)^{1-\epsilon}dj\right)^{\frac{1}{1-\epsilon}}.$$

The import price index is given by

$$P^i_F = \left(\int_0^1 (P^i_k)^{1-\gamma}dk\right)^{\frac{1}{1-\gamma}},$$

with $P^i_k = \left(\int_0^1 P^i_k(j)^{1-\epsilon}dj\right)^{\frac{1}{1-\epsilon}}$ denoting country $i$'s PPI.

Under the Law of One Price, we have $P^i_{H,t} = P^k_{H,t}$ where $P^k_{H,t}$ is country $k$'s domestic PPI in terms of country $k$'s currency. Therefore, $P^i_F = P^*$, where $P^* = \left(\int_0^1 (P^k_H)^{1-\gamma}dk\right)^{\frac{1}{1-\gamma}}$, is the world price index. As is standard, we define the terms of trade by $S^i_t = \frac{P^i_F}{P^i_{H,t}}$, which allows us to write the home CPI as

$$P^i_t = P^i_{H,t}\left(1-\alpha+\alpha(S^i_t)^{1-\eta}\right)^{\frac{1}{1-\eta}}.$$

The real exchange rate with country $k$ is given by $Q^i_{k,t} = \frac{P^k}{P^i_t}$ where $P^k$ is country $k$'s CPI. The effective real exchange rate is therefore $Q^i_t = \frac{P^*}{P^i_t}$.

The UIP condition is

$$1 + i^i_t = (1 + \tau^{cap,i}_t)(1 + i^*).$$

We define the Pareto weights $\Theta^i_t$ by

$$C^{i,o}_t = \Theta^i_t C^{*,o} Q^i_t.$$

The UIP condition implies a direct mapping between capital controls and these Pareto weights

$$\frac{\Theta^i_{t+1}}{\Theta^i_t} = 1 + \tau^{cap,i}_t.$$

From now on, we simply describe capital controls in terms of the Pareto weights $\Theta^i_t$.

### D.1.5 Steady State

We postulate a symmetric steady state such that all countries $i \in [0,1]$ within the monetary union are symmetric so that $S^i = \Theta^i = 1$ and for simplicity set $A^i = A^k = A^* = 1$ . The social planner sets the labor tax $\tau^L$ so that the steady state allocation is efficient. In particular, the centralized problem to pin down the efficient steady state allocation can be written as

$$\max (1-\chi)(1-v)\ln(C^{i,o}) + \chi(1-v)\ln(C^{i,r}) - \frac{\chi}{1+\phi}(N^{i,r})^{1+\phi} - \frac{1-\chi}{1+\phi}(N^{i,o})^{1+\phi} + v\ln(G^i), \quad (56)$$

subject to technology and resource constraints

$$\begin{aligned} Y^i &= N^i \\ Y^i &= C^i + G^i \\ C^i &= \chi C^{i,r} + (1-\chi)C^{i,o} \\ N^i &= \chi N^{i,r} + (1-\chi)N^{i,o}. \end{aligned}$$

The first-order conditions with respect to $C^{i,o}, C^{i,r}, N^{i,o}$ and $N^{i,r}$ immediately imply that agents should be symmetric. That is, the efficient steady state is characterized by $C^{i,o} = C^{i,r} = C^i$ and $N^{i,o} = N^{i,r} = N^i$. The remaining first-order conditions can be rearranged so that

$$\begin{aligned} (1-v)G^i &= vC^i \\ (N^i)^\phi &= \frac{1-v}{C^i}. \end{aligned}$$

The first of these, together with the resource constraint, implies that

$$v = \frac{G^i}{Y^i}, \quad 1 - v = \frac{C^i}{Y^i}. \tag{57}$$

The second, using the technology constraint, yields

$$N^i = 1, \quad Y^i = 1. \tag{58}$$

It is straightforward to show, as in Gali-Monacelli (2008) for example, that the efficient steady state can be decentralized with an employment tax that solves $\frac{\epsilon}{\epsilon-1}(1+\tau^L) = 1$, which implies $\tau^L = -\frac{1}{\epsilon}$.

Let $Z^i_{H,t}$ denote profits of domestic firms, defined as

$$P^i_t Z^i_{H,t} = P^i_{H,t} Y^i_t - W^i_t N^i_t.$$

In steady state, we have

$$Z^i_H = (S^i)^{-\alpha} Y^i - \frac{1}{1-v} C^{i,o} (N^{i,o})^{\phi} N^i = Y^i - (1-v) Y^i \frac{1}{1-v} = 0, \tag{59}$$

where the second line follows by setting $S^i = 1$ and $N^{i,o} = N^i = 1$.

It follows from our discussion in the previous subsection that in steady state, when trade is balanced and there are no transfers, we have

$$\tau^{i,r} = \tau^{i,o} = -(S^i)^{-\alpha} G^i = -vY^i \tag{60}$$

which is consistent with the government budget constraint. This furthermore implies that

$$T^{i,r} = T^{i,o} + \frac{1}{1-\chi} Z^i_H = T^{i,o} = -vY^i. \tag{61}$$

#### D.1.6 Linearized Equilibrium

As in the baseline model of Sections(4)-(5) and for the same reasons, we move to continuous time.

**Notation.** We begin by defining some useful notation. For all variables $X_t$, we drop the time subscript to denote the steady state by $X$. For $X_t \in \{T^{i,o}_t, T^{i,r}_t, \tau^{i,o}_t, \tau^{i,r}_t, NFA^i_0, NX^i_t\}$, we define $x_t = X_t - X$ and $\tilde{x}_t = \tilde{X}_t - X$, where $\tilde{X}_t$ denotes the natural allocation.[54] For all other variables, we define $x_t = \ln(X_t) - \ln(X)$ and $\tilde{x}_t = \ln(\tilde{X}_t) - \ln(X)$.

Furthermore, it will be convenient to work with the allocation in gaps from the natural allocation. To that end, we define for all variables $\hat{x}_t = x_t - \tilde{x}_t$. We will furthermore normalize our allocation by writing it in gaps from union aggregates. For all country $i$ variables, we let $\hat{\hat{x}}^i_t = \hat{x}^i_t - \hat{x}^*_t$, where

[54] We will abuse notation and let $\tau^{i,o}_t$ and $\tau^{i,r}_t$ refer to both the level of rebates and the normalized deviation from steady state. Throughout, it will be obvious which of the two we refer to.

$\hat{x}^*_t = \int_0^1 \hat{x}^i_t di$ is the union aggregate of the log-linearized variable $x^i_t$ in gaps from the natural.[55]

Finally since inflation under flexible prices is not well defined, we define $\hat{\pi}^i = \pi^i$ and $\hat{\hat{\pi}}^i = \pi^i - \pi^*$ for both PPI and CPI inflation.

**Labor supply.** We start with the households' labor supply condition,

$$(1-v)\frac{W^i_t}{P^i_t} = C^{i,o}_t (N^{i,o}_t)^\phi = C^{i,r}_t (N^{i,r}_t)^\phi. \tag{62}$$

Linearizing, we have

$$w^i_t - p^i_t = c^{i,o}_t + \phi n^{i,o}_t = c^{i,r}_t + \phi n^{i,r}_t. \tag{63}$$

Therefore, we have

$$\begin{aligned}
&\text{(Natural)} && \tilde{c}^{i,o}_t + \phi \tilde{n}^{i,o}_t = \tilde{c}^{i,r}_t + \phi \tilde{n}^{i,r}_t \\
&\text{(Gaps from natural)} && \hat{c}^{i,o}_t + \phi \hat{n}^{i,o}_t = \hat{c}^{i,r}_t + \phi \hat{n}^{i,r}_t \\
&\text{(Gaps from union)} && \hat{\hat{c}}^{i,o}_t + \phi \hat{\hat{n}}^{i,o}_t = \hat{\hat{c}}^{i,r}_t + \phi \hat{\hat{n}}^{i,r}_t.
\end{aligned}$$

**Backus-Smith.** We have

$$C^{i,o}_t = \Theta^i_t C^{*,o}_t Q^i_t, \tag{64}$$

where $Q^i_t = (S^i_t)^{1-\alpha}$.[56] Linearizing, we have

$$c^{i,o}_t = \theta^i_t + c^{*,o}_t + (1-\alpha)s^i_t. \tag{65}$$

Therefore, we have

$$\begin{aligned}
&\text{(Natural)} && \tilde{c}^{i,o}_t = \tilde{\theta}^i_t + (1-\alpha)\tilde{s}^i_t \\
&\text{(Gaps from natural)} && \hat{c}^{i,o}_t = \hat{\theta}^i_t + c^{*,o}_t + (1-\alpha)\hat{s}^i_t \\
&\text{(Gaps from union)} && \hat{\hat{c}}^{i,o}_t = \hat{\hat{\theta}}^i_t + (1-\alpha)\hat{\hat{s}}^i_t.
\end{aligned}$$

**Aggregation.** We have

$$c^i_t = \chi c^{i,r}_t + (1-\chi)c^{i,o}_t \tag{66}$$

and

$$n^i_t = \chi n^{i,r}_t + (1-\chi)n^{i,o}_t. \tag{67}$$

**Production.** We have

[55] We note here that under idiosyncratic shocks, we will have $\tilde{x}^*_t = \int_0^1 \tilde{x}^i_t di = 0$ for all natural variables. That is, any idiosyncratic shock to measure-0 country $i$ will not affect the union's natural allocation.

[56] $C^{*,o}_t = \int_0^1 C^{i,o}_t di$ denotes the union aggregate of optimizers' consumption. For double-superscripts (country and agent type), we write the country index ($i$ or $*$) first, followed by the agent index ($o$ or $r$).

$$Y_t^i = A_t^i N_t^i. \tag{68}$$

Linearizing, we have

$$y_t^i = a_t^i + n_t^i. \tag{69}$$

Therefore, we have

$$\begin{aligned}
&\text{(Natural)} && \tilde{y}_t^i = a_t^i + \tilde{n}_t^i \\
&\text{(Gaps from natural)} && \hat{y}_t^i = \hat{n}_t^i \\
&\text{(Gaps from union)} && \hat{\hat{y}}_t^i = \hat{\hat{n}}_t^i.
\end{aligned}$$

**HtM budget constraint.** We have

$$P_t^i T_t^{i,r} + W_t^i N_t^{i,r} = P_t^i C_t^{i,r}, \tag{70}$$

or

$$T_t^{i,r} + \frac{1}{1-v} C_t^{i,r} (N_t^{i,r})^{1+\phi} = C_t^{i,r}, \tag{71}$$

Linearizing, we have

$$\frac{T^{i,r}}{Y^i} + t_t^{i,r} + (1+\phi) n_t^{i,r} + v c_t^{i,r} + v = 0, \tag{72}$$

and since $\frac{T^{i,r}}{Y^i} = -v$, we have

$$t_t^{i,r} + (1+\phi) n_t^{i,r} + v c_t^{i,r} = 0. \tag{73}$$

Therefore, we have

$$\begin{aligned}
&\text{(Natural)} && \tilde{t}_t^{i,r} + (1+\phi)\tilde{n}_t^{i,r} + v\tilde{c}_t^{i,r} = 0 \\
&\text{(Gaps from natural)} && \hat{t}_t^{i,r} + (1+\phi)\hat{n}_t^{i,r} + v\hat{c}_t^{i,r} = 0 \\
&\text{(Gaps from union)} && \hat{\hat{t}}_t^{i,r} + (1+\phi)\hat{\hat{n}}_t^{i,r} + v\hat{\hat{c}}_t^{i,r} = 0.
\end{aligned}$$

**Aggregate demand.** We have

$$Y_t^i = (1-\alpha) C_t^i (S_t^i)^\alpha + \alpha \frac{C_t^{i,o} (S_t^i)^\alpha}{\Theta_t^i} \int_0^1 \frac{\Theta_t^j C_t^j}{C_t^{j,o}} dj + G_t^i. \tag{74}$$

Linearizing, we have

$$\frac{1}{1-v} y_t^i = (1-\alpha)\big(\theta_t^i + c_t^i + c_t^{*,o} - c_t^{i,o}\big) + s_t^i + \alpha c_t^* + \frac{v}{1-v} g_t^i. \tag{75}$$

Therefore, we have

$$
\begin{aligned}
&\text{(Natural)} && \frac{1}{1-v}\tilde{y}_t^i = \tilde{s}_t^i + (1-\alpha)\left(\tilde{\theta}_t^i + \tilde{c}_t^i - \tilde{c}_t^{i,o}\right) + \frac{v}{1-v}\tilde{g}_t^i \\
&\text{(Gaps from natural)} && \frac{1}{1-v}\hat{y}_t^i = \hat{s}_t^i + \alpha c_t^* + (1-\alpha)\left(\hat{\theta}_t^i + \hat{c}_t^i + c_t^{*,o} - \hat{c}_t^{i,o}\right) + \frac{v}{1-v}\hat{g}_t^i \\
&\text{(Gaps from union)} && \frac{1}{1-v}\hat{\hat{y}}_t^i = \hat{\hat{s}}_t^i + (1-\alpha)\left(\hat{\hat{\theta}}_t^i + \hat{\hat{c}}_t^i - \hat{\hat{c}}_t^{i,o}\right) + \frac{v}{1-v}\hat{\hat{g}}_t^i.
\end{aligned}
$$

**NFA.** Log-linearizing (52), we find

$$
NFA_0^i = -\int_0^\infty e^{-\rho t}\left(y_t^i - \alpha(1-v)s_t^i - (1-v)c_t^i - v g_t^i\right)dt, \tag{76}
$$

From (76), we have

$$
\begin{aligned}
&\text{(Natural)} && N\tilde{F}A_0^i = -\int_0^\infty e^{-\rho t}\left(\tilde{y}_t^i - \alpha(1-v)\tilde{s}_t^i - (1-v)\tilde{c}_t^i - v\tilde{g}_t^i\right)dt = 0 \\
&\text{(Gaps from natural)} && N\hat{F}A_0^i = -\int_0^\infty e^{-\rho t}\left(\hat{y}_t^i - \alpha(1-v)\hat{s}_t^i - (1-v)\hat{c}_t^i - v\hat{g}_t^i\right)dt \\
&\text{(Gaps from union)} && N\hat{\hat{F}}A_0^i = -\int_0^\infty e^{-\rho t}\left(\hat{\hat{y}}_t^i - \alpha(1-v)\hat{\hat{s}}_t^i - (1-v)\hat{\hat{c}}_t^i - v\hat{\hat{g}}_t^i\right)dt.
\end{aligned}
$$

**Transfer conversion.** We have

$$
\tau_t^{i,r} = T_t^{i,r} - Z_{H,t}^i = T_t^{i,r} - Y_t^i(S_t^i)^{-\alpha} + \frac{1}{1-v}C_t^{i,o}(N_t^{i,o})^{\phi}N_t^i. \tag{77}
$$

Linearizing, we have

$$
\tau_t^{i,r} = t_t^{i,r} - \left(y_t^i - \alpha s_t^i - c_t^{i,o} - \phi n_t^{i,o} - n_t^i\right). \tag{78}
$$

Therefore, we have

$$
\begin{aligned}
&\text{(Natural)} && \tilde{\tau}_t^{i,r} = \tilde{t}_t^{i,r} - \left(\tilde{y}_t^i - \alpha\tilde{s}_t^i - \tilde{c}_t^{i,o} - \phi\tilde{n}_t^{i,o} - \tilde{n}_t^i\right) \\
&\text{(Gaps from natural)} && \hat{\tau}_t^{i,r} = \hat{t}_t^{i,r} - \left(\hat{y}_t^i - \alpha\hat{s}_t^i - \hat{c}_t^{i,o} - \phi\hat{n}_t^{i,o} - \hat{n}_t^i\right) \\
&\text{(Gaps from union)} && \hat{\hat{\tau}}_t^{i,r} = \hat{\hat{t}}_t^{i,r} - \left(\hat{\hat{y}}_t^i - \alpha\hat{\hat{s}}_t^i - \hat{\hat{c}}_t^{i,o} - \phi\hat{\hat{n}}_t^{i,o} - \hat{\hat{n}}_t^i\right).
\end{aligned}
$$

Similar conditions hold for $\tau_t^{i,o}$.

**Government budget constraint.** We have

$$
0 = P_{H,t}^i G_t^i + \chi P_t^i \tau_t^{i,r} + (1-\chi)P_t^i\tau_t^{i,o}, \tag{79}
$$

which we can linearize as

$$
\alpha v s_t^i = \chi\tau_t^{i,r} + (1-\chi)\tau_t^{i,o} + v g_t^i. \tag{80}
$$

Therefore, we have

$$\begin{aligned}
&\text{(Natural)} && \alpha v\tilde{s}_t^i = \chi\tilde{\tau}_t^{i,r} + (1-\chi)\tilde{\tau}_t^{i,o} + v\tilde{g}_t^i \\
&\text{(Gaps from natural)} && \alpha v\hat{s}_t^i = \chi\hat{\tau}_t^{i,r} + (1-\chi)\hat{\tau}_t^{i,o} + v\hat{g}_t^i \\
&\text{(Gaps from union)} && \alpha v\hat{\hat{s}}_t^i = \chi\hat{\hat{\tau}}_t^{i,r} + (1-\chi)\hat{\hat{\tau}}_t^{i,o} + v\hat{\hat{g}}_t^i.
\end{aligned}$$

**Symmetric fiscal union transfers.** We want to rebate the transfers received by optimizers from the union to HtM agents so that each agent receives the same amount. This condition is given by

$$NFA_0^i = \int_0^\infty e^{-\rho t}[\tau_t^{i,r} + (S_t^i)^{-\alpha}G_t^i]dt. \tag{81}$$

Linearizing yields

$$NFA_0^i = \int_0^\infty e^{-\rho t}[\tau_t^{i,r} + vg_t^i - v\alpha s_t^i]dt. \tag{82}$$

This implies

$$\begin{aligned}
&\text{(Natural)} && N\tilde{F}A_0^i = \int_0^\infty e^{-\rho t}[\tilde{\tau}_t^{i,r} + v\tilde{g}_t^i - v\alpha\tilde{s}_t^i]dt = 0 \\
&\text{(Gaps from natural)} && N\hat{F}A_0^i = \int_0^\infty e^{-\rho t}[\hat{\tau}_t^{i,r} + v\hat{g}_t^i - v\alpha\hat{s}_t^i]dt = 0 \\
&\text{(Gaps from union)} && N\hat{\hat{F}}A_0^i = \int_0^\infty e^{-\rho t}[\hat{\hat{\tau}}_t^{i,r} + v\hat{\hat{g}}_t^i - v\alpha\hat{\hat{s}}_t^i]dt = 0.
\end{aligned}$$

**Supply side with flexible prices.** Under the natural, the supply side simplifies to the single equation

$$\tilde{y}_t^i = \tilde{c}_t^i + \alpha\tilde{s}_t^i + (1+\phi)\tilde{n}_t^i. \tag{83}$$

**Supply side with sticky prices.** Let $\lambda = \rho_\delta(\rho + \rho_\delta)$, where $\rho_\delta$ is the arrival rate of price changes. Using the supply-side equilibrium conditions in continuous time, we can write the Phillips Curve as

$$\begin{aligned}
&\text{(Gaps from natural)} && \dot{\hat{\pi}}_{H,t}^i = \rho\hat{\pi}_{H,t}^i - \lambda\left(\phi\hat{n}_t^i + \hat{c}_t^i + \alpha\hat{s}_t^i\right) \\
&\text{(Gaps from union)} && \dot{\hat{\hat{\pi}}}_{H,t}^i = \rho\hat{\hat{\pi}}_{H,t}^i - \lambda\left(\phi\hat{\hat{n}}_t^i + \hat{\hat{c}}_t^i + \alpha\hat{\hat{s}}_t^i\right).
\end{aligned}$$

## D.2 Allocations Given Exogenous Policies

We begin this section by characterizing and solving for the natural allocation. We then summarize the allocation with policy in gaps from the natural, which will be our exclusive focus for all subsequent analysis. Furthermore, we derive the (IS) and (NKPC) equations, an initial condition with which we can solve the dynamical system later on, as well as country $i$'s generalized budget constraint. Finally, we solve analytically the allocation under sticky prices but without policy, which we will later use for several of our numerical exercises.

### D.2.1 Natural Allocation

We define the natural as the allocation with flexible prices and with passive policy. As discussed in the main text, we allow government spending and taxes to move with the technology shock to ensure that the flexible price allocation is efficient. At the natural allocation, taxes are uniform across agents and the government budget is balanced in every period. This implies $\tilde{g}_t^i = a_t^i$ and $\tilde{\theta}_t^i = \tilde{\theta}^i$ for all $i$. Furthermore, there are no fiscal union transfers so that $N\tilde{F}A_0^i = 0$. This implies that $\tilde{\tau}_t^{i,r}$ and $\tilde{\tau}_t^{i,o}$ merely denote tax rebates that are equally imposed on both agents to balance the per-period government budget constraint. We note here again that under idiosyncratic shocks to any measure-0 country $i$, the union-wide natural allocation remains unaffected and $\tilde{x}_t^* = 0$ for all variables $\tilde{x}_t$.

The natural allocation is given by the system of equations

$$\tilde{y}_t^i = a_t^i + \tilde{n}_t^i \tag{84}$$

$$\tilde{c}_t^i = \chi \tilde{c}_t^{i,r} + (1-\chi)\tilde{c}_t^{i,o} \tag{85}$$

$$\tilde{n}_t^i = \chi \tilde{n}_t^{i,r} + (1-\chi)\tilde{n}_t^{i,o} \tag{86}$$

$$\tilde{c}_t^{i,o} + \phi \tilde{n}_t^{i,o} = \tilde{c}_t^{i,r} + \phi \tilde{n}_t^{i,r} \tag{87}$$

$$\tilde{c}_t^{i,o} = \tilde{\theta}_t^i + (1-\alpha)\tilde{s}_t^i \tag{88}$$

$$0 = \tilde{t}_t^{i,r} + (1+\phi)\tilde{n}_t^{i,r} + v\tilde{c}_t^{i,r} \tag{89}$$

$$\frac{1}{1-v}\tilde{y}_t^i = \tilde{s}_t^i + (1-\alpha)\left(\tilde{\theta}_t^i + \tilde{c}_t^i - \tilde{c}_t^{i,o}\right) + \frac{v}{1-v}\tilde{g}_t^i \tag{90}$$

$$\tilde{y}_t^i = \tilde{c}_t^i + \alpha \tilde{s}_t^i + (1+\phi)\tilde{n}_t^i \tag{91}$$

as well as the following conditions for transfers and the NFA

$$\alpha v \tilde{s}_t^i = \chi \tilde{\tau}_t^{i,r} + (1-\chi)\tilde{\tau}_t^{i,o} + v\tilde{g}_t^i \tag{92}$$

$$\tilde{\tau}_t^{i,o} = \tilde{t}_t^{i,o} + \frac{\chi}{1-\chi}\left(\tilde{y}_t^i - \alpha \tilde{s}_t^i - \tilde{c}_t^{i,o} - \phi \tilde{n}_t^{i,o} - \tilde{n}_t^i\right) \tag{93}$$

$$\tilde{\tau}_t^{i,r} = \tilde{t}_t^{i,r} - \left(\tilde{y}_t^i - \alpha \tilde{s}_t^i - \tilde{c}_t^{i,o} - \phi \tilde{n}_t^{i,o} - \tilde{n}_t^i\right) \tag{94}$$

$$0 = -\int_0^\infty e^{-\rho t}\left(\tilde{y}_t^i - \alpha(1-v)\tilde{s}_t^i - (1-v)\tilde{c}_t^i - v\tilde{g}_t^i\right)dt \tag{95}$$

$$0 = \int_0^\infty e^{-\rho t}\left(\tilde{\tau}_t^{i,r} + v\tilde{g}_t^i - \alpha v \tilde{s}_t^i\right)dt = 0. \tag{96}$$

We highlight two important features of the natural allocation: First, for any level of steady state government spending $v \in [0,1]$, the natural allocation is $\chi$-invariant and $\tilde{\theta}^i = 0$. Second, for any level of steady state government spending $v \in [0,1]$ and for any share of hand-to-mouth agents $\chi \in [0,1]$, the natural allocation is symmetric, with $\tilde{n}_t^{i,r} = \tilde{n}_t^{i,o}$ and $\tilde{c}_t^{i,r} = \tilde{c}_t^{i,o}$ for all $t$.

We will now solve the natural allocation in closed form. We use $\tilde{g}_t^i = a_t^i$. Solving out for $\tilde{\theta}_t^i$, $\tilde{n}_t^i = \tilde{y}_t^i - a_t^i$, $\tilde{c}_t^i = \chi\tilde{c}_t^{i,r} + (1-\chi)\tilde{c}_t^{i,o}$, and $\tilde{c}_t^{i,o} = \tilde{\theta}_t^i + (1-\alpha)\tilde{s}_t^i$, we find

$$\begin{aligned}
\tilde{y}_t^i - a_t^i &= \chi\tilde{n}_t^{i,r} + (1-\chi)\tilde{n}_t^{i,o} \\
\phi\tilde{n}_t^{i,o} + (1-\alpha)\tilde{s}_t^i &= \tilde{c}_t^{i,r} + \phi\tilde{n}_t^{i,r} \\
\alpha v\tilde{s}_t^i + (1+\phi)\tilde{n}_t^{i,r} + v\tilde{c}_t^{i,r} &= \tilde{s}_t^i + \phi\tilde{n}_t^{i,o} - (1-v)a_t^i \\
\frac{1}{1-v}\tilde{y}_t^i &= [1-\chi(1-\alpha)^2]\tilde{s}_t^i + \chi(1-\alpha)\tilde{c}_t^{i,r} + \frac{v}{1-v}a_t^i \\
(1+\phi)a_t^i &= \chi\tilde{c}_t^{i,r} + [(1-\chi)(1-\alpha)+\alpha]\tilde{s}_t^i + \phi\tilde{y}_t^i.
\end{aligned}$$

From here, we can solve for $\tilde{c}_t^{i,r}$ and $\tilde{n}_t^{i,r}$, yielding

$$\begin{aligned}
\tilde{c}_t^{i,r} &= \frac{\phi}{1-\chi+(1-v)\phi}\tilde{y}_t^i - \frac{v\phi}{1-\chi+(1-v)\phi}a_t^i + \frac{(1-\alpha)(1-\chi)-(1-v)\alpha\phi}{1-\chi+(1-v)\phi}\tilde{s}_t^i \\
\tilde{n}_t^{i,r} &= \frac{(1-v)\phi}{1-\chi+(1-v)\phi}\tilde{y}_t^i - \frac{(1-v)(1-\chi+\phi)}{1-\chi+(1-v)\phi}a_t^i + \frac{(1-\chi)(1-v)}{1-\chi+(1-v)\phi}\tilde{s}_t^i.
\end{aligned}$$

Plugging these into the aggregate demand relation, we can solve for the terms of trade and find that $\tilde{s}_t^i = \frac{1}{1-v}\tilde{y}_t^i - \frac{v}{1-v}a_t^i$. Using these to solve for the output gap, finally, we find

$$\tilde{y}_t^i = a_t^i, \tag{97}$$

and similarly

$$\tilde{s}_t^i = a_t^i. \tag{98}$$

Solving the remainder of the allocation is straightforward using these two results. Finally, we note here that

$$\begin{aligned}
\dot{\tilde{s}}_t^i &= -\dot{p}_{H,t}^i \\
&= -\psi e^{-\psi t}a_0^i.
\end{aligned} \tag{99}$$

We will make use of this result later on.

### D.2.2 Allocation with Policy

We now present the allocation in terms of gaps from the natural. Recall that $\hat{x}_t^i = \ln(X_t^i) - \ln(\tilde{X}_t^i)$ for all variables $X_t^i$, where $\tilde{X}_t^i$ denotes the natural allocation.

The linear model with policy is given by

$$\hat{c}_t^{i,o} + \phi\hat{n}_t^{i,o} = \hat{c}_t^{i,r} + \phi\hat{n}_t^{i,r} \tag{100}$$

$$\hat{c}_t^{i,o} = \hat{\theta}_t^i + (1-\alpha)\hat{s}_t^i \tag{101}$$

$$\hat{c}_t^i = \chi\hat{c}_t^{i,r} + (1-\chi)\hat{c}_t^{i,o} \tag{102}$$

$$\hat{n}_t^i = \chi \hat{n}_t^{i,r} + (1-\chi)\hat{n}_t^{i,o} \tag{103}$$
$$\hat{y}_t^i = \hat{n}_t^i \tag{104}$$
$$\hat{t}_t^{i,r} = -(1+\phi)\hat{n}_t^{i,r} - v\hat{c}_t^{i,r} \tag{105}$$
$$\frac{1}{1-v}\hat{y}_t^i = \hat{s}_t^i + (1-\alpha)\left(\hat{\theta}_t^i + \hat{c}_t^i - \hat{c}_t^{i,o}\right) + \frac{v}{1-v}\hat{g}_t^i \tag{106}$$
$$\hat{\tau}_t^{i,r} = \hat{t}_t^{i,r} - \left(\hat{y}_t^i - \alpha\hat{s}_t^i - \hat{c}_t^{i,o} - \phi\hat{n}_t^{i,o} - \hat{n}_t^i\right) \tag{107}$$
$$\alpha v\hat{s}_t^i = \chi\hat{\tau}_t^{i,r} + (1-\chi)\hat{\tau}_t^{i,o} + v\hat{g}_t^i \tag{108}$$
$$N\hat{F}A_0^i = \int_0^\infty e^{-\rho t}\left[\hat{\tau}_t^{i,r} + v\hat{g}_t^i - v\alpha\hat{s}_t^i\right]dt \tag{109}$$
$$N\hat{F}A_0^i = -\int_0^\infty e^{-\rho t}\left[\hat{y}_t^i - \alpha(1-v)\hat{s}_t^i - (1-v)\hat{c}_t^i - v\hat{g}_t^i\right]dt \tag{110}$$
$$\dot{\hat{\pi}}_{H,t}^i = \rho\hat{\pi}_{H,t}^i - \lambda\left(\phi\hat{n}_t^i + \hat{c}_t^i + \alpha\hat{s}_t^i\right) \tag{111}$$
$$\pi_t^i = \pi_{H,t}^i + \alpha\dot{s}_t^i \tag{112}$$
$$\dot{c}_t^{i,o} = i_t^i - \pi_t^i - \rho. \tag{113}$$

Note that for all variables we have $\int_0^1 \tilde{x}_t^i di = 0$, so that $\tilde{x}_t^i = \hat{x}_t^i$. We used this simplification above.

We furthermore need an initial value condition to close the system and pin down the solution of this first-order linear system of ODEs. We will derive the initial condition below.

Before doing so, it will be convenient to solve this system in terms of $\hat{y}_t^i$, $\hat{\tau}_t^{i,r}$, $\hat{\theta}_t^i$ and $\hat{g}_t^i$. We start by solving out for $\hat{n}_t^i$, $\hat{c}_t^i$, and $\hat{c}_t^{i,o}$. This yields

$$\begin{aligned}
\hat{\theta}_t^i + (1-\alpha)\hat{s}_t^i + \phi\hat{n}_t^{i,o} &= \hat{c}_t^{i,r} + \phi\hat{n}_t^{i,r} \\
\hat{y}_t^i &= \chi\hat{n}_t^{i,r} + (1-\chi)\hat{n}_t^{i,o} \\
\hat{t}_t^{i,r} &= -(1+\phi)\hat{n}_t^{i,r} - v\hat{c}_t^{i,r} \\
\frac{1}{1-v}\hat{y}_t^i &= (1-\alpha)(1-\chi)\hat{\theta}_t^i + \chi(1-\alpha)\hat{c}_t^{i,r} + [1-\chi(1-\alpha)^2]\hat{s}_t^i + \frac{v}{1-v}\hat{g}_t^i \\
\hat{t}_t^{i,r} &= \hat{\tau}_t^{i,r} + \hat{y}_t^i - \alpha\hat{s}_t^i - \hat{c}_t^{i,o} - \phi\hat{n}_t^{i,o} - \hat{n}_t^i.
\end{aligned}$$

Next, we solve out for $\hat{t}_t^{i,r}$ using the last equation in the previous block, and for $\hat{n}_t^{i,o} = \frac{1}{1-\chi}\hat{y}_t^i - \frac{\chi}{1-\chi}\hat{n}_t^{i,r}$. We are interested in the following two equations,

$$\begin{aligned}
\hat{\theta}_t^i + (1-\alpha)\hat{s}_t^i + \frac{\phi}{1-\chi}\hat{y}_t^i &= \hat{c}_t^{i,r} + \frac{\phi}{1-\chi}\hat{n}_t^{i,r} \\
\hat{\tau}_t^{i,r} - \hat{s}_t^i - \hat{\theta}_t^i - \frac{\phi}{1-\chi}\hat{y}_t^i + (1+\phi+\frac{\chi\phi}{1-\chi})\hat{n}_t^{i,r} &= -v\hat{c}_t^{i,r},
\end{aligned}$$

which we can now use to solve for $\hat{c}_t^{i,r}$ and $\hat{n}_t^{i,r}$. In particular, this yields

$$\hat{n}_t^{i,r} = \frac{1-\chi}{1-\chi+(1-v)\phi}\left((1-v)\frac{\phi}{1-\chi}\hat{y}_t^i + (1-v)\hat{\theta}_t^i + [1-v(1-\alpha)]\hat{s}_t^i - \hat{\tau}_t^{i,r}\right) \tag{114}$$

and

$$\hat{\hat{c}}_t^{i,r} = \frac{1}{1-\chi+(1-v)\phi}\Bigg((1-\chi)\hat{\hat{\theta}}_t^i + \phi\hat{\hat{y}}_t^i + [(1-\alpha)(1-\chi)-\alpha\phi]\hat{\hat{s}}_t^i + \phi\hat{\hat{\tau}}_t^{i,r}\Bigg). \tag{115}$$

Finally, we use the linearized aggregate demand relation to solve for $\hat{\hat{s}}_t^i$. We find

$$\begin{aligned}
\hat{\hat{s}}_t^i =& \left(1-(1-\alpha)\chi\frac{(1-v)\phi+\alpha v\phi}{1-\chi+(1-v)\phi}\right)^{-1}\Bigg[\left(\frac{1}{1-v}-\frac{(1-\alpha)\chi\phi}{1-\chi+(1-v)\phi}\right)\hat{\hat{y}}_t^i - \frac{v}{1-v}\hat{\hat{g}}_t^i \qquad (116)\\
&-(1-\alpha)(1-\chi)\frac{1+\phi-v\phi}{1-\chi+(1-v)\phi}\hat{\hat{\theta}}_t^i - \frac{(1-\alpha)\chi\phi}{1-\chi+(1-v)\phi}\hat{\hat{\tau}}_t^{i,r}\Bigg]\\
\equiv& \nu_y\hat{\hat{y}}_t^i + \nu_g\hat{\hat{g}}_t^i + \nu_\theta\hat{\hat{\theta}}_t^i + \nu_\tau\hat{\hat{\tau}}_t^{i,r}. \qquad (117)
\end{aligned}$$

Therefore, we can for the future introduce the following notation

$$\begin{aligned}
\hat{\hat{c}}_t^{i,r} &= \psi_y\hat{\hat{y}}_t^i + \psi_\theta\hat{\hat{\theta}}_t^i + \psi_\tau\hat{\hat{\tau}}_t^{i,r} + \psi_g\hat{\hat{g}}_t^i &(118)\\
\hat{\hat{n}}_t^{i,r} &= \eta_y\hat{\hat{y}}_t^i + \eta_g\hat{\hat{g}}_t^i + \eta_\theta\hat{\hat{\theta}}_t^i + \eta_\tau\hat{\hat{\tau}}_t^{i,r} &(119)\\
\hat{\hat{c}}_t^i &= \zeta_y\hat{\hat{y}}_t^i + \zeta_\theta\hat{\hat{\theta}}_t^i + \zeta_\tau\hat{\hat{\tau}}_t^{i,r} + \zeta_g\hat{\hat{g}}_t^i &(120)
\end{aligned}$$

where

$$\begin{aligned}
\psi_y &= \frac{\phi}{1-\chi+(1-v)\phi} + \left[1-\alpha-\frac{(1-v)\phi+\alpha v\phi}{1-\chi+(1-v)\phi}\right]\nu_y\\
\psi_\theta &= \frac{1-\chi}{1-\chi+(1-v)\phi} + \left[1-\alpha-\frac{(1-v)\phi+\alpha v\phi}{1-\chi+(1-v)\phi}\right]\nu_\theta\\
\psi_\tau &= \frac{\phi}{1-\chi+(1-v)\phi} + \left[1-\alpha-\frac{(1-v)\phi+\alpha v\phi}{1-\chi+(1-v)\phi}\right]\nu_\tau\\
\psi_g &= \left[1-\alpha-\frac{(1-v)\phi+\alpha v\phi}{1-\chi+(1-v)\phi}\right]\nu_g
\end{aligned}$$

and

$$\begin{aligned}
\eta_y &= \frac{1-\chi}{1-\chi+(1-v)\phi}\left[(1-v)\frac{\phi}{1-\chi} + [1-v(1-\alpha)]\nu_y\right]\\
\eta_g &= \frac{1-\chi}{1-\chi+(1-v)\phi}[1-v(1-\alpha)]\nu_g\\
\eta_\theta &= \frac{1-\chi}{1-\chi+(1-v)\phi}\left[1-v+[1-v(1-\alpha)]\nu_\theta\right]\\
\eta_\tau &= \frac{1-\chi}{1-\chi+(1-v)\phi}\left[-1+[1-v(1-\alpha)]\nu_\tau\right],
\end{aligned}$$

and

$$\zeta_y = \chi\psi_y + (1-\chi)(1-\alpha)\nu_y$$

$$
\begin{aligned}
\zeta_g &= \chi\psi_g + (1-\chi)(1-\alpha)\nu_g \\
\zeta_\tau &= \chi\psi_\tau + (1-\chi)(1-\alpha)\nu_\tau \\
\zeta_\theta &= \chi\psi_\theta + (1-\chi)[1+(1-\alpha)\nu_\theta].
\end{aligned}
$$

### D.2.3 IS equation

We start from the linearized aggregate demand equation and use slightly more convenient notation to write

$$\Lambda_y \hat{\hat{y}}_t^i = \Lambda_s \hat{\hat{s}}_t^i + \Lambda_g \hat{\hat{g}}_t^i + \Lambda_\theta \hat{\hat{\theta}}_t^i + \Lambda_\tau \hat{\hat{\tau}}_t^{i,r}, \tag{121}$$

where

$$
\begin{aligned}
\Lambda_y &= \frac{1}{1-v} - \frac{(1-\alpha)\chi\phi}{1-\chi+(1-v)\phi} \\
\Lambda_s &= 1-(1-\alpha)\chi\frac{(1-v)\phi+\alpha v\phi}{1-\chi+(1-v)\phi} \\
\Lambda_\theta &= (1-\alpha)(1-\chi)\frac{1+\phi-v\phi}{1-\chi+(1-v)\phi} \\
\Lambda_g &= \frac{v}{1-v} \\
\Lambda_\tau &= \frac{(1-\alpha)\chi\phi}{1-\chi+(1-v)\phi},
\end{aligned}
$$

and where we have the easy conversions: $\nu_y = \frac{\Lambda_y}{\Lambda_s}$, $\nu_g = -\frac{\Lambda_g}{\Lambda_s}$,$\nu_\theta = -\frac{\Lambda_\theta}{\Lambda_s}$, and $\nu_\tau = -\frac{\Lambda_\tau}{\Lambda_s}$. Differentiating, we have

$$\Lambda_y \dot{\hat{\hat{y}}}_t^i = \Lambda_s \dot{\hat{\hat{s}}}_t^i + \Lambda_g \dot{\hat{\hat{g}}}_t^i + \Lambda_\theta \dot{\hat{\hat{\theta}}}_t^i + \Lambda_\tau \dot{\hat{\hat{\tau}}}_t^{i,r},$$

To derive the (IS) equation, all that is left to do is to find an expression for $\dot{\hat{\hat{s}}}_t^i$. From the definition of the terms of trade, we can directly write

$$\dot{\hat{\hat{s}}}_t^i = -(\hat{\hat{\pi}}_{H,t}^i + \dot{\bar{s}}_t^i). \tag{122}$$

Therefore, the fully general (IS) equation is given by

$$\Lambda_y \dot{\hat{\hat{y}}}_t^i = -\Lambda_s(\hat{\hat{\pi}}_{H,t}^i + \dot{\bar{s}}_t^i) + \Lambda_g \dot{\hat{\hat{g}}}_t^i + \Lambda_\theta \dot{\hat{\hat{\theta}}}_t^i + \Lambda_\tau \dot{\hat{\hat{\tau}}}_t^{i,r}. \tag{123}$$

### D.2.4 NKPC

We start with the general Phillips Curve

$$\dot{\pi}_{H,t}^i = \rho\pi_{H,t}^i - \lambda\left(\phi n_t^i + c_t^i + \alpha s_t^i - a_t^i\right). \tag{124}$$

Noting that

$$\phi\tilde{n}_t^i + \tilde{c}_t^i + \alpha\tilde{s}_t^i - a_t^i = 0 \tag{125}$$

under the natural allocation, and taking deviations from union aggregates, we can write the NKPC in gap notation as

$$\dot{\hat{\pi}}_{H,t}^i = \rho\hat{\pi}_{H,t}^i - \lambda\big(\phi\hat{n}_t^i + \hat{c}_t^i + \alpha\hat{s}_t^i\big), \tag{126}$$

Using expressions we derived earlier, we have

$$\dot{\hat{\pi}}_{H,t}^i = \rho\hat{\pi}_{H,t}^i - \lambda\phi\hat{y}_t^i - \lambda\big(\zeta_y\hat{y}_t^i + \zeta_\theta\hat{\theta}_t^i + \zeta_\tau\hat{\tau}_t^{i,r} + \zeta_g\hat{g}_t^i\big) - \lambda\alpha\big(\nu_y\hat{y}_t^i + \nu_g\hat{g}_t^i + \nu_\theta\hat{\theta}_t^i + \nu_\tau\hat{\tau}_t^{i,r}\big),$$

or, collecting terms,

$$\dot{\hat{\pi}}_{H,t}^i = \rho\hat{\pi}_{H,t}^i - \kappa_y\hat{y}_t^i - \kappa_\theta\hat{\theta}_t^i - \kappa_\tau\hat{\tau}_t^{i,r} - \kappa_g\hat{g}_t^i, \tag{127}$$

where

$$\begin{aligned}
\kappa_y &= \lambda(\phi + \zeta_y + \alpha\nu_y) \\
\kappa_g &= \lambda(\zeta_g + \alpha\nu_g) \\
\kappa_\tau &= \lambda(\zeta_\tau + \alpha\nu_\tau) \\
\kappa_\theta &= \lambda(\zeta_\theta + \alpha\nu_\theta).
\end{aligned}$$

### D.2.5 Initial Condition

As for the (IS) equation, we start with the linearized aggregate demand relation. We have

$$\Lambda_y\hat{y}_0^i = \Lambda_s\hat{s}_0^i + \Lambda_g\hat{g}_0^i + \Lambda_\theta\hat{\theta}_0^i + \Lambda_\tau\hat{\tau}_0^{i,r}. \tag{128}$$

We can obtain an initial condition for our dynamic system by noting that

$$s_0^i = \hat{s}_t^i + \tilde{s}_0^i = 0, \tag{129}$$

implying that $\hat{s}_t^i = -\tilde{s}_0^i$. Using this, we can write

$$\Lambda_y\hat{y}_0^i = -\Lambda_s\tilde{s}_0^i + \Lambda_g\hat{g}_0^i + \Lambda_\theta\hat{\theta}_0^i + \Lambda_\tau\hat{\tau}_0^{i,r}. \tag{130}$$

### D.2.6 Country-$i$ Budget Constraint

Finally, it remains to simplify the net foreign asset relations. Solving out for $N\hat{F}A_0^i$, we can write

$$\int_0^\infty e^{-\rho t}\big[\hat{\tau}_t^{i,r} + v\hat{g}_t^i - v\alpha\hat{s}_t^i\big]dt = -\int_0^\infty e^{-\rho t}\big[\hat{y}_t^i - \alpha(1-v)\hat{s}_t^i - (1-v)\hat{c}_t^i - v\hat{g}_t^i\big]dt.$$

We can simplify to get

$$0 = \int_0^\infty e^{-\rho t} \Big[ \hat{\hat{y}}_t^i + \hat{\hat{\tau}}_t^{i,r} - (1-v)\chi \Big( \psi_y \hat{\hat{y}}_t^i + \psi_\theta \hat{\hat{\theta}}_t^i + \psi_\tau \hat{\hat{\tau}}_t^{i,r} + \psi_g \hat{\hat{g}}_t^i \Big) - (1-v)(1-\chi)\hat{\hat{\theta}}_t^i$$
$$- [\alpha + (1-v)(1-\chi)(1-\alpha)] \Big( \nu_y \hat{\hat{y}}_t^i + \nu_g \hat{\hat{g}}_t^i + \nu_\theta \hat{\hat{\theta}}_t^i + \nu_\tau \hat{\hat{\tau}}_t^{i,r} \Big) \Big] dt.$$

Collecting terms, we can write

$$0 = \int_0^\infty e^{-\rho t} \Big[ \Gamma_y \hat{\hat{y}}_t^i + \Gamma_\tau \hat{\hat{\tau}}_t^{i,r} + \Gamma_\theta \hat{\hat{\theta}}_t^i + \Gamma_g \hat{\hat{g}}_t^i \Big] dt, \tag{131}$$

where

$$\begin{aligned}
\Gamma_y &= 1 - (1-v)\chi\psi_y - [\alpha + (1-v)(1-\chi)(1-\alpha)]\nu_y \\
\Gamma_\tau &= 1 - (1-v)\chi\psi_\tau - [\alpha + (1-v)(1-\chi)(1-\alpha)]\nu_\tau \\
\Gamma_\theta &= -(1-v)(1-\chi) - (1-v)\chi\psi_\theta - [\alpha + (1-v)(1-\chi)(1-\alpha)]\nu_\theta \\
\Gamma_g &= -(1-v)\chi\psi_g - [\alpha + (1-v)(1-\chi)(1-\alpha)]\nu_g.
\end{aligned}$$

### D.2.7 Allocation with Sticky Prices

Finally, we will solve the allocation presented above under sticky prices but without optimal policy intervention. In particular, we have $N\hat{\hat{F}}A_0^i = 0$ since there are no international transfers. This implies that $\hat{\hat{\tau}}_t^{i,r} = \hat{\hat{\tau}}_t^{i,o} = \alpha v \hat{\hat{s}}_t^i$.

Furthermore, we set $\hat{\hat{g}}_t^i = 0$ and $\hat{\hat{\theta}}_t^i = \hat{\hat{\theta}}^i$ in the absence of government spending policy and capital controls. Since we also have $\hat{\hat{s}}_t^i = \nu_y \hat{\hat{y}}_t^i + \nu_\tau \hat{\hat{\tau}}_t^{i,r} + \nu_\theta \hat{\hat{\theta}}^i$, we can write

$$\begin{aligned}
\hat{\hat{\tau}}_t^{i,r} &= \frac{\alpha v}{1 - \alpha v \nu_\tau} (\nu_y \hat{\hat{y}}_t^i + \nu_\theta \hat{\hat{\theta}}^i) \\
\dot{\hat{\hat{\tau}}}_t^{i,r} &= \frac{\alpha v}{1 - \alpha v \nu_\tau} \nu_y \dot{\hat{\hat{y}}}_t^i.
\end{aligned}$$

This allocation will be important for our numerical exercises where we compare optimal policy to this no-policy benchmark.

**NFA.** The general country $i$ budget constraint simplifies to

$$0 = \int_0^\infty e^{-\rho t} \Big[ \Gamma_y \hat{\hat{y}}_t^i + \Gamma_\tau \hat{\hat{\tau}}_t^{i,r} + \Gamma_\theta \hat{\hat{\theta}}^i \Big] dt \tag{132}$$

$$0 = \int_0^\infty e^{-\rho t} \Big[ \Big( \Gamma_y + \Gamma_\tau \frac{\alpha v}{1 - \alpha v \nu_\tau} \nu_y \Big) \hat{\hat{y}}_t^i + \Big( \Gamma_\theta + \Gamma_\tau \frac{\alpha v}{1 - \alpha v \nu_\tau} \nu_\theta \Big) \hat{\hat{\theta}}^i \Big] dt. \tag{133}$$

One can easily verify that $\Gamma_y + \Gamma_\tau \frac{\alpha v}{1 - \alpha v \nu_\tau} \nu_y = 0$. The budget constraint then directly implies that $\hat{\hat{\theta}}^i = 0$ for the no-policy allocation.

**Allocation.** The simplified (IS) equation can then be written as

$$\dot{\hat{\tilde{y}}}^i_t = -G'(\hat{\tilde{\pi}}^i_{H,t} + \dot{\tilde{s}}^i_t), \tag{134}$$

where $G' = \frac{\Lambda_s}{\Lambda_y - \Lambda_\tau \frac{\alpha v}{1-\alpha v \nu_\tau} \nu_y}$. The (NKPC) becomes

$$\dot{\hat{\tilde{\pi}}}^i_{H,t} = \rho \hat{\tilde{\pi}}^i_{H,t} - \hat{\kappa}_y \hat{\tilde{y}}^i_t, \tag{135}$$

with $\hat{\kappa}_y = \kappa_y + \kappa_\tau \frac{\alpha v}{1-\alpha v \nu_\tau} \nu_y$. And finally, the initial condition can be written as

$$\hat{\tilde{y}}^i_0 = -\bar{G}' \tilde{s}^i_0 \tag{136}$$

with $\bar{G}' = \frac{\Lambda_s + \alpha v \Lambda_\tau}{\Lambda_y}$.
We can write the dynamical system describing the no-policy allocation as

$$\dot{X}^i_t = AX^i_t + B^i_t, \tag{137}$$

where

$$A = \begin{pmatrix} \rho & -\hat{\kappa}_y \\ -G' & 0 \end{pmatrix}, \qquad B^i_t = \begin{pmatrix} 0 \\ -G' \dot{\tilde{s}}^i_t \end{pmatrix}, \qquad X^i_0 = \begin{pmatrix} \hat{\tilde{\pi}}^i_{H,0} \\ -\bar{G}' \tilde{s}^i_0 \end{pmatrix},$$

We will solve for $\hat{\tilde{\pi}}^i_{H,0}$ to ensure the stability of this system. It is straightforward to verify that the only negative eigenvalue of $A$ is

$$\nu = \frac{1}{2}\left(\rho - \sqrt{\rho^2 + 4G'\hat{\kappa}_y}\right). \tag{138}$$

The corresponding eigenvector is $X_\nu = (-\frac{\nu}{G'}, 1)'$.

We know that the solution to this system is given by

$$X^i_t = e^{At}\left[X^i_0 + \int_0^t e^{-As}\left(-G'\dot{\tilde{s}}^i_t E_2\right) ds\right], \tag{139}$$

where $E_1 = (1,0)'$ and $E_2 = (0,1)'$ are projection matrices. We restrict attention to the parameter subspace where $A$ is nonsingular. Furthermore, we know that

$$\dot{\tilde{s}}^i_t = -\psi a^i_0 e^{-\psi t}, \tag{140}$$

where $a^i_0$ is the initial technology shock. This implies

$$X^i_t = e^{At}\left[X^i_0 + \psi G' a^i_0 \int_0^t e^{-As-\psi s} E_2 ds\right] \tag{141}$$

$$=e^{At}\left[X_0^i+\psi G' a_0^i (A+\psi I)^{-1}E_2\right]-\psi G' a_0^i e^{-\psi t}(A+\psi I)^{-1}E_2. \tag{142}$$

**Stability.** For stability, we require the term in square brackets be spanned on the stable manifold, which is satisfied if and only if there exists an $\alpha_\nu \in \mathbb{C}$ such that

$$\alpha_\nu X_\nu = X_0^i+\psi G' a_0^i (A+\psi I)^{-1}E_2. \tag{143}$$

This condition implies the two stability equations

$$-\frac{\nu}{G'}\alpha_\nu = \hat{\tilde{\pi}}_{H,0}^i+\psi G' a_0^i E_1'(A+\psi I)^{-1}E_2 \tag{144}$$

$$\alpha_\nu = -\bar{G}'\tilde{s}_0^i+\psi G' a_0^i E_2'(A+\psi I)^{-1}E_2, \tag{145}$$

from which we can solve explicitly for $\alpha_\nu$ and $\hat{\tilde{\pi}}_{H,0}^i$. Note that we have

$$E_1'(A+\psi I)^{-1}E_2 = \frac{\tilde{\kappa}_y}{\hat{\psi}}$$
$$E_2'(A+\psi I)^{-1}E_2 = \frac{\rho+\psi}{\hat{\psi}},$$

where $\hat{\psi}=(\rho+\psi)\psi - G'\hat{\kappa}_y$. We can now rewrite the two stability conditions as

$$-\frac{\nu}{G'}\alpha_\nu = \hat{\tilde{\pi}}_{H,0}^i+\psi G'\frac{\hat{\kappa}_y}{\hat{\psi}}a_0^i \tag{146}$$

$$\alpha_\nu = -\bar{G}'\tilde{s}_0^i+\psi G'\frac{\rho+\psi}{\hat{\psi}}a_0^i. \tag{147}$$

We can now characterize the closed-form solution of our dynamical system, which is given by

$$X_t^i = \alpha_\nu e^{\nu t}X_\nu - \psi G' a_0^i e^{-\psi t}(A+\psi I)^{-1}E_2, \tag{148}$$

from which we can back out explicit solutions for inflation and output:

$$\hat{\tilde{\pi}}_{H,t}^i = -\frac{\nu}{G'}\alpha_\nu e^{\nu t} - \psi G'\frac{\hat{\kappa}_y}{\hat{\psi}}e^{-\psi t}a_0^i \tag{149}$$

$$\hat{\tilde{y}}_t^i = \alpha_\nu e^{\nu t} - \psi G'\frac{\rho+\psi}{\hat{\psi}}e^{-\psi t}a_0^i. \tag{150}$$

We note here that the no-policy benchmark allocation is $\chi$-invariant. To gain some intuition for this result, note that under Cole-Obstfeld and with profit redistribution the positive allocation is symmetric, with $\hat{\tilde{c}}_t^{i,o}=\hat{\tilde{c}}_t^{i,r}$, and trade is balanced in every period, $\hat{\tilde{NX}}_t^i=0$. That is, HtM agents and optimizers always consume the same amount and face the same budget constraint, in a sense, since there is no inter-temporal substitution.

## D.3 Loss Function and Planning Problem

Before we can set up the planning problem for a currency union with hand-to-mouth agents, we have to derive the loss function. Our loss function features three distinct policy instruments: ex-post transfers, capital controls and government spending. It also leaves room for several derivative policy instruments which we will discuss later. We follow the linear-quadratic approach discussed in Benigno and Woodford (2012). We continue to assume that there are no aggregate shocks but in this section, to derive the loss function, we need to temporarily allow for idiosyncratic shocks in all countries.

### D.3.1 Notation

Once again, we reiterate our notational conventions. For all $X_t^i \notin \{T_t^{i,o}, T_t^{i,r}, \tau_t^{i,o}, \tau_t^{i,r}, NFA_t^i, NX_t^i\}$, we have $x_t^i = \ln(X_t^i) - \ln(X^i)$, where $X^i$ is the steady state value of $X_t^i$; furthermore, we define $\tilde{x}_t^i = \ln(\tilde{X}_t^i) - \ln(X^i)$, $\hat{x}_t^i = \ln(X_t^i) - \ln(\tilde{X}_t^i)$ and $\hat{\hat{x}}_t^i = \hat{x}_t^i - \int_0^1 \hat{x}_t^i di \equiv \hat{x}_t^i - \hat{x}_t^*$. Our goal is to approximate welfare in terms of $\hat{\hat{x}}_t^i$ variables.

By definition, we have $X_t^i = X^i e^{x_t^i}$. Taylor expansions of the function $x_t^i \mapsto X_t^i$ around $x_t^i = 0$ yield:

To $1^{st}$ order: $X_t^i \approx X^i + X^i x_t^i = X^i(1 + x_t^i)$.

To $2^{nd}$ order: $X_t^i \approx X^i + X^i x_t^i + \frac{1}{2} X^i (x_t^i)^2 = X^i(1 + x_t^i + \frac{1}{2}(x_t^i)^2)$.

So for example, the second-order approximation of $\ln(X_t^i) = \ln(X^i) + x_t^i$ around $x_t^i = 0$ is exactly $\ln(X^i) + x_t^i$. Similarly, approximating $(X_t^i)^\phi = (X^i)^\phi e^{\phi x_t^i}$ around $\phi x_t^i = 0$, we have

$$(X_t^i)^\phi \approx (X^i)^\phi + \frac{(X^i)^\phi e^0}{1} + \frac{(X^i)^\phi e^0}{2} (\phi x_t^i)^2 = (X^i)^\phi \big(1 + \phi x_t^i + \frac{\phi^2}{2} (x_t^i)^2\big). \tag{151}$$

### D.3.2 Starting with Household Utility

We begin by taking as our welfare criterion the households' utility function. In particular, we solve a coordinated problem, with the social planner's objective function of the form

$$U = \int_0^\infty \int_0^1 e^{-\rho t} \bigg[ \chi (1 - \upsilon) \ln(C_t^{i,r}) - \frac{\chi}{1+\phi} (N_t^{i,r})^{1+\phi} \tag{152}$$
$$+ (1-\chi)(1-\upsilon) \ln(C_t^{i,o}) - \frac{1-\chi}{1+\phi} (N_t^{i,o})^{1+\phi} + \upsilon \ln(G_t^i) \bigg] di dt.$$

Approximating the RHS to second order, we have

$$U = \int_0^\infty \int_0^1 e^{-\rho t}\Big[\chi(1-v)\ln(C^{i,r}) + \chi(1-v)c_t^{i,r} + (1-\chi)(1-v)\ln(C^{i,o}) + (1-\chi)(1-v)c_t^{i,o}$$
$$- \frac{\chi}{1+\phi}(N^{i,r})^{1+\phi}\big(1+(1+\phi)n_t^{i,r} + \frac{(1+\phi)^2}{2}(n_t^{i,r})^2\big)$$
$$- \frac{1-\chi}{1+\phi}(N^{i,o})^{1+\phi}\big(1+(1+\phi)n_t^{i,o} + \frac{(1+\phi)^2}{2}(n_t^{i,o})^2\big) + v\ln(G^i) + vg_t^i\Big]didt.$$

Recall the steady state relations: $Y^i = 1$, $C^{i,r} = C^{i,o} = (1-v)$, $G^i = v$, and $(N^{i,o})^{1+\phi} = (N^{i,r})^{1+\phi} = 1$. Hence, we have

$$U = \int_0^\infty \int_0^1 e^{-\rho t}\Big[(1-v)\ln(1-v) - \frac{1}{1+\phi} + v\ln(v)$$
$$+ (1-v)[\chi c_t^{i,r} + (1-\chi)c_t^{i,o}] - [\chi n_t^{i,r} + (1-\chi)n_t^{i,o}] + vg_t^i$$
$$- \frac{\chi(1+\phi)}{2}(n_t^{i,r})^2 - \frac{(1-\chi)(1+\phi)}{2}(n_t^{i,o})^2\Big]didt.$$

Let $Z^i \equiv (1-v)\ln(1-v) - \frac{1}{1+\phi} + v\ln(v)$. Also note that $\chi c_t^{i,r} + (1-\chi)c_t^{i,o} = c_t^i$ and $\chi n_t^{i,r} + (1-\chi)n_t^{i,o} = n_t^i$. Finally, we use the linearity of the integral operator, noting that $\int_0^1 x_t^i di \equiv \int_0^1 (\hat{\tilde{x}}_t^i + \hat{x}_t^* + \tilde{x}_t^i)di = \hat{x}_t^*$, to write

$$U = \int_0^\infty \int_0^1 e^{-\rho t}\Big[Z^i + (1-v)\hat{c}_t^* + v\hat{g}_t^* - n_t^i - \frac{\chi(1+\phi)}{2}(n_t^{i,r})^2 - \frac{(1-\chi)(1+\phi)}{2}(n_t^{i,o})^2\Big]didt. \quad (153)$$

### D.3.3 Linear-Quadratic Framework

Having completed a second-order approximation of the social planner's welfare criterion, we will now try to express said metric in terms of the variables $\hat{y}_t^i$, $\hat{\pi}_{H,t}^i$, $\hat{\tau}_t^{i,r}$, $\hat{\theta}_t^i$, and $\hat{g}_t^i$, where the latter three represent the full set of policy instruments available to the planner. In this particular case, a proper linear-quadratic approximation rests on the following two key steps:

i. Approximate the A-D relation to second order to find an expression for $(1-v)\hat{c}_t^* + v\hat{g}_t^*$.

ii. Express $\hat{n}_t^i$ in terms of inflation.

We start with (2): With Calvo-pricing, we have the distorted production function

$$Y_t^i \int_0^1 \left(\frac{P_{H,t}^i(j)}{P_{H,t}^i}\right)^{-\epsilon} dj = A_t^i N_t^i.$$

Therefore, we can approximate $n_t^i = y_t^i + z_t^i - a_t^i$ to second order, where

$$z_t^i = \ln\left(\int_0^1 \left(\frac{P_{H,t}^i(j)}{P_{H,t}^i}\right)^{-\epsilon} dj\right) \approx \frac{\epsilon}{2}\int_0^1 (P_{H,t}^i(j) - P_{H,t}^i)^2 dj = \alpha_\pi(\pi_{H,t}^i)^2, \quad (154)$$

for $\alpha_\pi = \frac{\epsilon}{2\lambda}$ and $\lambda = \rho_\delta(\rho + \rho_\delta)$. Note that this definition of $\alpha_\pi$ differs slightly from that used in the main body of the paper. We found this notation slightly more convenient for the analysis in the appendix. Hence, we have

$$n_t^i = y_t^i + \alpha_\pi(\pi_{H,t}^i)^2 - a_t^i. \tag{155}$$

The technology term is invariant to policy and so we drop it. It would later naturally drop out as we rewrite the loss in gaps from the natural allocation.

Next, we tackle the more difficult step (1): Recall that the A-D relation can be written as

$$Y_t^i - G_t^i = S_t^i C^{*,o}\left((1-\alpha)\frac{\Theta_t^i C_t^i}{C_t^{i,o}} + \alpha \int_0^1 \frac{\Theta_t^j C_t^j}{C_t^{j,o}} dj\right).$$

The linear-quadratic procedure requires that we approximate all those constraints to second order that we use to substitute out terms in the loss function. We start by approximating the LHS to second order. We have

$$\begin{aligned} LHS \approx &(1 + y_t^i + \frac{1}{2}(y_t^i)^2) - v(1 + g_t^i + \frac{1}{2}(g_t^i)^2) \\ =&(1-v) + [\hat{y}_t^i + \tilde{y}_t^i] + \frac{1}{2}[\hat{y}_t^i + \tilde{y}_t^i]^2 - v[\hat{g}_t^i + \tilde{g}_t^i] - \frac{v}{2}[\hat{g}_t^i + \tilde{g}_t^i]^2. \end{aligned}$$

Integrating (using linearity of the integral), noting that $\tilde{y}_t^* = 0$ and $\tilde{g}_t^i = a_t^i$ because we only consider idiosyncratic shocks (and country $i$ is of measure 0) and we defined the natural allocation without policy intervention, respectively, we have

$$LHS \approx (1-v)Y^i + Y^i \hat{y}_t^* + \frac{Y^i}{2}\int_0^1 [\hat{y}_t^i + \tilde{y}_t^i]^2 di - vY^i \hat{g}_t^* - \frac{vY^i}{2}\int_0^1 [\hat{g}_t^i + \tilde{g}_t^i]^2 di. \tag{156}$$

For the RHS, we have

$$\begin{aligned} RHS \approx &(1-\alpha)\frac{S^i C^{*,o}\Theta^i C^i}{C^{i,o}}\left(1 + s_t^i + c_t^{*,o} + \theta_t^i + c_t^i - c_t^{i,o} + \frac{1}{2}[s_t^i + c_t^{*,o} + \theta_t^i + c_t^i - c_t^{i,o}]^2\right) \\ &+ \alpha \int_0^1 \frac{\Theta^j C^j S^i C^{*,o}}{C^{j,o}}\left(1 + s_t^i + c_t^{*,o} + \theta_t^j + c_t^j - c_t^{j,o} + \frac{1}{2}[s_t^i + c_t^{*,o} + \theta_t^j + c_t^j - c_t^{j,o}]^2\right)dj \\ =&(1-\alpha)(1-v)Y^i\left(1 + s_t^i + c_t^{*,o} + \theta_t^i + c_t^i - c_t^{i,o} + \frac{1}{2}[s_t^i + c_t^{*,o} + \theta_t^i + c_t^i - c_t^{i,o}]^2\right) \\ &+ \alpha(1-v)Y^i\left(1 + \theta_t^* + c_t^* + s_t^* + \frac{1}{2}\int_0^1 [s_t^i + c_t^{*,o} + \theta_t^j + c_t^j - c_t^{j,o}]^2 dj\right). \end{aligned}$$

Putting these together, we can write

$$\begin{aligned} &\hat{y}_t^* - v\hat{g}_t^* + \frac{1}{2}\int_0^1 [\hat{y}_t^i + \tilde{y}_t^i]^2 di - \frac{v}{2}\int_0^1 [\hat{g}_t^i + \tilde{g}_t^i]^2 di \\ =&(1-v)c_t^* + (1-v)s_t^* + (1-v)\theta_t^* \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1-v}{2}\int_0^1 (\theta_t^i + c_t^i - c_t^{i,o} + c_t^{*,o})^2 di \\
&+ \frac{(1-\alpha)(1-v)}{2}\int_0^1 \Big( (s_t^i)^2 + 2s_t^i(\theta_t^i + c_t^i - c_t^{i,o} + c_t^{*,o}) \Big) di \\
&+ \frac{\alpha(1-v)}{2}\int_0^1\int_0^1 \Big( (s_t^i)^2 + 2s_t^i(\theta_t^j + c_t^j - c_t^{j,o} + c_t^{*,o}) \Big) dj di.
\end{aligned}
$$

Re-writing,

$$
\begin{aligned}
&\hat{y}_t^* - v\hat{g}_t^* + \frac{1}{2}\int_0^1 (y_t^i)^2 di - \frac{v}{2}\int_0^1 (g_t^i)^2 di \\
=&(1-v)c_t^* + (1-v)s_t^* + (1-v)\theta_t^* \\
&+ \frac{1-v}{2}\int_0^1 \Big( (\theta_t^i)^2 + (c_t^i)^2 + (c_t^{i,o})^2 + (c_t^{*,o})^2 + 2\theta_t^i c_t^i + 2\theta_t^i c_t^{*,o} - 2\theta_t^i c_t^{i,o} - 2c_t^i c_t^{i,o} + 2c_t^i c_t^{*,o} - 2c_t^{i,o} c_t^{*,o} \Big) di \\
&+ (1-\alpha)(1-v)s_t^* c_t^{*,o} + \frac{(1-\alpha)(1-v)}{2}\int_0^1 (s_t^i)^2 di + (1-\alpha)(1-v)\int_0^1 s_t^i(\theta_t^i + c_t^i - c_t^{i,o} + c_t^{*,o}) di \\
&+ \frac{\alpha(1-v)}{2}\int_0^1 (s_t^i)^2 di + \alpha(1-v)(\theta_t^* + c_t^*)\int_0^1 s_t^i di.
\end{aligned}
$$

Note that by definition, we have for all union variables $x_t^* = \hat{x}_t^* + \tilde{x}_t^* = \hat{x}_t^*$ because $\tilde{x}_t^* = 0$ (we use idiosyncratic shocks and country $i$ is of measure 0). Therefore, we also have $\hat{s}_t^* = s_t^* = 0$ and $\hat{\theta}_t^* = \theta_t^* = 0$. Hence, we have

$$
\begin{aligned}
&\hat{y}_t^* - v\hat{g}_t^* + \frac{1}{2}\int_0^1 (y_t^i)^2 di - \frac{v}{2}\int_0^1 (g_t^i)^2 di \\
=&(1-v)\hat{c}_t^* + \frac{1-v}{2}(\hat{c}_t^{*,o})^2 - (1-v)(\hat{c}_t^{*,o})^2 + (1-v)\hat{c}_t^*\hat{c}_t^{*,o} \\
&+ \frac{1-v}{2}\int_0^1 \Big( (\theta_t^i)^2 + (c_t^i)^2 + (c_t^{i,o})^2 + 2\theta_t^i c_t^i - 2\theta_t^i c_t^{i,o} - 2c_t^i c_t^{i,o} \Big) di \\
&+ \frac{1-v}{2}\int_0^1 (s_t^i)^2 di + (1-\alpha)(1-v)\int_0^1 s_t^i(\theta_t^i + c_t^i - c_t^{i,o}) di.
\end{aligned}
$$

And so finally, we have

$$
\begin{aligned}
(1-v)\hat{c}_t^* + v\hat{g}_t^* =&\hat{y}_t^* + \frac{1}{2}\int_0^1 (y_t^i)^2 di - \frac{v}{2}\int_0^1 (g_t^i)^2 di - (1-v)\hat{c}_t^*\hat{c}_t^{*,o} \\
&+ \frac{1-v}{2}(\hat{c}_t^{*,o})^2 - \frac{1-v}{2}\int_0^1 \Big( (\theta_t^i)^2 + (c_t^i)^2 + (c_t^{i,o})^2 + 2\theta_t^i c_t^i - 2\theta_t^i c_t^{i,o} - 2c_t^i c_t^{i,o} \Big) di \\
&- \frac{1-v}{2}\int_0^1 (s_t^i)^2 di - (1-\alpha)(1-v)\int_0^1 s_t^i(\theta_t^i + c_t^i - c_t^{i,o}) di.
\end{aligned} \tag{157}
$$

Therefore, we can write the welfare criterion as

$$
U = \int_0^\infty e^{-\rho t}\Big[ -\alpha_\pi (\pi_{H,t}^i)^2 + U_t^i \Big] dt, \tag{158}
$$

where $\int_0^1 \hat{y}_t^* - y_t^i di = 0$ and thus drops out, and so

$$U_t^i = Z^i + \frac{1-v}{2}(\hat{c}_t^{*,o})^2 - (1-v)\hat{c}_t^*\hat{c}_t^{*,o} \tag{159}$$
$$+ \frac{1}{2}(y_t^i)^2 - \frac{v}{2}(g_t^i)^2 - \frac{1-v}{2}(s_t^i)^2 - (1-\alpha)(1-v)s_t^i(\theta_t^i + c_t^i - c_t^{i,o})$$
$$- \frac{1-v}{2}\Big((\theta_t^i)^2 + (c_t^i)^2 + (c_t^{i,o})^2 + 2\theta_t^i c_t^i - 2\theta_t^i c_t^{i,o} - 2c_t^i c_t^{i,o}\Big)$$
$$- \frac{\chi(1+\phi)}{2}(n_t^{i,r})^2 - \frac{(1-\chi)(1+\phi)}{2}(n_t^{i,o})^2. \tag{160}$$

### D.3.4 Gap Notation

We have now finished deriving a second-order approximation of the social planner's welfare criterion. We want to express this equation entirely in gap notation because we later want to write all optimal policy problems in gap notation as well.

To that end, we convert variables according to $x_t^i = \hat{\hat{x}}_t^i + \hat{x}_t^* + \tilde{x}_t^i$. After some algebra, we find

$$\begin{aligned}
U_t^i =& Z^i + \hat{Z}_t^* + \tilde{Z}_t^i + \frac{1}{2}(\hat{\hat{y}}_t^i)^2 + \hat{\hat{y}}_t^i \tilde{y}_t^i - \frac{v}{2}(\hat{\hat{g}}_t^i)^2 - v\hat{\hat{g}}_t^i\tilde{g}_t^i - \frac{1-v}{2}(\hat{\hat{s}}_t^i)^2 - (1-v)\hat{\hat{s}}_t^i\tilde{s}_t^i \\
&- (1-\alpha)(1-v)\hat{\hat{s}}_t^i[\hat{\hat{\theta}}_t^i + \tilde{\theta}_t^i + \hat{\hat{c}}_t^i + \tilde{c}_t^i - \hat{\hat{c}}_t^{i,o} - \tilde{c}_t^{i,o}] - (1-\alpha)(1-v)\tilde{s}_t^i[\hat{\hat{\theta}}_t^i + \hat{\hat{c}}_t^i - \hat{\hat{c}}_t^{i,o}] \\
&- \frac{1-v}{2}\Big((\hat{\hat{\theta}}_t^i)^2 + 2\hat{\hat{\theta}}_t^i\tilde{\theta}_t^i + (\hat{\hat{c}}_t^i)^2 + 2\hat{\hat{c}}_t^i\tilde{c}_t^i + (\hat{\hat{c}}_t^{i,o})^2 + 2\hat{\hat{c}}_t^{i,o}\tilde{c}_t^{i,o} + 2\hat{\hat{\theta}}_t^i\hat{\hat{c}}_t^i + 2\hat{\hat{\theta}}_t^i\tilde{c}_t^i + 2\tilde{\theta}_t^i\hat{\hat{c}}_t^i \\
&\qquad - 2\hat{\hat{\theta}}_t^i\hat{\hat{c}}_t^{i,o} - 2\hat{\hat{\theta}}_t^i\tilde{c}_t^{i,o} - 2\tilde{\theta}_t^i\hat{\hat{c}}_t^{i,o} - 2\hat{\hat{c}}_t^i\hat{\hat{c}}_t^{i,o} - 2\hat{\hat{c}}_t^i\tilde{c}_t^{i,o} - 2\tilde{c}_t^i\hat{\hat{c}}_t^{i,o}\Big) \\
&- \frac{\chi(1+\phi)}{2}(\hat{\hat{n}}_t^{i,r})^2 - \frac{\chi(1+\phi)}{2}\hat{\hat{n}}_t^{i,r}\tilde{n}_t^{i,r} - \frac{(1-\chi)(1+\phi)}{2}(\hat{\hat{n}}_t^{i,o})^2 - \frac{(1-\chi)(1+\phi)}{2}\hat{\hat{n}}_t^{i,o}\tilde{n}_t^{i,o},
\end{aligned}$$

where

$$\begin{aligned}
\hat{Z}_t^* =& \frac{1}{2}(\hat{y}_t^*)^2 - \frac{v}{2}(\hat{g}_t^*)^2 - \frac{1-v}{2}(\hat{c}_t^*)^2 - \frac{\chi(1+\phi)}{2}(\hat{n}_t^{*,r})^2 - \frac{(1-\chi)(1+\phi)}{2}(\hat{n}_t^{*,o})^2 \\
\tilde{Z}_t^i =& \frac{1}{2}(\tilde{y}_t^i)^2 - \frac{v}{2}(\tilde{g}_t^i)^2 - \frac{1-v}{2}(\tilde{s}_t^i)^2 - (1-\alpha)(1-v)\tilde{s}_t^i[\tilde{\theta}_t^i + \hat{c}_t^* + \tilde{c}_t^i - \hat{c}_t^{*,o} - \tilde{c}_t^{i,o}] - \frac{1-v}{2}(\tilde{\theta}_t^i)^2 - \frac{1-v}{2}(\tilde{c}_t^i)^2 \\
&- \frac{1-v}{2}(\tilde{c}_t^{i,o})^2 - (1-v)\tilde{\theta}_t^i(\hat{c}_t^* + \tilde{c}_t^i) + (1-v)\tilde{\theta}_t^i(\hat{c}_t^{*,o} + \tilde{c}_t^{i,o}) + (1-v)[\tilde{c}_t^i\tilde{c}_t^{i,o} + \hat{c}_t^*\tilde{c}_t^{i,o} + \tilde{c}_t^i\hat{c}_t^{*,o}] \\
&- \frac{\chi(1+\phi)}{2}(\tilde{n}_t^{i,r})^2 - \frac{(1-\chi)(1+\phi)}{2}(\tilde{n}_t^{i,o})^2.
\end{aligned}$$

We also use gap notation for the inflation term in the loss function, noting that $\pi_{H,t}^i = \hat{\pi}_{H,t}^i = \hat{\hat{\pi}}_{H,t}^i + \hat{\pi}_{H,t}^*$. As in the body of the paper, it is easy to verify that the social planner can fully stabilize union-wide inflation so that $\hat{\pi}_{H,t}^* = 0$. Alternatively, we could have added terms in $\hat{\pi}_{H,t}^*$ to the term $\hat{Z}_t^*$ which will be entirely inconsequential for all subsequent analysis since the planning problem can be disaggregated by country, as in the body of the paper, and we will focus exclusively on idiosyncratic shocks in country $i$.

### D.3.5 Final Steps

What remains now is to use the first-order approximation of the equilibrium conditions to solve out terms so that we can write the welfare criterion entirely as a function of $\hat{\hat{y}}_t^i$, $\hat{\hat{\pi}}_{H,t}^i$, $\hat{\hat{\tau}}_t^{i,r}$, $\hat{\hat{\theta}}_t^i$, and $\hat{\hat{g}}_t^i$. This is consistent with the linear-quadratic approach.

Drawing on our earlier derivations, we use the following substitutions:

$$
\begin{aligned}
\hat{\hat{n}}_t^{i,r} &= \eta_y \hat{\hat{y}}_t^i + \eta_g \hat{\hat{g}}_t^i + \eta_\theta \hat{\hat{\theta}}_t^i + \eta_\tau \hat{\hat{\tau}}_t^{i,r} \\
\hat{\hat{n}}_t^{i,o} &= \left(\frac{1}{1-\chi} - \frac{\chi}{1-\chi}\eta_y\right)\hat{\hat{y}}_t^i - \frac{\chi}{1-\chi}\eta_g \hat{\hat{g}}_t^i - \frac{\chi}{1-\chi}\eta_\theta \hat{\hat{\theta}}_t^i - \frac{\chi}{1-\chi}\eta_\tau \hat{\hat{\tau}}_t^{i,r} \\
\hat{\hat{c}}_t^{i,r} &= \psi_y \hat{\hat{y}}_t^i + \psi_\theta \hat{\hat{\theta}}_t^i + \psi_\tau \hat{\hat{\tau}}_t^{i,r} + \psi_g \hat{\hat{g}}_t^i \\
\hat{\hat{c}}_t^{i,o} &= (1-\alpha)\nu_y \hat{\hat{y}}_t^i + [1+(1-\alpha)\nu_\theta]\hat{\hat{\theta}}_t^i + (1-\alpha)\nu_g \hat{\hat{g}}_t^i + (1-\alpha)\nu_\tau \hat{\hat{\tau}}_t^{i,r} \\
\hat{\hat{c}}_t^i &= \zeta_y \hat{\hat{y}}_t^i + \zeta_\theta \hat{\hat{\theta}}_t^i + \zeta_\tau \hat{\hat{\tau}}_t^{i,r} + \zeta_g \hat{\hat{g}}_t^i.
\end{aligned}
$$

After a lot of algebra, the social planner's welfare criterion can be written as

$$
\begin{aligned}
U = \int_0^\infty e^{-\rho t}\int_0^1 \Big[ &-\alpha_\pi(\hat{\hat{\pi}}_{H,t}^i)^2 + Z^i + \hat{Z}_t^* + \tilde{Z}_t^i + \alpha_{yy}(\hat{\hat{y}}_t^i)^2 + \alpha_{\theta\theta}(\hat{\hat{\theta}}_t^i)^2 + \alpha_{\tau\tau}(\hat{\hat{\tau}}_t^{i,r})^2 + \alpha_{gg}(\hat{\hat{g}}_t^i)^2 \\
&+ \alpha_{y\theta}\hat{\hat{y}}_t^i\hat{\hat{\theta}}_t^i + \alpha_{yg}\hat{\hat{y}}_t^i\hat{\hat{g}}_t^i + \alpha_{y\tau}\hat{\hat{y}}_t^i\hat{\hat{\tau}}_t^{i,r} + \alpha_{g\theta}\hat{\hat{g}}_t^i\hat{\hat{\theta}}_t^i + \alpha_{g\tau}\hat{\hat{g}}_t^i\hat{\hat{\tau}}_t^{i,r} + \alpha_{\tau\theta}\hat{\hat{\tau}}_t^{i,r}\hat{\hat{\theta}}_t^i \\
&+ \alpha_{y,t}^i\hat{\hat{y}}_t^i + \alpha_{g,t}^i\hat{\hat{g}}_t^i + \alpha_{\tau,t}^i\hat{\hat{\tau}}_t^i + \alpha_{\theta,t}^i\hat{\hat{\theta}}_t^i\Big] di dt,
\end{aligned} \tag{161}
$$

where

$$
\begin{aligned}
\alpha_{yy} =& \frac{1}{2} - \frac{1-v}{2}\nu_y^2 - (1-\alpha)(1-v)\nu_y(\zeta_y - (1-\alpha)\nu_y) - \frac{1-v}{2}\zeta_y^2 - \frac{1-v}{2}(1-\alpha)^2\nu_y^2 \\
&+ (1-v)(1-\alpha)\nu_y\zeta_y - \frac{\chi(1+\phi)}{2}\eta_y^2 - \frac{(1-\chi)(1+\phi)}{2}\left(\frac{1}{1-\chi} - \frac{\chi}{1-\chi}\eta_y\right)^2 \\
\alpha_{\theta\theta} =& -\frac{1-v}{2}\nu_\theta^2 - (1-\alpha)(1-v)\nu_\theta(\zeta_\theta - (1-\alpha)\nu_\theta) - \frac{1-v}{2} - \frac{1-v}{2}\zeta_\theta^2 - \frac{1-v}{2}[(1-\alpha)\nu_\theta + 1]^2 \\
&- (1-v)(\zeta_\theta - [1+(1-\alpha)\nu_\theta]) + (1-v)[1+(1-\alpha)\nu_\theta]\zeta_\theta - \frac{\chi(1+\phi)}{2}\eta_\theta^2 - \frac{(1-\chi)(1+\phi)}{2}\left(\frac{\chi}{1-\chi}\eta_\theta\right)^2 \\
\alpha_{\tau\tau} =& -\frac{1-v}{2}\nu_\tau^2 - (1-\alpha)(1-v)\nu_\tau(\zeta_\tau - (1-\alpha)\nu_\tau) - \frac{1-v}{2}\zeta_\tau^2 - \frac{1-v}{2}(1-\alpha)^2\nu_\tau^2 \\
&+ (1-v)(1-\alpha)\nu_\tau\zeta_\tau - \frac{\chi(1+\phi)}{2}\eta_\tau^2 - \frac{(1-\chi)(1+\phi)}{2}\left(\frac{\chi}{1-\chi}\eta_\tau\right)^2 \\
\alpha_{gg} =& -\frac{v}{2} - \frac{1-v}{2}\nu_g^2 - (1-\alpha)(1-v)\nu_g(\zeta_g - (1-\alpha)\nu_g) - \frac{1-v}{2}\zeta_g^2 - \frac{1-v}{2}(1-\alpha)^2\nu_g^2 \\
&+ (1-v)(1-\alpha)\nu_g\zeta_g - \frac{\chi(1+\phi)}{2}\eta_g^2 - \frac{(1-\chi)(1+\phi)}{2}\left(\frac{\chi}{1-\chi}\eta_g\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{y\theta} =& -(1-v)\nu_y\nu_\theta - (1-\alpha)(1-v)\big(\nu_y[\zeta_\theta - (1-\alpha)\nu_\theta] + \nu_\theta[\zeta_y - (1-\alpha)\nu_y]\big) \\
&- (1-v)\zeta_y\zeta_\theta - (1-v)(1-\alpha)\nu_y[1+(1-\alpha)\nu_\theta] - (1-v)(\zeta_y - (1-\alpha)\nu_y)
\end{aligned}
$$

$$+ (1-v)(1-\alpha)\nu_y\zeta_\theta + (1-v)[1+(1-\alpha)\nu_\theta]\zeta_y - \chi(1+\phi)\eta_y\eta_\theta - (1+\phi)\chi\eta_\theta(\frac{1}{1-\chi} - \frac{\chi}{1-\chi}\eta_y)$$

$$\begin{aligned}
\alpha_{yg} =& -(1-v)\nu_y\nu_g - (1-\alpha)(1-v)\big(\nu_y[\zeta_g - (1-\alpha)\nu_g] + \nu_g[\zeta_y - (1-\alpha)\nu_y]\big) \\
& - (1-v)\zeta_y\zeta_g - (1-v)(1-\alpha)^2\nu_y\nu_g + (1-v)(1-\alpha)\nu_y\zeta_g + (1-v)(1-\alpha)\nu_g\zeta_y \\
& - \chi(1+\phi)\eta_y\eta_g - (1+\phi)\chi\eta_g(\frac{1}{1-\chi} - \frac{\chi}{1-\chi}\eta_y) \\
\alpha_{y\tau} =& -(1-v)\nu_y\nu_\tau - (1-\alpha)(1-v)\big(\nu_y[\zeta_\tau - (1-\alpha)\nu_\tau] + \nu_\tau[\zeta_y - (1-\alpha)\nu_y]\big) \\
& - (1-v)\zeta_y\zeta_\tau - (1-v)(1-\alpha)^2\nu_y\nu_\tau + (1-v)(1-\alpha)\nu_y\zeta_\tau + (1-v)(1-\alpha)\nu_\tau\zeta_y \\
& - \chi(1+\phi)\eta_y\eta_\tau - (1+\phi)\chi\eta_\tau(\frac{1}{1-\chi} - \frac{\chi}{1-\chi}\eta_y) \\
\alpha_{g\theta} =& -(1-v)\nu_g\nu_\theta - (1-\alpha)(1-v)\big(\nu_g[\zeta_\theta - (1-\alpha)\nu_\theta] + \nu_\theta[\zeta_g - (1-\alpha)\nu_g]\big) \\
& - (1-v)\zeta_g\zeta_\theta - (1-v)(1-\alpha)\nu_g[1+(1-\alpha)\nu_\theta] - (1-v)(\zeta_g - (1-\alpha)\nu_g) \\
& + (1-v)[1+(1-\alpha)\nu_\theta]\zeta_g + (1-v)(1-\alpha)\nu_g\zeta_\theta - \chi(1+\phi)\eta_g\eta_\theta - (1+\phi)\frac{\chi^2}{1-\chi}\eta_\theta\eta_g \\
\alpha_{g\tau} =& -(1-v)\nu_g\nu_\tau - (1-\alpha)(1-v)\big(\nu_g[\zeta_\tau - (1-\alpha)\nu_\tau] + \nu_\tau[\zeta_g - (1-\alpha)\nu_g]\big) \\
& - (1-v)\zeta_g\zeta_\tau - (1-v)(1-\alpha)^2\nu_g\nu_\tau + (1-v)(1-\alpha)\nu_g\zeta_\tau + (1-v)(1-\alpha)\nu_\tau\zeta_g \\
& - \chi(1+\phi)\eta_g\eta_\tau - (1+\phi)\frac{\chi^2}{1-\chi}\eta_g\eta_\tau \\
\alpha_{\tau\theta} =& -(1-v)\nu_\tau\nu_\theta - (1-\alpha)(1-v)\big(\nu_\tau[\zeta_\theta - (1-\alpha)\nu_\theta] + \nu_\theta[\zeta_\tau - (1-\alpha)\nu_\tau]\big) \\
& - (1-v)\zeta_\tau\zeta_\theta - (1-v)(1-\alpha)\nu_\tau[1+(1-\alpha)\nu_\theta] - (1-v)(\zeta_\tau - (1-\alpha)\nu_g\tau) \\
& + (1-v)[1+(1-\alpha)\nu_\theta]\zeta_\tau + (1-v)(1-\alpha)\nu_\tau\zeta_\theta - \chi(1+\phi)\eta_\tau\eta_\theta - (1+\phi)\frac{\chi^2}{1-\chi}\eta_\theta\eta_\tau
\end{aligned}$$

$$\begin{aligned}
\alpha^i_{y,t} =& a^i_t - (1-v)\nu_y a^i_t - (1-\alpha)(1-v)[\zeta_y - (1-\alpha)\nu_y]a^i_t \\
\alpha^i_{g,t} =& -(1-v)\nu_g a^i_t - (1-\alpha)(1-v)[\zeta_g - (1-\alpha)\nu_g]a^i_t - v a^i_t \\
\alpha^i_{\tau,t} =& -(1-v)\nu_\tau a^i_t - (1-\alpha)(1-v)[\zeta_\tau - (1-\alpha)\nu_\tau]a^i_t \\
\alpha^i_{\theta,t} =& -(1-v)\nu_\theta a^i_t - (1-\alpha)(1-v)[\zeta_\theta - (1-\alpha)\nu_\theta]a^i_t.
\end{aligned}$$

It turns out that we generally have $\alpha^i_{y,t} = \alpha^i_{g,t} = \alpha^i_{\tau,t} = \alpha^i_{\theta,t} = 0$ for all $t$ and $i$.

### D.3.6 Loss Function

We will find it more convenient later on to work with the loss function instead of the welfare function. The former is the negative of the latter. Moreover, as in the body of the paper, we will be interested in the loss function in gaps from its natural counterpart. Note in (161) that for the loss function under the natural allocation, all terms drop out except for $Z^i + \hat{Z}^*_t + \tilde{Z}^i_t$. Therefore, we have the following loss function in gaps from the natural

$$\begin{aligned}
\mathbb{L} &= -\left(U - \tilde{U}\right) \qquad (162)\\
&= \int_0^\infty \int_0^1 e^{-\rho t}\Big[\alpha_\pi(\hat{\tilde{\pi}}^i_{H,t})^2 - \alpha_{yy}(\hat{\tilde{y}}^i_t)^2 - \alpha_{\theta\theta}(\hat{\tilde{\theta}}^i_t)^2 - \alpha_{\tau\tau}(\hat{\tilde{\tau}}^{i,r}_t)^2 - \alpha_{gg}(\hat{\tilde{g}}^i_t)^2 \\
&\qquad - \alpha_{y\theta}\hat{\tilde{y}}^i_t\hat{\tilde{\theta}}^i_t - \alpha_{yg}\hat{\tilde{y}}^i_t\hat{\tilde{g}}^i_t - \alpha_{y\tau}\hat{\tilde{y}}^i_t\hat{\tilde{\tau}}^{i,r}_t - \alpha_{g\theta}\hat{\tilde{g}}^i_t\hat{\tilde{\theta}}^i_t - \alpha_{g\tau}\hat{\tilde{g}}^i_t\hat{\tilde{\tau}}^{i,r}_t - \alpha_{\tau\theta}\hat{\tilde{\tau}}^{i,r}_t\hat{\tilde{\theta}}^i_t\Big] didt \\
&\equiv \int_0^\infty \int_0^1 e^{-\rho t}\Big[\alpha_\pi(\hat{\tilde{\pi}}^i_{H,t})^2 - \hat{\tilde{U}}^i_t\Big] didt.
\end{aligned}$$

### D.3.7 General Planning Problem

In summary, in this section we have derived all elements that comprise the general planning problem for country $i$, featuring all three policy instruments. Assembling them now, we can write the fully general country-$i$ planning problem as follows:

$$\min_{\{\hat{\tilde{\theta}}^i_t, \hat{\tilde{g}}^i_t, \hat{\tilde{\tau}}^{i,r}_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t}\Big[\alpha_\pi(\hat{\tilde{\pi}}^i_{H,t})^2 - \hat{\tilde{U}}^i_t\Big] dt \qquad (163)$$

subject to

$$\begin{aligned}
\Lambda_y \dot{\hat{\tilde{y}}}^i_t &= -\Lambda_s(\hat{\tilde{\pi}}^i_{H,t} + \dot{\tilde{s}}^i_t) + \Lambda_g \dot{\hat{\tilde{g}}}^i_t + \Lambda_\theta \dot{\hat{\tilde{\theta}}}^i_t + \Lambda_\tau \dot{\hat{\tilde{\tau}}}^{i,r}_t \\
\dot{\hat{\tilde{\pi}}}^i_{H,t} &= \rho\hat{\tilde{\pi}}^i_{H,t} - \kappa_y \hat{\tilde{y}}^i_t - \kappa_\theta \hat{\tilde{\theta}}^i_t - \kappa_\tau \hat{\tilde{\tau}}^{i,r}_t - \kappa_g \hat{\tilde{g}}^i_t \\
\Lambda_y \hat{\tilde{y}}^i_0 &= -\Lambda_s \tilde{s}^i_0 + \Lambda_g \hat{\tilde{g}}^i_0 + \Lambda_\theta \hat{\tilde{\theta}}^i_0 + \Lambda_\tau \hat{\tilde{\tau}}^{i,r}_0 \\
0 &= \int_0^\infty e^{-\rho t}\Big[\Gamma_y \hat{\tilde{y}}^i_t + \Gamma_\tau \hat{\tilde{\tau}}^{i,r}_t + \Gamma_\theta \hat{\tilde{\theta}}^i_t + \Gamma_g \hat{\tilde{g}}^i_t\Big] dt.
\end{aligned}$$

We note again that the coordinated union-wide planning problem coincides with the disaggregated problem, where the social planner solves the above optimal control problem for each country $i \in [0,1]$ separately. This follows from the analogous result derived in the main body of the paper. In the remainder of this appendix, we will therefore only consider the disaggregated planning problem of country $i$ under an idiosyncratic productivity shock.

The constraints in this planning problem are the relevant implementability conditions. All other equilibrium conditions can be used to simply back out the equilibrium variables that are not featured in the planning problem.

## D.4 Optimal Transfers

We now consider the problem where the social planner only has access to ex-post transfers, as in body of the paper. In particular, we have $\hat{\tilde{\theta}}^i_t = \hat{\tilde{\theta}}^i$ for all $t$ and $\hat{\tilde{g}}^i_t = 0$.

Country $i$'s planning problem can then be written as

$$\min_{\hat{\bar{\theta}}^i, \{\hat{\bar{\tau}}_t^{i,r}\}_{t\geq 0}} \int_0^\infty e^{-\rho t} \left[ \alpha_\pi (\hat{\bar{\pi}}_{H,t}^i)^2 - \hat{U}_t^i |_{\hat{\bar{\theta}}_t^i = \hat{\bar{\theta}}^i, \hat{\bar{g}}_t^i = 0} \right] dt, \tag{164}$$

subject to

$$\begin{aligned}
\dot{\hat{\bar{y}}}_t^i &= -D'(\hat{\bar{\pi}}_{H,t}^i + \dot{\tilde{s}}_t^i) + D\hat{v}_t^i \\
\dot{\hat{\bar{\tau}}}_t^{i,r} &= \hat{v}_t^i \\
\dot{\hat{\bar{\pi}}}_{H,t}^i &= \rho \hat{\bar{\pi}}_{H,t}^i - \kappa_y \hat{\bar{y}}_t^i - \kappa_\theta \hat{\bar{\theta}}^i - \kappa_\tau \hat{\bar{\tau}}_t^{i,r}
\end{aligned}$$

where $D' = \frac{\Lambda_s}{\Lambda_y}$ and $D = \frac{\Lambda_\tau}{\Lambda_y}$, and also subject to the intial condition

$$\Lambda_y \hat{\bar{y}}_0^i = -\Lambda_s \tilde{s}_0^i + \Lambda_\theta \hat{\bar{\theta}}^i + \Lambda_\tau \hat{\bar{\tau}}_0^{i,r} \tag{165}$$

which will be used to ensure the stability of the dynamical system, and the budget constraint

$$0 = \int_0^\infty e^{-\rho t} \left[ \Gamma_y \hat{\bar{y}}_t^i + \Gamma_\tau \hat{\bar{\tau}}_t^{i,r} + \Gamma_\theta \hat{\bar{\theta}}^i \right] dt \tag{166}$$

which will be used to solve for $\hat{\bar{\theta}}^i$. In particular, our strategy will be to solve the optimal allocation and policy intervention as a function of $\hat{\bar{\theta}}^i$ analytically, and then numerically solve for the optimal $\hat{\bar{\theta}}^i$ in a second stage.

Also note that we have replaced $\dot{\hat{\bar{\tau}}}_t^{i,r}$ in the (IS) equation with another variable. This is necessary to bring the planning problem into the form of a standard optimal control problem. Finally, we will integrate the initial condition and the budget constraint, an isoperimetric constraint in this case, into the optimal control problem using Lagrange multipliers. In particular, let

- $x_t^i = \{\hat{\bar{\pi}}_{H,t}^i, \hat{\bar{y}}_t^i, \hat{\bar{\tau}}_t^{i,r}\}$ denote the vector of state variables,
- $u_t^i = \{\hat{v}_t^i\}$ denote the vector of control variables, and
- $\mu_t^i = \{\mu_{\pi,t}^i, \mu_{y,t}^i, \mu_{\tau,t}^i\}$ denote the vector of costates.

Then we can summarize country $i$'s optimal control problem with its current-value Hamiltonian, which is given by

$$\begin{aligned}
H(x_t^i, u_t^i, \mu_t^i) =& \alpha_\pi (\pi_{H,t}^i)^2 - \hat{U}_t^i |_{\hat{\bar{\theta}}_t^i = \hat{\bar{\theta}}^i, \hat{\bar{g}}_t^i = 0} \\
&+ \lambda \left[ \Gamma_y \hat{\bar{y}}_t^i + \Gamma_\tau \hat{\bar{\tau}}_t^{i,r} + \Gamma_\theta \hat{\bar{\theta}}^i \right] \\
&+ \Delta \left[ \Lambda_y \hat{\bar{y}}_0^i + \Lambda_s \tilde{s}_0^i - \Lambda_\theta \hat{\bar{\theta}}^i - \Lambda_\tau \hat{\bar{\tau}}_0^{i,r} \right] \\
&+ \mu_{y,t}^i \left[ D\hat{v}_t^i - D'(\hat{\bar{\pi}}_{H,t}^i + \dot{\tilde{s}}_t^i) \right]
\end{aligned} \tag{167}$$

$$+ \mu^i_{\tau,t}\hat{\dot{v}}^i_t$$
$$+ \mu^i_{\pi,t}\left[\rho\hat{\hat{\pi}}^i_{H,t} - \kappa_y\hat{\hat{y}}^i_t - \kappa_\theta\hat{\hat{\theta}}^i - \kappa_\tau\hat{\hat{\tau}}^{i,r}_t\right].$$

### D.4.1 Optimality Conditions

Since we have transformed the planning problem into a standard optimal control problem, the optimality conditions are given by

[(1)]

i. $\partial_u H^i_t = 0$ (Optimality)

ii. $\rho\mu^i_{x,t} - \dot{\mu}^i_{x,t} = \partial_x H^i_t$ (Multiplier)

iii. $\dot{x}^i_t = \partial_{\mu_x} H^i_t$ (State equations)

iv. $\mu^i_{x,0} = \partial_x H^i_0$ (Initial conditions),

where we abuse notation slightly to let $x$ and $u$ stand in for each element of the respective vectors. $\partial_x$ denotes the partial derivative with respect to $x$. This yields the following nine first-order conditions:

**Optimality**:

$$D\mu^i_{y,t} + \mu^i_{\tau,t} = 0. \tag{168}$$

**Multiplier**:

$$\dot{\mu}^i_{\pi,t} = D'\mu^i_{y,t} - 2\alpha_\pi\hat{\hat{\pi}}^i_{H,t} \tag{169}$$
$$\rho\mu^i_{y,t} - \dot{\mu}^i_{y,t} = \lambda\Gamma_y - \kappa_y\mu^i_{\pi,t} - 2\alpha_{yy}\hat{\hat{y}}^i_t - \alpha_{y\theta}\hat{\hat{\theta}}^i - \alpha_{y\tau}\hat{\hat{\tau}}^{i,r}_t \tag{170}$$
$$\rho\mu^i_{\tau,t} - \dot{\mu}^i_{\tau,t} = \lambda\Gamma_\tau - \kappa_\tau\mu^i_{\pi,t} - 2\alpha_{\tau\tau}\hat{\hat{\tau}}^{i,r}_t - \alpha_{\tau\theta}\hat{\hat{\theta}}^i - \alpha_{y\tau}\hat{\hat{y}}^i_t \tag{171}$$

**State**:

$$\dot{\hat{\hat{y}}}^i_t = -D'(\hat{\hat{\pi}}^i_{H,t} + \dot{\hat{s}}^i_t) + D\hat{\dot{v}}^i_t$$
$$\dot{\hat{\hat{\tau}}}^{i,r}_t = \hat{\dot{v}}^i_t$$
$$\dot{\hat{\hat{\pi}}}^i_{H,t} = \rho\hat{\hat{\pi}}^i_{H,t} - \kappa_y\hat{\hat{y}}^i_t - \kappa_\theta\hat{\hat{\theta}}^i - \kappa_\tau\hat{\hat{\tau}}^{i,r}_t$$

**Initial conditions**:

$$\mu^i_{\pi,0} = 0 \tag{172}$$
$$\mu^i_{y,0} = \Delta\Lambda_y \tag{173}$$
$$\mu^i_{\tau,0} = -\Delta\Lambda_\tau. \tag{174}$$

We will now reduce the size of the above dynamical system by solving out for $\mu^i_{\pi,t}$ and $\mu^i_{\tau,t}$. Using equations (168), (170) and (171), we can write

$$(\kappa_\tau + D\kappa_y)\mu^i_{\pi,t} = (\Gamma_y D + \Gamma_\tau)\lambda - (2\alpha_{\tau\tau} + D\alpha_{y\tau})\hat{\tau}^{i,r}_t - (2D\alpha_{yy} + \alpha_{y\tau})\hat{y}^i_t - (\alpha_{\tau\theta} + D\alpha_{y\theta})\hat{\theta}^i. \tag{175}$$

Taking the derivative with respect to time and using (169), we have

$$(\kappa_\tau + D\kappa_y)(D'\mu^i_{y,t} - 2\alpha_\pi \hat{\pi}^i_{H,t}) + (2\alpha_{\tau\tau} + D\alpha_{y\tau})\dot{\hat{\tau}}^{i,r}_t + (2D\alpha_{yy} + \alpha_{y\tau})\dot{\hat{y}}^i_t = 0,$$

from which we can solve for the rate of change of transfers using the (IS) equation:

$$\dot{\hat{\tau}}^{i,r}_t = -\frac{(\kappa_\tau + D\kappa_y)(D'\mu^i_{y,t} - 2\alpha_\pi \hat{\pi}^i_{H,t}) - D'(2D\alpha_{yy} + \alpha_{y\tau})(\hat{\pi}^i_{H,t} + \dot{s}^i_t)}{2\alpha_{\tau\tau} + D\alpha_{y\tau} + D(2D\alpha_{yy} + \alpha_{y\tau})} \tag{176}$$

Using the (IS) equation again, we have

$$\dot{\hat{y}}^i_t = -D'(\hat{\pi}^i_{H,t} + \dot{s}^i_t) - D\frac{(\kappa_\tau + D\kappa_y)(D'\mu^i_{y,t} - 2\alpha_\pi \hat{\pi}^i_{H,t}) - D'(2D\alpha_{yy} + \alpha_{y\tau})(\hat{\pi}^i_{H,t} + \dot{s}^i_t)}{2\alpha_{\tau\tau} + D\alpha_{y\tau} + D(2D\alpha_{yy} + \alpha_{y\tau})}. \tag{177}$$

Finally, using (170) and our solution for $\mu^i_{\pi,t}$ above, we find

$$\dot{\mu}^i_{y,t} = \rho\mu^i_{y,t} + \lambda\frac{\kappa_y\Gamma_\tau - \kappa_\tau\Gamma_y}{\kappa_\tau + D\kappa_y} - \hat{\tau}^{i,r}_t\frac{2\kappa_y\alpha_{\tau\tau} - \kappa_\tau\alpha_{y\tau}}{\kappa_\tau + D\kappa_y} - \hat{y}^i_t\frac{\alpha_{y\tau}\kappa_y - 2\kappa_\tau\alpha_{yy}}{\kappa_\tau + D\kappa_y} - \hat{\theta}^i\frac{\kappa_y\alpha_{\tau\theta} - \kappa_\tau\alpha_{y\theta}}{\kappa_\tau + D\kappa_y}. \tag{178}$$

Therefore, we can write the reduced system of optimality conditions as

$$\dot{\mu}^i_{y,t} = \rho\mu^i_{y,t} + K_\tau\hat{\tau}^{i,r}_t + K_y\hat{y}^i_t + K_\theta\hat{\theta}^i + K_\lambda\lambda \tag{179}$$

$$\dot{\hat{y}}^i_t = J_{\mu_y}\mu^i_{y,t} + J_\pi\hat{\pi}^i_{H,t} + J_s\dot{s}^i_t \tag{180}$$

$$\dot{\hat{\tau}}^{i,r}_t = H_{\mu_y}\mu^i_{y,t} + H_\pi\hat{\pi}^i_{H,t} + H_s\dot{s}^i_t \tag{181}$$

$$\dot{\hat{\pi}}^i_{H,t} = \rho\hat{\pi}^i_{H,t} - \kappa_y\hat{y}^i_t - \kappa_\theta\hat{\theta}^i - \kappa_\tau\hat{\tau}^{i,r}_t. \tag{182}$$

### D.4.2 Solving the Dynamic System

In particular, the optimality conditions yield a system of linear ODEs which, letting $X^i_t = \{\hat{\pi}^i_{H,t}, \hat{y}^i_t, \hat{\tau}^{i,r}_t, \mu^i_{y,t}\}$, we can express as

$$\dot{X}^i_t = AX^i_t + B^i_t, \tag{183}$$

where

$$A = \begin{pmatrix} \rho & -\kappa_y & -\kappa_\tau & 0 \\ J_\pi & 0 & 0 & J_{\mu_y} \\ H_\pi & 0 & 0 & H_{\mu_y} \\ 0 & K_y & K_\tau & \rho \end{pmatrix}, \qquad B^i_t = \begin{pmatrix} -\kappa_\theta\hat{\theta}^i \\ \bar{J}^i_t \\ \bar{H}^i_t \\ \bar{K}^i_t \end{pmatrix}, \qquad X^i_0 = \begin{pmatrix} \hat{\pi}^i_{H,0} \\ \hat{y}^i_0 \\ \hat{\tau}^{i,r}_0 \\ \Delta\Lambda_y \end{pmatrix},$$

where $\bar{J}_t^i = J_s \hat{\tilde{s}}_t^i$, $\bar{H}_t^i = H_s \hat{\tilde{s}}_t^i$, and $\bar{K}_t^i = K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda$.

The solution to this dynamical system is, as is well known, given by

$$X_t^i = e^{At}\left[X_0^i + \int_0^t e^{-As} B_s^i ds\right],$$

which we can rewrite as

$$X_t^i = e^{At}\left[X_0^i + \int_0^t e^{-As}\Big( - \kappa_\theta \hat{\hat{\theta}}^i E_1 + \bar{J}_s^i E_2 + \bar{H}_s^i E_3 + \bar{K}_s^i E_4 ds\right],$$

where $E_i$ is the $4 \times 1$ zero vector with a 1 in the $i^{th}$ position. We can write

$$\begin{aligned} X_t^i = e^{At}\Bigg[X_0^i - \kappa_\theta \hat{\hat{\theta}}^i \int_0^t e^{-As} E_1 ds + (K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda) \int_0^t e^{-As} E_4 ds \\ + W_2 \int_0^t e^{-As-\psi s} E_2 ds + W_3 \int_0^t e^{-As-\psi s} E_3 ds\Bigg], \end{aligned}$$

where $W_2 = -\psi a_0^i J_s$, and $W_3 = -\psi a_0^i H_s$.

Assuming that the economy is in the parameter subspace where $A$ is nonsingular, we can use the results $\int_0^t e^{-As} ds = A^{-1}(I - e^{-At})$ and $\int_0^t e^{-(A+\psi I)s} ds = (A + \psi I)^{-1}(I - e^{-(A+\psi I)t})$ to solve the integral, so that

$$\begin{aligned} X_t^i = e^{At}\Bigg[X_0^i - \kappa_\theta \hat{\hat{\theta}}^i A^{-1}(I - e^{-At}) E_1 + (K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda) A^{-1}(I - e^{-At}) E_4 \\ + W_2 (A + \psi I)^{-1}(I - e^{-(A+\psi I)t}) E_2 + W_3 (A + \psi I)^{-1}(I - e^{-(A+\psi I)t}) E_3\Bigg]. \end{aligned}$$

From this, we finally arrive at the solution

$$\begin{aligned} X_t^i =& e^{At}\left[X_0^i - \kappa_\theta \hat{\hat{\theta}}^i A^{-1} E_1 + (K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda) A^{-1} E_4 + (A + \psi I)^{-1}\big(W_2 E_2 + W_3 E_3\big)\right] \\ &+ \kappa_\theta \hat{\hat{\theta}}^i A^{-1} E_1 - (K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda) A^{-1} E_4 - (A + \psi I)^{-1} e^{-\psi t}\big(W_2 E_2 + W_3 E_3\big) \end{aligned} \tag{184}$$

### D.4.3 Stability

There is a unique solution if and only if $A$ has two negative eigenvalues. For stability, we then need

$$X_0^i - \kappa_\theta \hat{\hat{\theta}}^i A^{-1} E_1 + (K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda) A^{-1} E_4 + (A + \psi I)^{-1}\big(W_2 E_2 + W_3 E_3\big)$$

to be in the stable manifold, the subspace of the state space spanned by the eigenvectors associated with the negative eigenvalues. Let $V_j$, $j \in \{1, 2\}$, be the eigenvector of $A$ associated with the negative eigenvalue $\lambda_j$. Then there must be $\alpha_j \in \mathbb{C}, j \in \{1,2\}$ such that

$$0 = \alpha_1 V_1 + \alpha_2 V_2 + X_0^i - \kappa_\theta \hat{\hat{\theta}}^i A^{-1} E_1 + (K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda) A^{-1} E_4 + (A + \psi I)^{-1} \big(W_2 E_2 + W_3 E_3\big). \tag{185}$$

In particular, to guarantee stability we need to solve for the vector $Z = (\alpha_1, \alpha_2, \lambda, \Delta, \hat{\hat{\pi}}^i_{H,0}, \hat{\hat{y}}^i_0, \hat{\hat{\tau}}^{i,r}_0)' \in \mathbb{C}^7$. Therefore, we need seven linearly independent conditions involving the desired variables. Equation (185) yields four of these conditions. To see this, left-multiply (185) by $E_i'$ for each $i \in \{1,2,3,4\}$.

The fifth stability condition will be the initial condition, $\Lambda_y \hat{\hat{y}}^i_0 = -\Lambda_s \hat{\hat{s}}^i_0 + \Lambda_\tau \hat{\hat{\tau}}^{i,r}_0 + \Lambda_\theta \hat{\hat{\theta}}^i$. The sixth condition we obtain by writing

$$\begin{aligned}(\kappa_\tau + D\kappa_y)\mu^i_{\pi,0} &= 0 \\ &= (\Gamma_y D + \Gamma_\tau)\lambda - (2\alpha_{\tau\tau} + D\alpha_{y\tau})\hat{\hat{\tau}}^{i,r}_0 - (2D\alpha_{yy} + \alpha_{y\tau})\hat{\hat{y}}^i_0 - (\alpha_{\tau\theta} + D\alpha_{y\theta})\hat{\hat{\theta}}^i.\end{aligned} \tag{186}$$

The seventh and final condition we obtain from country $i$'s budget constraint. In particular, we have

$$\begin{aligned}0 &= \int_0^\infty e^{-\rho t}\Big[\Gamma_y \hat{\hat{y}}^i_t + \Gamma_\tau \hat{\hat{\tau}}^{i,r}_t + \Gamma_\theta \hat{\hat{\theta}}^i\Big] dt \\ &= \frac{\Gamma_\theta}{\rho}\hat{\hat{\theta}}^i + \int_0^\infty e^{-\rho t}(\Gamma_y E_2 + \Gamma_\tau E_3)' X^i_t dt.\end{aligned}$$

We can write

$$\begin{aligned}X^i_t &= e^{At}\big(-\alpha_1 V_1 - \alpha_2 V_2\big) + \kappa_\theta \hat{\hat{\theta}}^i A^{-1} E_1 - (K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda) A^{-1} E_4 - (A+\psi I)^{-1} e^{-\psi t}\big(W_2 E_2 + W_3 E_3\big) \\ &= -\alpha_1 e^{\lambda_1 t} V_1 - \alpha_2 e^{\lambda_2 t} V_2 + \kappa_\theta \hat{\hat{\theta}}^i A^{-1} E_1 - (K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda) A^{-1} E_4 - (A+\psi I)^{-1} e^{-\psi t}\big(W_2 E_2 + W_3 E_3\big).\end{aligned}$$

Plugging this in and solving the integrals, we obtain our final stability condition

$$\begin{aligned}0 =& \frac{\Gamma_\theta}{\rho}\hat{\hat{\theta}}^i + \frac{\alpha_1}{\lambda_1 - \rho}(\Gamma_y E_2 + \Gamma_\tau E_3)' V_1 + \frac{\alpha_2}{\lambda_2 - \rho}(\Gamma_y E_2 + \Gamma_\tau E_3)' V_2 \\ &+ \frac{\kappa_\theta \hat{\hat{\theta}}^i}{\rho}(\Gamma_y E_2 + \Gamma_\tau E_3)' A^{-1} E_1 - \frac{K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda}{\rho}(\Gamma_y E_2 + \Gamma_\tau E_3)' A^{-1} E_4 \\ &- \frac{1}{\rho + \psi}(\Gamma_y E_2 + \Gamma_\tau E_3)'(A + \psi I)^{-1}[W_2 E_2 + W_3 E_3].\end{aligned} \tag{187}$$

Putting everything together, we can write $MZ = N$, where

$$M = \begin{pmatrix} E_1'V_1 & E_1'V_2 & K_\lambda E_1'A^{-1}E_4 & 0 & 1 & 0 & 0 \\ E_2'V_1 & E_2'V_2 & K_\lambda E_2'A^{-1}E_4 & 0 & 0 & 1 & 0 \\ E_3'V_1 & E_3'V_2 & K_\lambda E_3'A^{-1}E_4 & 0 & 0 & 0 & 1 \\ E_4'V_1 & E_4'V_2 & K_\lambda E_4'A^{-1}E_4 & \Lambda_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_y & -\Lambda_\tau \\ 0 & 0 & \Gamma_\tau + D\Gamma_y & 0 & 0 & -(2\alpha_{yy}D + \alpha_{y\tau}) & -(2\alpha_{\tau\tau} + D\alpha_{y\tau}) \\ \frac{(\Gamma_y E_2 + \Gamma_\tau E_3)'}{\lambda_1 - \rho}V_1 & \frac{(\Gamma_y E_2 + \Gamma_\tau E_3)'}{\lambda_2 - \rho}V_2 & -\frac{K_\lambda}{\rho}(\Gamma_y E_2 + \Gamma_\tau E_3)'A^{-1}E_4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$N = \begin{pmatrix} -E_1'W \\ -E_2'W \\ -E_3'W \\ -E_4'W \\ -\Lambda_s \tilde{s}_0^i + \Lambda_\theta \hat{\bar{\theta}}^i \\ (\alpha_{\tau\theta} + D\alpha_{y\theta})\hat{\bar{\theta}}^i \\ -\frac{\Gamma_\theta}{\rho}\hat{\bar{\theta}}^i - \frac{\kappa_\theta \hat{\bar{\theta}}^i}{\rho}(\Gamma_y E_2 + \Gamma_\tau E_3)'A^{-1}E_1 + \frac{K_\theta \hat{\bar{\theta}}^i}{\rho}(\Gamma_y E_2 + \Gamma_\tau E_3)'A^{-1}E_4 + \frac{(\Gamma_y E_2 + \Gamma_\tau E_3)'}{\rho + \psi}(A + \psi I)^{-1}[W_2 E_2 + W_3 E_3] \end{pmatrix}$$

where $W = -\kappa_\theta \hat{\bar{\theta}}^i A^{-1}E_1 + K_\theta \hat{\bar{\theta}}^i A^{-1}E_4 + (A + \psi I)^{-1}[W_2 E_2 + W_3 E_3]$. Given these matrices, we can compute the vector $Z$ numerically by setting

$$Z = M^{-1}N \tag{188}$$

over the parameter subspace on which $M$ is invertible.

## D.5 Optimal Capital Controls

We now consider the case where the social planner only has access to capital controls. That is, $\hat{\tilde{g}}_t^i = 0$ for all $t$ and $NFA_0^i = 0$. The planner's choice of capital controls will be encoded in a time-varying $\hat{\bar{\theta}}_t^i$.

This directly implies that $N\hat{\tilde{F}}A_0^i = 0$ in the absence of transfers. Furthermore, since we assume that the government does not engage in additional redistribution, we have $\tau_t^{i,r} = -(S_t^i)^{-\alpha} G_t^i$. Similarly, $\tau_t^{i,r} = \tau_t^{i,o}$ because the government does not discriminate between different types of agents. Hence, we have

$$\hat{\tilde{\tau}}_t^{i,r} = \alpha v \hat{\tilde{s}}_t^i. \tag{189}$$

Plugging in for $\hat{\tilde{s}}_t^i = \nu_y \hat{\tilde{y}}_t^i + \nu_g \hat{\tilde{g}}_t^i + \nu_\tau \hat{\tilde{\tau}}_t^i + \nu_\theta \hat{\bar{\theta}}_t^i$, with $\hat{\tilde{g}}_t^i = 0$, we find that

$$\hat{\tilde{\tau}}_t^{i,r} = \frac{\alpha v}{1 - \alpha v \nu_\tau}\left(\nu_y \hat{\tilde{y}}_t^i + \nu_\theta \hat{\bar{\theta}}_t^i\right) \tag{190}$$

$$\dot{\hat{\tilde{\tau}}}_t^{i,r} = \frac{\alpha v}{1 - \alpha v \nu_\tau}\left(\nu_y \dot{\hat{\tilde{y}}}_t^i + \nu_\theta \dot{\hat{\bar{\theta}}}_t^i\right). \tag{191}$$

**IS equation.** This allows us to rewrite the general (IS) equation for this particular allocation as

$$\Lambda_y \dot{\hat{\tilde{y}}}_t^i = -\Lambda_s(\hat{\tilde{\pi}}_{H,t}^i + \dot{\tilde{s}}_t^i) + \frac{\alpha \upsilon \nu_y}{1-\alpha\upsilon\nu_\tau}\Lambda_\tau \dot{\hat{\tilde{y}}}_t^i + \left(\Lambda_\theta + \frac{\alpha\upsilon\nu_\theta}{1-\alpha\upsilon\nu_\tau}\Lambda_\tau\right)\dot{\hat{\tilde{\theta}}}_t^i, \tag{192}$$

or, more conveniently,

$$\dot{\hat{\tilde{y}}}_t^i = -G'(\hat{\tilde{\pi}}_{H,t}^i + \dot{\tilde{s}}_t^i) + G\dot{\hat{\tilde{\theta}}}_t^i, \tag{193}$$

where

$$G = \frac{\Lambda_\theta + \frac{\alpha\upsilon\nu_\theta}{1-\alpha\upsilon\nu_\tau}\Lambda_\tau}{\Lambda_y - \frac{\alpha\upsilon\nu_y}{1-\alpha\upsilon\nu_\tau}\Lambda_\tau}$$
$$G' = \frac{\Lambda_s}{\Lambda_y - \frac{\alpha\upsilon\nu_y}{1-\alpha\upsilon\nu_\tau}\Lambda_\tau}.$$

**NKPC.** Similarly, we can rewrite the Phillips Curve noting that we now have

$$\dot{\hat{\tilde{\pi}}}_{H,t}^i = \rho\hat{\tilde{\pi}}_{H,t}^i - \kappa_y \hat{\tilde{y}}_t^i - \kappa_\theta \hat{\tilde{\theta}}_t^i - \frac{\alpha\upsilon}{1-\alpha\upsilon\nu_\tau}\kappa_\tau\left(\nu_y\hat{\tilde{y}}_t^i + \nu_\theta\hat{\tilde{\theta}}_t^i\right), \tag{194}$$

or, more conveniently,

$$\dot{\hat{\tilde{\pi}}}_{H,t}^i = \rho\hat{\tilde{\pi}}_{H,t}^i - \hat{\kappa}_y \hat{\tilde{y}}_t^i - \hat{\kappa}_\theta \hat{\tilde{\theta}}_t^i, \tag{195}$$

where

$$\hat{\kappa}_y = \kappa_y + \frac{\alpha\upsilon}{1-\alpha\upsilon\nu_\tau}\kappa_\tau\nu_y$$
$$\hat{\kappa}_\theta = \kappa_\theta + \frac{\alpha\upsilon}{1-\alpha\upsilon\nu_\tau}\kappa_\tau\nu_\theta.$$

**Initial condition.** Following similar conversions relative to the general specification presented in the last section, we find that

$$\hat{\tilde{y}}_0^i = -\bar{G}'\tilde{s}_0^i + \bar{G}\hat{\tilde{\theta}}_0^i, \tag{196}$$

for

$$\bar{G} = \frac{\Lambda_\theta}{\Lambda_y}$$
$$\bar{G}' = \frac{\Lambda_s + \alpha\upsilon\Lambda_\tau}{\Lambda_y}.$$

**NFA.** Finally, we record here the simplified country-$i$ budget constraint when only capital controls are used. We have

$$0 = \int_0^\infty e^{-\rho t}\left(\hat{\Gamma}_y\hat{\tilde{y}}_t^i + \hat{\Gamma}_\theta\hat{\tilde{\theta}}_t^i\right)dt, \tag{197}$$

where

$$\hat{\Gamma}_y = \Gamma_y + \frac{\alpha v}{1-\alpha v \nu_\tau}\nu_y \Gamma_\tau$$
$$\hat{\Gamma}_\theta = \Gamma_\theta + \frac{\alpha v}{1-\alpha v \nu_\tau}\nu_\theta \Gamma_\tau.$$

We note at this point that $\hat{\Gamma}_y = 0$ over the entire parameter space. We will continue to carry the term around in this section, but we want to highlight the important implication of this result: informally, present-value capital controls must always average to zero over time. Formally, of course, we must have

$$0 = \int_0^\infty e^{-\rho t}\hat{\Gamma}_\theta \hat{\bar{\theta}}_t^i dt = \int_0^\infty e^{-\rho t}\hat{\bar{\theta}}_t^i dt. \tag{198}$$

**Disaggregated control problem.** We are now in a position to transform country-$i$'s disaggregated planning problem to the form of a standard optimal control problem. In particular, the planning problem is given by

$$\min_{\{\hat{\bar{\theta}}_t^i\}_{t\geq 0}} \int_0^\infty e^{-\rho t}\left[\alpha_\pi(\hat{\bar{\pi}}_{H,t}^i)^2 - \hat{\bar{U}}_t^i\big|_{\hat{\bar{g}}_t^i=0,\hat{\bar{\tau}}_t^{i,r}=\alpha v \hat{\bar{s}}_t^i}\right]dt, \tag{199}$$

subject to

$$\dot{\hat{\bar{y}}}_t^i = -G'(\hat{\bar{\pi}}_{H,t}^i + \dot{\bar{s}}_t^i) + G\dot{\hat{\bar{\theta}}}_t^i$$
$$\dot{\hat{\bar{\pi}}}_{H,t}^i = \rho\hat{\bar{\pi}}_{H,t}^i - \hat{\kappa}_y\hat{\bar{y}}_t^i - \hat{\kappa}_\theta\hat{\bar{\theta}}_t^i$$
$$\hat{\bar{y}}_0^i = -\bar{G}'\bar{s}_0^i + \bar{G}\hat{\bar{\theta}}_0^i$$
$$0 = \int_0^\infty e^{-\rho t}\left(\hat{\Gamma}_y\hat{\bar{y}}_t^i + \hat{\Gamma}_\theta\hat{\bar{\theta}}_t^i\right)dt.$$

As in the previous section for optimal transfers, the conversion from planning two optimal control problem requires to important steps. First, we include the initial condition and the isoperimetric budget constraint in the Hamiltonian using Lagrange multipliers. Second, we have to substitute out the time derivative $\dot{\hat{\bar{\theta}}}_t^i$ in the (IS) equation and replace it with placeholder $\hat{\bar{v}}_t^i$. We record country $i$'s disaggregated optimal control problem in the form of the following Hamiltonian:

$$\begin{aligned} H_t^i(x_t^i, u_t^i, \mu_t^i) =& \alpha_\pi(\pi_{H,t}^i)^2 - \hat{\bar{U}}_t^i\big|_{\hat{\bar{g}}_t^i=0,\hat{\bar{\tau}}_t^{i,r}=\alpha v\hat{\bar{s}}_t^i} \\ &+ \lambda\left[\hat{\Gamma}_y\hat{\bar{y}}_t^i + \hat{\Gamma}_\theta\hat{\bar{\theta}}_t^i\right] \\ &+ \Delta\left[\hat{\bar{y}}_0^i + \bar{G}'\bar{s}_0^i - \bar{G}\hat{\bar{\theta}}_0^i\right] \\ &+ \mu_{y,t}^i\left[G\hat{\bar{v}}_t^i - G'(\hat{\bar{\pi}}_{H,t}^i + \dot{\bar{s}}_t^i)\right] \\ &+ \mu_{\theta,t}^i\hat{\bar{v}}_t^i \end{aligned}$$

$$+ \mu^i_{\pi,t}\Big[\rho\hat{\pi}^i_{H,t} - \hat{\kappa}_y\hat{y}^i_t - \hat{\kappa}_\theta\hat{\theta}^i_t\Big].$$

As before, $x^i_t$ describes the vector of state variables, $u^i_t$ the vector of control variables, and $\mu^i_t$ the vector of costates.

**Loss function.** Finally, to work with the above Hamiltonian we have to record the precise form that the loss function takes when the planner can only use capital controls. It is straightforward to verify that we have

$$\hat{U}^i_t\big|_{\hat{g}^i_t=0,\hat{\tau}^{i,r}_t=\alpha\upsilon\hat{s}^i_t} = \hat{\alpha}_{yy}(\hat{y}^i_t)^2 + \hat{\alpha}_{\theta\theta}(\hat{\theta}^i_t)^2 + \hat{\alpha}_{y\theta}\hat{y}^i_t\hat{\theta}^i_t, \tag{200}$$

where

$$\begin{aligned}
\hat{\alpha}_{yy} &= \alpha_{yy} + \big(\frac{\alpha\upsilon}{1-\alpha\upsilon\nu_\tau}\big)^2\nu_y^2\alpha_{\tau\tau} + \frac{\alpha\upsilon}{1-\alpha\upsilon\nu_\tau}\nu_y\alpha_{y\tau} \\
\hat{\alpha}_{\theta\theta} &= \alpha_{\theta\theta} + \big(\frac{\alpha\upsilon}{1-\alpha\upsilon\nu_\tau}\big)^2\nu_\theta^2\alpha_{\tau\tau} + \frac{\alpha\upsilon}{1-\alpha\upsilon\nu_\tau}\nu_\theta\alpha_{\tau\theta} \\
\hat{\alpha}_{y\theta} &= 2\alpha_{\tau\tau}\big(\frac{\alpha\upsilon}{1-\alpha\upsilon\nu_\tau}\big)^2\nu_y\nu_\tau + \alpha_{y\theta} + \frac{\alpha\upsilon}{1-\alpha\upsilon\nu_\tau}\nu_\theta\alpha_{y\tau} + \frac{\alpha\upsilon}{1-\alpha\upsilon\nu_\tau}\nu_y\alpha_{\tau\theta}.
\end{aligned}$$

### D.5.1 Optimality Conditions

The Hamiltonian is associated with the following first-order conditions.

**Optimality**:

$$G\mu^i_{y,t} + \mu^i_{\tau,t} = 0. \tag{201}$$

**Multiplier**:

$$\dot{\mu}^i_{\pi,t} = G'\mu^i_{y,t} - 2\alpha_\pi\hat{\pi}^i_{H,t} \tag{202}$$

$$\rho\mu^i_{y,t} - \dot{\mu}^i_{y,t} = \lambda\hat{\Gamma}_y - \hat{\kappa}_y\mu^i_{\pi,t} - 2\hat{\alpha}_{yy}\hat{y}^i_t - \hat{\alpha}_{y\theta}\hat{\theta}^i_t \tag{203}$$

$$\rho\mu^i_{\theta,t} - \dot{\mu}^i_{\theta,t} = \lambda\hat{\Gamma}_\theta - \hat{\kappa}_\theta\mu^i_{\pi,t} - 2\hat{\alpha}_{\theta\theta}\hat{\theta}^i_t - \hat{\alpha}_{y\theta}\hat{y}^i_t \tag{204}$$

**State**:

$$\begin{aligned}
\dot{\hat{y}}^i_t &= -G'(\hat{\pi}^i_{H,t} + \hat{s}^i_t) + G\hat{v}^i_t \\
\dot{\hat{\theta}}^i_t &= \hat{v}^i_t \\
\dot{\hat{\pi}}^i_{H,t} &= \rho\hat{\pi}^i_{H,t} - \hat{\kappa}_y\hat{y}^i_t - \hat{\kappa}_\theta\hat{\theta}^i_t
\end{aligned}$$

**Initial conditions**:

$$\mu^i_{\pi,0} = 0 \tag{205}$$

$$\mu^i_{y,0} = \Delta \tag{206}$$

$$\mu^i_{\theta,0} = -\Delta\bar{G}. \tag{207}$$

We will now reduce the size of the above dynamical system by solving out for $\mu^i_{\pi,t}$ and $\mu^i_{\theta,t}$. Using equations (201), (203) and (204), we can write

$$(\hat{\kappa}_\theta + G\hat{\kappa}_y)\mu^i_{\pi,t} =(\hat{\Gamma}_y G + \hat{\Gamma}_\theta)\lambda - (2\hat{\alpha}_{\theta\theta} + G\hat{\alpha}_{y\theta})\hat{\tilde{\theta}}^i_t - (2G\hat{\alpha}_{yy} + \hat{\alpha}_{y\theta})\hat{\tilde{y}}^i_t. \tag{208}$$

Taking the derivative with respect to time and using (202), we have

$$(\hat{\kappa}_\theta + G\hat{\kappa}_y)(G'\mu^i_{y,t} - 2\alpha_\pi \hat{\tilde{\pi}}^i_{H,t}) + (2\hat{\alpha}_{\theta\theta} + G\hat{\alpha}_{y\theta})\dot{\hat{\tilde{\theta}}}^i_t + (2G\hat{\alpha}_{yy} + \hat{\alpha}_{y\theta})\dot{\hat{\tilde{y}}}^i_t = 0,$$

from which we can solve for optimal capital controls using the (IS) equation:

$$\dot{\hat{\tilde{\theta}}}^i_t = -\frac{(\hat{\kappa}_\theta + G\hat{\kappa}_y)(G'\mu^i_{y,t} - 2\alpha_\pi \hat{\tilde{\pi}}^i_{H,t}) - G'(2G\hat{\alpha}_{yy} + \hat{\alpha}_{y\theta})(\hat{\tilde{\pi}}^i_{H,t} + \dot{\tilde{s}}^i_t)}{2\hat{\alpha}_{\theta\theta} + G\hat{\alpha}_{y\theta} + G(2G\hat{\alpha}_{yy} + \hat{\alpha}_{y\theta})} \tag{209}$$

Using the (IS) equation again, we have

$$\dot{\hat{\tilde{y}}}^i_t = -G'(\hat{\tilde{\pi}}^i_{H,t} + \dot{\tilde{s}}^i_t) - G\frac{(\hat{\kappa}_\theta + G\hat{\kappa}_y)(G'\mu^i_{y,t} - 2\alpha_\pi \hat{\tilde{\pi}}^i_{H,t}) - G'(2G\hat{\alpha}_{yy} + \hat{\alpha}_{y\theta})(\hat{\tilde{\pi}}^i_{H,t} + \dot{\tilde{s}}^i_t)}{2\hat{\alpha}_{\theta\theta} + G\hat{\alpha}_{y\theta} + G(2G\hat{\alpha}_{yy} + \hat{\alpha}_{y\theta})}. \tag{210}$$

Finally, using (203) and our solution for $\mu^i_{\pi,t}$ above, we find

$$\dot{\mu}^i_{y,t} =\rho\mu^i_{y,t} + \lambda\frac{\hat{\kappa}_y\hat{\Gamma}_\theta - \hat{\kappa}_\theta\hat{\Gamma}_y}{\hat{\kappa}_\theta + G\hat{\kappa}_y} - \hat{\tilde{\theta}}^i_t\frac{2\hat{\kappa}_y\hat{\alpha}_{\theta\theta} - \hat{\kappa}_\theta\hat{\alpha}_{y\theta}}{\hat{\kappa}_\theta + G\hat{\kappa}_y} - \hat{\tilde{y}}^i_t\frac{\hat{\alpha}_{y\theta}\hat{\kappa}_y - 2\hat{\kappa}_\theta\hat{\alpha}_{yy}}{\hat{\kappa}_\theta + G\hat{\kappa}_y}. \tag{211}$$

Therefore, we can write the reduced system of optimality conditions just like in the case of optimal transfers as the dynamical system

$$\dot{\mu}^i_{y,t} = \rho\mu^i_{y,t} + \hat{K}_\theta\hat{\tilde{\theta}}^i_t + \hat{K}_y\hat{\tilde{y}}^i_t + \hat{K}_\lambda\lambda \tag{212}$$
$$\dot{\hat{\tilde{y}}}^i_t = \hat{J}_{\mu_y}\mu^i_{y,t} + \hat{J}_\pi\hat{\tilde{\pi}}^i_{H,t} + \hat{J}_s\dot{\tilde{s}}^i_t \tag{213}$$
$$\dot{\hat{\tilde{\theta}}}^i_t = \hat{H}_{\mu_y}\mu^i_{y,t} + \hat{H}_\pi\hat{\tilde{\pi}}^i_{H,t} + \hat{H}_s\dot{\tilde{s}}^i_t \tag{214}$$
$$\dot{\hat{\tilde{\pi}}}^i_{H,t} = \rho\hat{\tilde{\pi}}^i_{H,t} - \hat{\kappa}_y\hat{\tilde{y}}^i_t - \hat{\kappa}_\theta\hat{\tilde{\theta}}^i_t \tag{215}$$

subject to the initial conditions $\mu^i_{y,0} = \Delta$, $\hat{\tilde{y}}^i_0 = -\bar{G}'\tilde{s}^i_0 + \bar{G}\hat{\tilde{\theta}}^i_0$.

### D.5.2 Solving the Dynamic System

Viewing the system of optimality conditions as system of linear ODEs, letting $X^i_t = \{\hat{\tilde{\pi}}^i_{H,t}, \hat{\tilde{y}}^i_t, \hat{\tilde{\theta}}^i_t, \mu^i_{y,t}\}$, we can write as

$$\dot{X}^i_t = AX^i_t + B^i_t, \tag{216}$$

where

$$A = \begin{pmatrix} \rho & -\hat{\kappa}_y & -\hat{\kappa}_\theta & 0 \\ \hat{J}_\pi & 0 & 0 & \hat{J}_{\mu_y} \\ \hat{H}_\pi & 0 & 0 & \hat{H}_{\mu_y} \\ 0 & \hat{K}_y & \hat{K}_\theta & \rho \end{pmatrix}, \quad B_t^i = \begin{pmatrix} 0 \\ \hat{\hat{J}}_t^i \\ \hat{\hat{H}}_t^i \\ \hat{\hat{K}}_t^i \end{pmatrix}, \quad X_0^i = \begin{pmatrix} \hat{\hat{\pi}}_{H,0}^i \\ \hat{\hat{y}}_0^i \\ \hat{\hat{\theta}}_0^i \\ \Delta \end{pmatrix},$$

where $\hat{\hat{J}}_t^i = \hat{J}_s \dot{\hat{s}}_t^i$, $\hat{\hat{H}}_t^i = \hat{H}_s \dot{\hat{s}}_t^i$, and $\hat{\hat{K}}_t^i = \hat{K}_\theta \hat{\hat{\theta}}_t^i + \hat{K}_\lambda \lambda$.

The solution to this dynamical system is given by

$$X_t^i = e^{At}\left[X_0^i + \int_0^t e^{-As} B_s^i ds\right],$$

which we can rewrite as

$$X_t^i = e^{At}\left[X_0^i + \int_0^t e^{-As}\left(\hat{\hat{J}}_s^i E_2 + \hat{\hat{H}}_s^i E_3 + \hat{\hat{K}}_s^i E_4 ds\right],\right.$$

where $E_i$ is the $4 \times 1$ zero vector with a 1 in the $i^{th}$ position. We rewrite the solution for $X_t^i$ and get

$$\begin{aligned} X_t^i = e^{At}\Bigg[ & X_0^i + \hat{K}_\lambda \lambda \int_0^t e^{-As} E_4 ds \\ & + W_2 \int_0^t e^{-As-\psi s} E_2 ds + W_3 \int_0^t e^{-As-\psi s} E_3 ds \Bigg], \end{aligned}$$

where now $W_2 = -\psi a_0^i \hat{J}_s$ and $W_3 = -\psi a_0^i \hat{H}_s$.

Following the same steps as in the previous section for optimal transfers, we arrive at the following final expression for the solution of the dynamical system of optimality conditions:

$$\begin{aligned} X_t^i = & e^{At}\left[X_0^i + \hat{K}_\lambda \lambda A^{-1} E_4 + (A+\psi I)^{-1}\big(W_2 E_2 + W_3 E_3\big)\right] \\ & - \hat{K}_\lambda \lambda A^{-1} E_4 - (A+\psi I)^{-1} e^{-\psi t}\big(W_2 E_2 + W_3 E_3\big). \end{aligned} \tag{217}$$

### D.5.3 Stability

There is a unique solution to this system of linear ODEs if and only if $A$ has two negative eigenvalues. For stability, we require the term in square brackets above to be in the stable manifold. This gives us four stability conditions, requiring that the term be spanned by the two $4 \times 1$ eigenvectors of $A$ associated with the negative eigenvalues. In particular, we have

$$\begin{aligned} 0 = & \alpha_1 V_1 + \alpha_2 V_2 + X_0^i - \kappa_\theta \hat{\hat{\theta}}^i A^{-1} E_1 + (K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda) A^{-1} E_4 \\ & + (A+\psi I)^{-1}\big(W_2 E_2 + W_3 E_3\big), \end{aligned} \tag{218}$$

where, as before, $V_i$ is associated with the negative eigenvalue of $A$ $\lambda_i$, and $\alpha_1, \alpha_2 \in \mathbb{C}$. To guarantee stability, we solve the vector $Z = (\alpha_1, \alpha_2, \lambda, \Delta, \hat{\hat{\pi}}_{H,0}^i, \hat{\hat{y}}_0^i, \hat{\hat{\tau}}_0^{i,r})' \in \mathbb{C}^7$ using seven linearly independent equations that satisfy said stability criterion as well as the initial conditions of the dynamical system. Equation (218) gives us four of these conditions if we left-multiply both sides by $E_i'$ for $i \in \{1,2,3,4\}$.

As before, the fifth stability condition will be the initial condition, $\hat{\tilde{y}}_0^i + \bar{G}'\tilde{s}_0^i - \bar{G}\hat{\tilde{\theta}}_0^i = 0$. Using the initial conditions we obtained from the first-order conditions, we can write

$$0 = (\hat{\Gamma}_y G + \hat{\Gamma}_\theta)\lambda - (2\hat{\alpha}_{\theta\theta} + G\hat{\alpha}_{y\theta})\hat{\tilde{\theta}}_0^i - (2G\hat{\alpha}_{yy} + \hat{\alpha}_{y\theta})\hat{\tilde{y}}_0^i. \tag{219}$$

Finally, using the budget constraint, we have

$$\begin{aligned} 0 &= \int_0^\infty e^{-\rho t}\left(\hat{\Gamma}_y \hat{\tilde{y}}_t^i + \hat{\Gamma}_\theta \hat{\tilde{\theta}}_t^i\right) dt \\ &= \int_0^\infty e^{-\rho t}(\hat{\Gamma}_y E_2 + \hat{\Gamma}_\theta E_3)' X_t^i dt. \end{aligned}$$

Solving as before yields

$$\begin{aligned} 0 =& \frac{\alpha_1}{\lambda_1 - \rho}(\hat{\Gamma}_y E_2 + \hat{\Gamma}_\theta E_3)' V_1 + \frac{\alpha_2}{\lambda_2 - \rho}(\hat{\Gamma}_y E_2 + \hat{\Gamma}_\theta E_3)' V_2 \\ &- \frac{\hat{K}_\lambda \lambda}{\rho}(\hat{\Gamma}_y E_2 + \hat{\Gamma}_\theta E_3)' A^{-1} E_4 - \frac{1}{\rho + \psi}(\hat{\Gamma}_y E_2 + \hat{\Gamma}_\theta E_3)'(A + \psi I)^{-1}(W_2 E_2 + W_3 E_3). \end{aligned} \tag{220}$$

Putting everything together, we can write $MZ = N$, where

$$M = \begin{pmatrix} E_1'V_1 & E_1'V_2 & \hat{K}_\lambda E_1' A^{-1} E_4 & 0 & 1 & 0 & 0 \\ E_2'V_1 & E_2'V_2 & \hat{K}_\lambda E_2' A^{-1} E_4 & 0 & 0 & 1 & 0 \\ E_3'V_1 & E_3'V_2 & \hat{K}_\lambda E_3' A^{-1} E_4 & 0 & 0 & 0 & 1 \\ E_4'V_1 & E_4'V_2 & \hat{K}_\lambda E_4' A^{-1} E_4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\bar{G} \\ 0 & 0 & \hat{\Gamma}_\theta + G\hat{\Gamma}_y & 0 & 0 & -(2\hat{\alpha}_{yy}G + \hat{\alpha}_{y\theta}) & -(2\hat{\alpha}_{\theta\theta} + G\hat{\alpha}_{y\theta}) \\ \frac{(\hat{\Gamma}_y E_2 + \hat{\Gamma}_\theta E_3)'}{\lambda_1 - \rho} V_1 & \frac{(\hat{\Gamma}_y E_2 + \hat{\Gamma}_\theta E_3)'}{\lambda_2 - \rho} V_2 & -\frac{\hat{K}_\lambda}{\rho}(\hat{\Gamma}_y E_2 + \hat{\Gamma}_\theta E_3)' A^{-1} E_4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $W = (A + \psi I)^{-1}(W_2 E_2 + W_3 E_3)$, then

$$N = \begin{pmatrix} -E_1'W \\ -E_2'W \\ -E_3'W \\ -E_4'W \\ -\bar{G}'\tilde{s}_0^i \\ 0 \\ \frac{1}{\rho+\psi}(\hat{\Gamma}_y E_2 + \hat{\Gamma}_\theta E_3)'(A + \psi I)^{-1}[W_2 E_2 + W_3 E_3] \end{pmatrix}.$$

Given these matrices, we can compute the vector $Z$ numerically by setting

$$Z = M^{-1}N \tag{221}$$

over the parameter subspace on which $M$ is invertible.

## D.6 Optimal Government Spending

Finally, we consider optimal government spending. We set $\hat{\hat{\theta}}^i_t = \hat{\hat{\theta}}^i$ for all $t$ and $NFA^i_0 = 0$. This again implies $N\hat{\hat{F}}A^i_0 = 0$. Hence, $\tau^{i,r}_t = -(S^i_t)^{-\alpha} G^i_t = \tau^{i,o}_t$ because the government does not discriminate between agents in the way it sets taxes. Thus, we have $\hat{\hat{\tau}}^{i,r}_t = \alpha v \hat{\hat{s}}^i_t - v\hat{\hat{g}}^i_t$ or, substituting in for the terms of trade,

$$\hat{\hat{\tau}}^{i,r}_t = \frac{\alpha v}{1-\alpha v \nu_\tau}\big(\nu_y \hat{\hat{y}}^i_t + (\nu_g - \frac{1}{\alpha})\hat{\hat{g}}^i_t + \nu_\theta \hat{\hat{\theta}}^i\big).$$

**NFA.** We start by considering country $i$'s budget constraint and record an important result: For all $\chi \in [0,1]$ and $v \in [0,1]$, we have $\hat{\hat{\theta}}^i = 0$ under optimal government spending. Moreover, under the Cole-Obstfeld calibration trade is balanced in every period and HtM agents and optimizers are entirely symmetric.

Under optimal government spending, the budget constraint becomes

$$0 = \int_0^\infty e^{-\rho t}\left[\left(\Gamma_y + \frac{\alpha v}{1-\alpha v \nu_\tau}\nu_y \Gamma_\tau\right)\hat{\hat{y}}^i_t + \left(\Gamma_\theta + \frac{\alpha v}{1-\alpha v\nu_\tau}\nu_\theta\Gamma_\tau\right)\hat{\hat{\theta}}^i + \left(\Gamma_g + \frac{\alpha v}{1-\alpha v \nu_\tau}(\nu_g - \frac{1}{\alpha})\Gamma_\tau\right)\hat{\hat{g}}^i_t\right]dt,$$

and we have

$$\hat{\Gamma}_y = \Gamma_y + \frac{\alpha v}{1-\alpha v\nu_\tau}\nu_y\Gamma_\tau = 0$$
$$\hat{\Gamma}_g = \Gamma_g + \frac{\alpha v}{1-\alpha v\nu_\tau}(\nu_g - \frac{1}{\alpha})\Gamma_\tau = 0.$$

Therefore, we must have $\hat{\hat{\theta}}^i = 0$. Since the term in square brackets is also equivalent to net exports, $\hat{\Gamma}_y \hat{\hat{y}}^i_t + \hat{\Gamma}_g \hat{\hat{g}}^i_t + \hat{\Gamma}_\theta \hat{\hat{\theta}}^i_t = N\hat{\hat{X}}^i_t$, we furthermore find that trade is balanced in every period. This implies that optimizers and HtM agents will have the same consumption profile, since the former do not save or borrow to smooth their consumption over time.

As a result, we can rewrite the expression for the rebate as follows:

$$\hat{\hat{\tau}}^{i,r}_t = \frac{\alpha v}{1-\alpha v\nu_\tau}\big(\nu_y \hat{\hat{y}}^i_t + (\nu_g - \frac{1}{\alpha})\hat{\hat{g}}^i_t\big). \tag{222}$$

And with that, we can now present the simplified dynamical system characterizing the allocation under optimal government spending.

**IS equation.** The (IS) equation becomes

$$\dot{\hat{\hat{y}}}^i_t = -G'(\hat{\hat{\pi}}^i_{H,t} + \dot{\hat{\hat{s}}}^i_t) + F\dot{\hat{\hat{g}}}^i_t, \tag{223}$$

where

$$F = \frac{\Lambda_g + \frac{\alpha v}{1-\alpha v \nu_\tau}\Lambda_\tau(\nu_g - \frac{1}{\alpha})}{\Lambda_y - \frac{\alpha v \nu_y}{1-\alpha v \nu_\tau}\Lambda_\tau}$$

**NKPC.** The Phillips Curve becomes

$$\dot{\hat{\pi}}^i_{H,t} = \rho\hat{\pi}^i_{H,t} - \hat{\kappa}_y \hat{y}^i_t - \hat{\kappa}_\theta \hat{\theta}^i - \hat{\kappa}_g \hat{g}^i_t, \tag{224}$$

where

$$\hat{\kappa}_g = \kappa_g + \frac{\alpha v}{1-\alpha v \nu_\tau}\kappa_\tau(\nu_g - \frac{1}{\alpha}).$$

**Initial condition.** Finally, we can rewrite the initial condition as

$$\Lambda_y \hat{y}^i_0 = -(\Lambda_s + \alpha v \Lambda_\tau)\tilde{s}^i_0 + (\Lambda_g - v\Lambda_\tau)\hat{g}^i_0 + \Lambda_\theta \hat{\theta}^i. \tag{225}$$

**Planning problem.** Therefore, the planning problem for country $i$ under optimal government spending becomes

$$\min_{\{\hat{g}^i_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t}\left[\alpha_\pi(\hat{\pi}^i_{H,t})^2 - \hat{U}^i_t|_{\hat{\theta}^i_t=0,\hat{\tau}^{i,r}_t=\alpha v \hat{s}^i_t}\right]dt, \tag{226}$$

subject to

$$\begin{aligned}
\dot{\hat{y}}^i_t &= -G'(\hat{\pi}^i_{H,t} + \dot{\tilde{s}}^i_t) + F\dot{\hat{g}}^i_t \\
\dot{\hat{\pi}}^i_{H,t} &= \rho\hat{\pi}^i_{H,t} - \hat{\kappa}_y \hat{y}^i_t - \hat{\kappa}_g \hat{g}^i_t \\
\Lambda_y \hat{y}^i_0 &= -(\Lambda_s + \alpha v \Lambda_\tau)\tilde{s}^i_0 + (\Lambda_g - v\Lambda_\tau)\hat{g}^i_0
\end{aligned}$$

**Control problem.** Again, the conversion from planning to optimal control problem requires two steps. First, we now only include the initial condition in the Hamiltonian using Lagrange multiplier $\Delta$, since the isoperimetric budget constraint has dropped out. Second, we substitute out the time derivative $\dot{\hat{g}}^i_t$ in the (IS) equation and replace it with placeholder $\hat{v}^i_t$. The following Hamiltonian emerges:

$$\begin{aligned}
H^i_t(x^i_t, u^i_t, \mu^i_t) =& \alpha_\pi(\hat{\pi}^i_{H,t})^2 - \hat{U}^i_t|_{\hat{\theta}^i_t=0,\hat{\tau}^{i,r}_t=\alpha v \hat{s}^i_t} \\
&+ \Delta\left[\Lambda_y \hat{y}^i_0 + (\Lambda_s + \alpha v \Lambda_\tau)\tilde{s}^i_0 - (\Lambda_g - v\Lambda_\tau)\hat{g}^i_0\right] \\
&+ \mu^i_{y,t}\left[F\hat{v}^i_t - G'(\hat{\pi}^i_{H,t} + \dot{\tilde{s}}^i_t)\right] \\
&+ \mu^i_{g,t}\hat{v}^i_t
\end{aligned}$$

$$+\mu^i_{\pi,t}\Big[\rho\hat{\hat{\pi}}^i_{H,t}-\hat{\kappa}_y\hat{\hat{y}}^i_t-\hat{\kappa}_g\hat{\hat{g}}^i_t\Big].$$

As before, $x^i_t$ describes the vector of state variables, $u^i_t$ the vector of control variables, and $\mu^i_t$ the vector of costates.

**Loss function.** Finally, we want to express the simplified loss function under government spending is characterized by

$$\hat{U}^i_t\big|_{\hat{\theta}^i_t=0,\hat{\tau}^{i,r}_t=\alpha v\hat{s}^i_t}=\hat{\alpha}_{yy}(\hat{\hat{y}}^i_t)^2+\hat{\alpha}_{gg}(\hat{\hat{g}}^i_t)^2+\hat{\alpha}_{yg}\hat{\hat{y}}^i_t\hat{\hat{g}}^i_t, \tag{227}$$

where

$$\begin{aligned}
\hat{\alpha}_{gg}&=\alpha_{gg}+\big(\frac{\alpha v}{1-\alpha v\nu_\tau}\big)^2(\nu_g-\frac{1}{\alpha})^2\alpha_{\tau\tau}+\frac{\alpha v}{1-\alpha v\nu_\tau}(\nu_g-\frac{1}{1-\alpha})\alpha_{g\tau}\\
\hat{\alpha}_{yg}&=2\alpha_{\tau\tau}\big(\frac{\alpha v}{1-\alpha v\nu_\tau}\big)^2\nu_y(\nu_g-\frac{1}{\alpha})+\alpha_{yg}+\frac{\alpha v}{1-\alpha v\nu_\tau}(\nu_g-\frac{1}{\alpha})\alpha_{y\tau}+\frac{\alpha v}{1-\alpha v\nu_\tau}\nu_y\alpha_{g\tau}
\end{aligned}$$

and the remaining coefficients are as before.

### D.6.1 Optimality Conditions

The Hamiltonian yields the following first-order conditions.

**Optimality**:

$$F\mu^i_{y,t}+\mu^i_{\tau,t}=0. \tag{228}$$

**Multiplier**:

$$\dot{\mu}^i_{\pi,t}=G'\mu^i_{y,t}-2\alpha_\pi\hat{\hat{\pi}}^i_{H,t} \tag{229}$$

$$\rho\mu^i_{y,t}-\dot{\mu}^i_{y,t}=-\hat{\kappa}_y\mu^i_{\pi,t}-2\hat{\alpha}_{yy}\hat{\hat{y}}^i_t-\hat{\alpha}_{yg}\hat{\hat{g}}^i_t \tag{230}$$

$$\rho\mu^i_{g,t}-\dot{\mu}^i_{g,t}=-\hat{\kappa}_g\mu^i_{\pi,t}-2\hat{\alpha}_{gg}\hat{\hat{g}}^i_t-\hat{\alpha}_{yg}\hat{\hat{y}}^i_t \tag{231}$$

**State**:

$$\begin{aligned}
\dot{\hat{\hat{y}}}^i_t&=-G'(\hat{\hat{\pi}}^i_{H,t}+\dot{\hat{s}}^i_t)+F\hat{v}^i_t\\
\dot{\hat{\hat{g}}}^i_t&=\hat{v}^i_t\\
\dot{\hat{\hat{\pi}}}^i_{H,t}&=\rho\hat{\hat{\pi}}^i_{H,t}-\hat{\kappa}_y\hat{\hat{y}}^i_t-\hat{\kappa}_g\hat{\hat{g}}^i_t
\end{aligned}$$

**Initial Conditions**:

$$\mu^i_{\pi,0}=0 \tag{232}$$

$$\mu^i_{y,0}=\Delta\Lambda_y \tag{233}$$

$$\mu^i_{g,0} = -\Delta(\Lambda_g - v\Lambda_\tau). \tag{234}$$

We will now simplify the above dynamical system by solving out for $\mu^i_{\pi,t}$ and $\mu^i_{\theta,t}$. Using equations (228), (230) and (231), we can write

$$(\hat{\kappa}_g + F\hat{\kappa}_y)\mu^i_{\pi,t} = -(2\hat{\alpha}_{gg} + F\hat{\alpha}_{yg})\hat{\tilde{g}}^i_t - (2F\hat{\alpha}_{yy} + \alpha_{yg})\hat{\tilde{y}}^i_t. \tag{235}$$

Taking the derivative with respect to time and using (229), we have

$$(\hat{\kappa}_g + F\hat{\kappa}_y)(G'\mu^i_{y,t} - 2\alpha_\pi\hat{\tilde{\pi}}^i_{H,t}) + (2\hat{\alpha}_{gg} + F\hat{\alpha}_{yg})\dot{\hat{\tilde{g}}}^i_t + (2F\hat{\alpha}_{yy} + \hat{\alpha}_{yg})\dot{\hat{\tilde{y}}}^i_t = 0,$$

from which we can solve for optimal capital controls using the (IS) equation:

$$\dot{\hat{\tilde{g}}}^i_t = -\frac{(\hat{\kappa}_g + F\hat{\kappa}_y)(G'\mu^i_{y,t} - 2\alpha_\pi\hat{\tilde{\pi}}^i_{H,t}) - G'(2F\hat{\alpha}_{yy} + \hat{\alpha}_{yg})(\hat{\tilde{\pi}}^i_{H,t} + \dot{\tilde{s}}^i_t)}{2\hat{\alpha}_{gg} + F\hat{\alpha}_{yg} + F(2F\hat{\alpha}_{yy} + \hat{\alpha}_{yg})} \tag{236}$$

Using the (IS) equation again, we have

$$\dot{\hat{\tilde{y}}}^i_t = -G'(\hat{\tilde{\pi}}_{H,t} + \dot{\tilde{s}}^i_t) - F\frac{(\hat{\kappa}_g + F\hat{\kappa}_y)(G'\mu^i_{y,t} - 2\alpha_\pi\hat{\tilde{\pi}}^i_{H,t}) - G'(2F\hat{\alpha}_{yy} + \hat{\alpha}_{yg})(\hat{\tilde{\pi}}^i_{H,t} + \dot{\tilde{s}}^i_t)}{2\hat{\alpha}_{gg} + F\hat{\alpha}_{yg} + F(2F\hat{\alpha}_{yy} + \hat{\alpha}_{yg})}. \tag{237}$$

Finally, using (230) and our solution for $\mu^i_{\pi,t}$ above, we find

$$\dot{\mu}^i_{y,t} = \rho\mu^i_{y,t} - \hat{\tilde{g}}^i_t\frac{2\hat{\kappa}_y\hat{\alpha}_{gg} - \hat{\kappa}_g\hat{\alpha}_{yg}}{\hat{\kappa}_g + F\hat{\kappa}_y} - \hat{\tilde{y}}^i_t\frac{\hat{\alpha}_{yg}\hat{\kappa}_y - 2\hat{\kappa}_g\hat{\alpha}_{yy}}{\hat{\kappa}_g + F\hat{\kappa}_y}. \tag{238}$$

Therefore, we can write the reduced system of optimality conditions as

$$\dot{\mu}^i_{y,t} = \rho\mu^i_{y,t} + \tilde{K}_y\hat{\tilde{y}}^i_t + \tilde{K}_g\hat{\tilde{g}}^i_t \tag{239}$$
$$\dot{\hat{\tilde{y}}}^i_t = \tilde{J}_{\mu_y}\mu^i_{y,t} + \tilde{J}_\pi\hat{\tilde{\pi}}^i_{H,t} + \tilde{J}_s\dot{\tilde{s}}^i_t \tag{240}$$
$$\dot{\hat{\tilde{\theta}}}^i_t = \tilde{H}_{\mu_y}\mu^i_{y,t} + \tilde{H}_\pi\hat{\tilde{\pi}}^i_{H,t} + \tilde{H}_s\dot{\tilde{s}}^i_t \tag{241}$$
$$\dot{\hat{\tilde{\pi}}}^i_{H,t} = \rho\hat{\tilde{\pi}}^i_{H,t} - \tilde{\kappa}_y\hat{\tilde{y}}^i_t - \tilde{\kappa}_g\hat{\tilde{g}}^i_t \tag{242}$$

subject to the initial conditions $\mu^i_{y,0} = \Delta\Lambda_y$ and $\Lambda_y\hat{\tilde{y}}^i_0 = -(\Lambda_s + \alpha v\Lambda_\tau)\tilde{s}^i_0 + (\Lambda_g - v\Lambda_\tau)\hat{\tilde{g}}^i_0$.

### D.6.2 Solving the Dynamic System

Let $X^i_t = \{\hat{\tilde{\pi}}^i_{H,t}, \hat{\tilde{y}}^i_t, \hat{\tilde{g}}^i_t, \mu^i_{y,t}\}$. Then we have

$$\dot{X}^i_t = AX^i_t + B^i_t, \tag{243}$$

where

$$A = \begin{pmatrix} \rho & -\hat{\kappa}_y & -\hat{\kappa}_\theta & 0 \\ \tilde{J}_\pi & 0 & 0 & \tilde{J}_{\mu_y} \\ \tilde{H}_\pi & 0 & 0 & \tilde{H}_{\mu_y} \\ 0 & \tilde{K}_y & \tilde{K}_\theta & \rho \end{pmatrix}, \qquad B_t^i = \begin{pmatrix} 0 \\ \tilde{\tilde{J}}_t^i \\ \tilde{\tilde{H}}_t^i \\ \tilde{\tilde{K}}_t^i \end{pmatrix}, \qquad X_0^i = \begin{pmatrix} \hat{\hat{\pi}}_{H,0}^i \\ \hat{\hat{y}}_0^i \\ \hat{\hat{\theta}}_0^i \\ \Delta\Lambda_y \end{pmatrix},$$

where $\tilde{\tilde{J}}_t^i = \tilde{J}_s \dot{\tilde{s}}_t^i$, $\tilde{\tilde{H}}_t^i = \tilde{H}_s \dot{\tilde{s}}_t^i$, and $\tilde{\tilde{K}}_t^i = \tilde{K}_\lambda \lambda$.

Therefore, following the same steps as in the previous two sections the solution of the dynamical system of optimality conditions can be written as

$$\begin{aligned} X_t^i =& e^{At}\Big[X_0^i + (A + \psi I)^{-1}\big(W_2 E_2 + W_3 E_3 + W_4 E_4\big)\Big] \\ & - (A + \psi I)^{-1} e^{-\psi t}\big(W_2 E_2 + W_3 E_3\big), \end{aligned} \tag{244}$$

where $W_2 = -\psi a_0^i \tilde{J}_s$ and $W_3 = -\psi a_0^i \tilde{H}_s$.

### D.6.3 Stability

As we argued in previous sections, there is a unique solution to this system of linear ODEs if and only if $A$ has two negative eigenvalues. For stability, we require the term in square brackets above to be in the stable manifold. In other words, we require the term in square brackets to be spanned by the two eigenvectors, $V_1$ and $V_2$, of $A$ that are associated with the two negative eigenvalues, which we will call $\lambda_1$ and $\lambda_2$. This gives us a system of four linearly independent equations

$$0 = \alpha_1 V_1 + \alpha_2 V_2 + X_0^i + (A + \psi I)^{-1}\big(W_2 E_2 + W_3 E_3 + W_4 E_4\big),$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$.

Unlike in the previous two sections, we now only have to solve for six variables which we summarize in the vector $Z = (\alpha_1, \alpha_2, \Delta, \hat{\hat{\pi}}_{H,0}^i, \hat{\hat{y}}_0^i, \hat{\hat{\tau}}_0^{i,r})' \in \mathbb{C}^6$. Equation (218) gives us four of these conditions if we left-multiply both sides by $E_i'$ for $i \in \{1, 2, 3, 4\}$.

The fifth stability condition will be the initial condition, $\Lambda_y \hat{\hat{y}}_0^i = -(\Lambda_s + \alpha v \Lambda_\tau)\tilde{s}_0^i + (\Lambda_g - v\Lambda_\tau)\hat{\hat{g}}_0^i$. The sixth and final stability condition we obtain from one of the initial conditions of the dynamical system of optimality conditions,

$$0 = -(2\hat{\alpha}_{gg} + F\hat{\alpha}_{yg})\hat{\hat{g}}_0^i - (2F\hat{\alpha}_{yy} + \alpha_{yg})\hat{\hat{y}}_0^i. \tag{245}$$

In matrix form, this system of linear equations can be written $MZ = N$, where

$$M = \begin{pmatrix} E_1'V_1 & E_1'V_2 & 0 & 1 & 0 & 0 \\ E_2'V_1 & E_2'V_2 & 0 & 0 & 1 & 0 \\ E_3'V_1 & E_3'V_2 & 0 & 0 & 0 & 1 \\ E_4'V_1 & E_4'V_2 & \Lambda_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_y & -(\Lambda_g - v\Lambda_\tau) \\ 0 & 0 & 0 & 0 & -(2\hat{\alpha}_{yy}F + \hat{\alpha}_{yg}) & -(2\hat{\alpha}_{gg} + F\hat{\alpha}_{yg}) \end{pmatrix}$$

and

$$N = \begin{pmatrix} -E_1'(A+\psi I)^{-1}(W_2E_2 + W_3E_3) \\ -E_2'(A+\psi I)^{-1}(W_2E_2 + W_3E_3) \\ -E_3'(A+\psi I)^{-1}(W_2E_2 + W_3E_3) \\ -E_4'(A+\psi I)^{-1}(W_2E_2 + W_3E_3) \\ -(\Lambda_s + \alpha v\Lambda_\tau)\tilde{s}_0^i \\ 0 \end{pmatrix}.$$

Given these matrices, we can compute the vector $Z$ numerically by setting

$$Z = M^{-1}N \tag{246}$$

over the parameter subspace on which $M$ is invertible.

## D.7 Optimal Redistribution

We can easily study redistribution as a distinct, fourth policy instrument in the present framework, even though we have not explicitly discussed it until now.

We have previously imposed that the government does not discriminate between agents when funding its own outlays. To introduce redistribution as a distinct policy instrument, we remove this constraint. In addition, we set $NFA_0^i = 0$, implying that country $i$ does not receive international transfers. Therefore, the only relevant constraint becomes

$$\chi\hat{\tilde{\tau}}_t^{i,r} + (1-\chi)\hat{\tilde{\tau}}_t^{i,o} = \alpha v\hat{\tilde{s}}_t^i, \tag{247}$$

since government spending is not used, $\hat{\tilde{g}}_t^i = 0$. Finally, the social planner cannot use capital controls, either, so that $\hat{\tilde{\theta}}_t^i = \hat{\tilde{\theta}}^i$ for all $t$.

Note that the only important difference between the characterization of optimal redistribution and optimal transfers is country $i$'s budget constraint. In particular, we now have $NFA_0^i = 0$ so that

$$0 = \int_0^\infty e^{-\rho t}\left(\hat{\tilde{y}}_t^i - (1-v)\alpha\hat{\tilde{s}}_t^i - (1-v)\hat{\tilde{c}}_t^i\right)dt, \tag{248}$$

where $\hat{NX}^i_t = \hat{\hat{y}}^i_t - (1-v)\alpha\hat{\hat{s}}^i_t - (1-v)\hat{\hat{c}}^i_t$. We can rewrite the budget constraint using earlier results so that

$$0 = \int_0^\infty e^{-\rho t}\left[\left(1-\alpha(1-v)\nu_y - (1-v)\zeta_y\right)\hat{\hat{y}}^i_t - (1-v)(\alpha\nu_\theta + \zeta_\theta)\hat{\hat{\theta}}^i - (1-v)(\alpha\nu_\tau + \zeta_\tau)\hat{\hat{\tau}}^{i,r}_t\right]dt,$$

or, more conveniently,

$$0 = \int_0^\infty e^{-\rho t}\left[\Gamma^*_y\hat{\hat{y}}^i_t + \Gamma^*_\theta\hat{\hat{\theta}}^i + \Gamma^*_\tau\hat{\hat{\tau}}^{i,r}_t\right]dt. \tag{249}$$

### D.7.1 Optimal Control Problem

Exploiting the similarity between transfers and redistribution, we can simply adopt the same loss function, IS equation, Phillips Curve and initial condition. The planning problem can therefore be written as

$$\min_{\hat{\hat{\theta}}^i, \{\hat{\hat{\tau}}^{i,r}_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t}\left[\alpha_\pi(\pi^i_{H,t})^2 - \hat{U}^i_t\big|_{\hat{\hat{\theta}}^i_t=\hat{\hat{\theta}}^i, \hat{\hat{g}}^i_t=0}\right]dt, \tag{250}$$

subject to

$$\begin{aligned}
\dot{\hat{\hat{y}}}^i_t &= -D'(\hat{\hat{\pi}}^i_{H,t} + \dot{\hat{s}}^i_t) + D\hat{\hat{v}}^i_t \\
\dot{\hat{\hat{\tau}}}^{i,r}_t &= \hat{\hat{v}}^i_t \\
\dot{\hat{\hat{\pi}}}^i_{H,t} &= \rho\hat{\hat{\pi}}^i_{H,t} - \kappa_y\hat{\hat{y}}^i_t - \kappa_\theta\hat{\hat{\theta}}^i - \kappa_\tau\hat{\hat{\tau}}^{i,r}_t
\end{aligned}$$

where $D' = \frac{\Lambda_s}{\Lambda_y}$ and $D = \frac{\Lambda_\tau}{\Lambda_y}$, and also subject to the intial condition $\Lambda_y\hat{\hat{y}}^i_0 = -\Lambda_s\tilde{s}^i_0 + \Lambda_\theta\hat{\hat{\theta}}^i + \Lambda_\tau\hat{\hat{\tau}}^{i,r}_0$. Instead of the budget constraint under transfers, however, we now use the budget constraint presented in (249). As before, the associated optimal control problem can be characterized via the Hamiltonian

$$\begin{aligned}
H(x^i_t, u^i_t, \mu^i_t) =& \alpha_\pi(\pi^i_{H,t})^2 - \hat{U}^i_t\big|_{\hat{\hat{\theta}}^i_t=\hat{\hat{\theta}}^i, \hat{\hat{g}}^i_t=0} \\
&+ \lambda\left[\Gamma^*_y\hat{\hat{y}}^i_t + \Gamma^*_\tau\hat{\hat{\tau}}^{i,r}_t + \Gamma^*_\theta\hat{\hat{\theta}}^i\right] \\
&+ \Delta\left[\Lambda_y\hat{\hat{y}}^i_0 + \Lambda_s\tilde{s}^i_0 - \Lambda_\theta\hat{\hat{\theta}}^i - \Lambda_\tau\hat{\hat{\tau}}^{i,r}_0\right] \\
&+ \mu^i_{y,t}\left[D\hat{\hat{v}}^i_t - D'(\hat{\hat{\pi}}^i_{H,t} + \dot{\hat{s}}^i_t)\right] \\
&+ \mu^i_{\tau,t}\hat{\hat{v}}^i_t \\
&+ \mu^i_{\pi,t}\left[\rho\hat{\hat{\pi}}^i_{H,t} - \kappa_y\hat{\hat{y}}^i_t - \kappa_\theta\hat{\hat{\theta}}^i - \kappa_\tau\hat{\hat{\tau}}^{i,r}_t\right],
\end{aligned} \tag{251}$$

where $x^i_t$, $u^i_t$, and $\mu^i_t$ denote the vectors of state variables, control variables, and costates, respectively.

Reducing the system of the first-order conditions associated with this control problem as before, we can summarize the optimal allocation using the dynamical system

$$\dot{X}_t^i = AX_t^i + B_t^i, \tag{252}$$

where $X_t^i = \{\hat{\hat{\pi}}_{H,t}^i, \hat{\hat{y}}_t^i, \hat{\hat{\tau}}_t^{i,r}, \mu_{y,t}^i\}$ and

$$A = \begin{pmatrix} \rho & -\kappa_y & -\kappa_\tau & 0 \\ J_\pi & 0 & 0 & J_{\mu_y} \\ H_\pi & 0 & 0 & H_{\mu_y} \\ 0 & K_y & K_\tau & \rho \end{pmatrix}, \qquad B_t^i = \begin{pmatrix} -\kappa_\theta \hat{\hat{\theta}}^i \\ \bar{J}_t^i \\ \bar{H}_t^i \\ \bar{K}_t^i \end{pmatrix}, \qquad X_0^i = \begin{pmatrix} \hat{\hat{\pi}}_{H,0}^i \\ \hat{\hat{y}}_0^i \\ \hat{\hat{\tau}}_0^{i,r} \\ \Delta\Lambda_y \end{pmatrix},$$

and where $\bar{J}_t^i = J_s \dot{\hat{s}}_t^i$, $\bar{H}_t^i = H_s \dot{\hat{s}}_t^i$, and $\bar{K}_t^i = K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda$.

The solution to this dynamical system is, as before, given by

$$\begin{aligned} X_t^i =& e^{At}\left[X_0^i - \kappa_\theta \hat{\hat{\theta}}^i A^{-1} E_1 + (K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda) A^{-1} E_4 + (A + \psi I)^{-1}\big(W_2 E_2 + W_3 E_3\big)\right] \\ &+ \kappa_\theta \hat{\hat{\theta}}^i A^{-1} E_1 - (K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda) A^{-1} E_4 - (A + \psi I)^{-1} e^{-\psi t}\big(W_2 E_2 + W_3 E_3\big) \end{aligned} \tag{253}$$

### D.7.2 Stability

To guarantee the stability of this system, we want to solve for the vector $Z = (\alpha_1, \alpha_2, \lambda, \Delta, \hat{\hat{\pi}}_{H,0}^i, \hat{\hat{y}}_0^i, \hat{\hat{\tau}}_0^{i,r})' \in \mathbb{C}^7$ using a set of stability conditions. We use the same first six stability conditions as for optimal transfers. The seventh stability condition, however, is now given by

$$\begin{aligned} 0 =& \frac{\Gamma_\theta^*}{\rho} \hat{\hat{\theta}}^i + \frac{\alpha_1}{\lambda_1 - \rho} (\Gamma_y^* E_2 + \Gamma_\tau^* E_3)' V_1 + \frac{\alpha_2}{\lambda_2 - \rho} (\Gamma_y^* E_2 + \Gamma_\tau^* E_3)' V_2 \\ &+ \frac{\kappa_\theta \hat{\hat{\theta}}^i}{\rho} (\Gamma_y^* E_2 + \Gamma_\tau^* E_3)' A^{-1} E_1 - \frac{K_\theta \hat{\hat{\theta}}^i + K_\lambda \lambda}{\rho} (\Gamma_y^* E_2 + \Gamma_\tau^* E_3)' A^{-1} E_4 \\ &- \frac{1}{\rho + \psi} (\Gamma_y^* E_2 + \Gamma_\tau^* E_3)' (A + \psi I)^{-1} [W_2 E_2 + W_3 E_3]. \end{aligned} \tag{254}$$

Putting everything together, we can write $MZ = N$, where

$$M = \begin{pmatrix} E_1' V_1 & E_1' V_2 & K_\lambda E_1' A^{-1} E_4 & 0 & 1 & 0 & 0 \\ E_2' V_1 & E_2' V_2 & K_\lambda E_2' A^{-1} E_4 & 0 & 0 & 1 & 0 \\ E_3' V_1 & E_3' V_2 & K_\lambda E_3' A^{-1} E_4 & 0 & 0 & 0 & 1 \\ E_4' V_1 & E_4' V_2 & K_\lambda E_4' A^{-1} E_4 & \Lambda_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_y & -\Lambda_\tau \\ 0 & 0 & \Gamma_\tau^* + D\Gamma_y^* & 0 & 0 & -(2\alpha_{yy} D + \alpha_{y\tau}) & -(2\alpha_{\tau\tau} + D\alpha_{y\tau}) \\ \frac{(\Gamma_y^* E_2 + \Gamma_\tau^* E_3)'}{\lambda_1 - \rho} V_1 & \frac{(\Gamma_y^* E_2 + \Gamma_\tau^* E_3)'}{\lambda_2 - \rho} V_2 & -\frac{K_\lambda}{\rho} (\Gamma_y^* E_2 + \Gamma_\tau^* E_3)' A^{-1} E_4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$N = \begin{pmatrix} -E_1'W \\ -E_2'W \\ -E_3'W \\ -E_4'W \\ -\Lambda_s \tilde{s}_0^i + \Lambda_\theta \hat{\tilde{\theta}}^i \\ (\alpha_{\tau\theta} + D\alpha_{y\theta})\hat{\tilde{\theta}}^i \\ -\frac{\Gamma_\theta^*}{\rho}\hat{\tilde{\theta}}^i - \frac{\kappa_\theta \hat{\tilde{\theta}}^i}{\rho}(\Gamma_y^* E_2 + \Gamma_\tau^* E_3)' A^{-1} E\_1 + \frac{K_\theta \hat{\tilde{\theta}}^i}{\rho}(\Gamma_y^* E_2 + \Gamma_\tau^* E_3)' A^{-1} E_4 + \frac{(\Gamma_y^* E_2 + \Gamma_\tau^* E_3)'}{\rho + \psi}(A + \psi I)^{-1}[W_2 E_2 + W_3 E_3] \end{pmatrix}$$

where $W = -\kappa_\theta \hat{\tilde{\theta}}^i A^{-1} E_1 + K_\theta \hat{\tilde{\theta}}^i A^{-1} E_4 + (A + \psi I)^{-1}[W_2 E_2 + W_3 E_3]$. Given these matrices, we can compute the vector $Z$ numerically by setting

$$Z = M^{-1} N \tag{255}$$

over the parameter subspace on which $M$ is invertible.

## D.8 Optimal Deficits

We now study a particular form of deficits in this economy. In particular, we allow the government to accrue debt over time and rebate these funds to households. With the presence of HtM agents, Ricardian equivalence breaks down and such a policy can be effective.

Instead of the per-period government budget constraint, we now impose an isoperimetric constraint of the form

$$0 = \int_0^\infty e^{-\rho t}\left(\chi \hat{\tilde{\tau}}_t^{i,r} + (1-\chi)\hat{\tilde{\tau}}_t^{i,o} + v\hat{\tilde{g}}_t^i - \alpha v \hat{\tilde{s}}_t^i\right)dt. \tag{256}$$

Implicitly, this constraint states that the government can now accumulate debt from one period to the next. In this section, we assume that $\hat{\tilde{g}}_t^i = 0$ for all $t$ because we want to narrow in on the question of budget deficits, independent from the question of optimal government spending. We furthermore assume that $\hat{\tilde{\theta}}_t^i = \hat{\tilde{\theta}}^i$, since the government cannot use capital controls, and $NFA_0^i = 0$, since we do not want to consider cross-border transfers here.

The second key assumption we make is that the government does not discriminate between agents when rebating its deficit funds, which implies that $\hat{\tilde{\tau}}_t^{i,r} = \hat{\tilde{\tau}}_t^{i,o}$. Therefore, the inter-temporal government budget constraint becomes

$$0 = \int_0^\infty e^{-\rho t}\left(\hat{\tilde{\tau}}_t^{i,r} - \alpha v \hat{\tilde{s}}_t^i\right)dt. \tag{257}$$

We now consider the problem where the social planner only has access to ex-post transfers, as in body of the paper. In particular, we have $\hat{\tilde{\theta}}_t^i = \hat{\tilde{\theta}}^i$ for all $t$ and $\hat{\tilde{g}}_t^i = 0$.

**IS, NKPC and initial condition.** The problem laid out above is almost identical to that of Section 4 with optimal fiscal union transfers. In fact, the only difference in the two planning problems comes from the budget constraint and the now new isoperimetric government budget constraint. In particular, it is easy to verify that we can adopt the same (IS) equation, (NKPC) and initial condition as we specified in Section 4. In particular, the (IS) equation is given by

$$\dot{\hat{\tilde{y}}}_t^i = -D'(\hat{\tilde{\pi}}_{H,t}^i + \dot{\tilde{s}}_t^i) + D\hat{\tilde{\tau}}_t^{i,r}.$$

The (NKPC) is given by

$$\dot{\hat{\tilde{\pi}}}_{H,t}^i = \rho\hat{\tilde{\pi}}_{H,t}^i - \kappa_y\hat{\tilde{y}}_t^i - \kappa_\theta\hat{\tilde{\theta}}^i - \kappa_\tau\hat{\tilde{\tau}}_t^{i,r},$$

and the initial condition is

$$\Lambda_y\hat{\tilde{y}}_0^i = -\Lambda_s\tilde{s}_0^i + \Lambda_\theta\hat{\tilde{\theta}}^i + \Lambda_\tau\hat{\tilde{\tau}}_0^{i,r}.$$

**NFA.** Finally, we rederive country $i$'s budget constraint. Recall that in the general derivation of the NFA condition in Section 3, we assumed that the government budget constraint held period-by-period. Under deficit spending, we have

$$\begin{aligned}
W_t^iN_t^{i,o} - P_t^iC_t^{i,o} + P_{H,t}^iY_t^i - W_t^iN_t^i + P_t^i\tau_t^{i,r} &= W_t^iN_t^{i,o} - P_t^iC_t^{i,o} + P_{H,t}^iY_t^i - W_t^iN_t^i + P_t^iT_t^{i,r} - P_t^iZ_{H,t}^i \\
&= W_t^iN_t^{i,o} - P_t^iC_t^{i,o} + P_t^iC_t^{i,r} - W_t^iN_t^{i,r} \\
&= P_t^i(C_t^{i,r} - C_t^{i,o}) - W_t^i(N_t^{i,r} - N_t^{i,o}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
NFA_0^i &= -\int_0^\infty e^{-\rho t}\frac{1}{P_F^i}\left[P_t^i(C_t^{i,r} - C_t^{i,o}) - W_t^i(N_t^{i,r} - N_t^{i,o})\right]dt \\
&= -\int_0^\infty e^{-\rho t}\left[(S_t^i)^{-(1-\alpha)}(C_t^{i,r} - C_t^{i,o}) - \frac{1}{1-v}C_t^{i,r}(N_t^{i,r})^\phi(S_t^i)^{-(1-\alpha)}(N_t^{i,r} - N_t^{i,o})\right]dt
\end{aligned}$$

since

$$\frac{W_t^i}{P_F^i} = \frac{W_t^i}{P_t^i}\frac{P_t^i}{P_F^i} = \frac{1}{1-v}C_t^{i,r}(N_t^{i,r})^\phi\frac{P_t^i}{P_{H,t}^i}\frac{P_{H,t}^i}{P_F^i} = \frac{1}{1-v}C_t^{i,r}(N_t^{i,r})^\phi(S_t^i)^{-(1-\alpha)}.$$

Linearizing, we find

$$NFA_0^i = -\int_0^\infty e^{-\rho t}\left[(1-v)(c_t^{i,r} - c_t^{i,o}) - (n_t^{i,r} - n_t^{i,o})\right]dt. \tag{258}$$

Adopting gap notation and noting that $N\hat{\tilde{F}}A_0^i = 0$,

$$\begin{aligned}
0 &= \int_0^\infty e^{-\rho t}\left[(1-v)(\hat{\tilde{c}}_t^{i,r} - \hat{\tilde{c}}_t^{i,o}) - (\hat{\tilde{n}}_t^{i,r} - \hat{\tilde{n}}_t^{i,o})\right]dt \\
&= \int_0^\infty e^{-\rho t}\left[(1-v)(\hat{\tilde{c}}_t^{i,r} - \hat{\tilde{\theta}}^i - (1-\alpha)\hat{\tilde{s}}_t^i) - (\hat{\tilde{n}}_t^{i,r} - \frac{1}{1-\chi}\hat{\tilde{n}}_t^i + \frac{\chi}{1-\chi}\hat{\tilde{n}}_t^{i,r})\right]dt \\
&= \int_0^\infty e^{-\rho t}\Big[(1-v)\big(\psi_y\hat{\tilde{y}}_t^i + (\psi_\theta - 1)\hat{\tilde{\theta}}^i + \psi_\tau\hat{\tilde{\tau}}_t^{i,r}\big) + \frac{1}{1-\chi}\hat{\tilde{y}}_t^i \\
&\quad - (1-\alpha)(1-v)\big(\nu_y\hat{\tilde{y}}_t^i + \nu_\theta\hat{\tilde{\theta}}^i + \nu_\tau\hat{\tilde{\tau}}_t^{i,r}\big) - \frac{1}{1-\chi}\big(\eta_y\hat{\tilde{y}}_t^i + \eta_\theta\hat{\tilde{\theta}}^i + \eta_\tau\hat{\tilde{\tau}}_t^{i,r}\big)\Big]dt.
\end{aligned}$$

To simplify notation, we write

$$0 = \int_0^\infty e^{-\rho t} \left[ \Gamma_y^{**} \hat{\tilde{y}}_t^i + \Gamma_\theta^{**} \hat{\tilde{\theta}}^i + \Gamma_\tau^{**} \hat{\tilde{\tau}}_t^{i,r} \right] dt. \tag{259}$$

**Partial collinearity of NFA and GBC.** Comparing the NFA condition with the isoperimetric government budget constraint, which we can write as

$$0 = \int_0^\infty e^{-\rho t} \left[ (1 - \alpha v \nu_\tau) \hat{\tilde{\tau}}_t^{i,r} - \alpha v \nu_y \hat{\tilde{y}}_t^i - \alpha v \nu_\theta \hat{\tilde{\theta}}^i \right] dt, \tag{260}$$

we note that we have

$$\frac{\Gamma_y^{**}}{-\alpha v \nu_y} = \frac{\Gamma_\tau^{**}}{1 - \alpha v \nu_\tau} \equiv \nu. \tag{261}$$

We can multiply (259) by $1/\nu$ and subtract it from (260). This yields

$$0 = \int_0^\infty e^{-\rho t} \left[ -\alpha v \nu_\theta \hat{\tilde{\theta}}^i - \frac{\Gamma_\theta^{**}}{\nu} \hat{\tilde{\theta}}^i \right] dt,$$

which immediately implies that we must have $\hat{\tilde{\theta}}^i = 0$.

**Planning problem.** Country $i$'s planning problem can therefore be written as

$$\min_{\hat{\tilde{\theta}}^i, \{\hat{\tilde{\tau}}_t^{i,r}\}_{t \geq 0}} \int_0^\infty e^{-\rho t} \left[ \alpha_\pi (\hat{\tilde{\pi}}_{H,t}^i)^2 - \hat{U}_t^i |_{\hat{\theta}_t^i = 0, \hat{g}_t^i = 0} \right] dt, \tag{262}$$

subject to

$$\begin{aligned}
\dot{\hat{\tilde{y}}}_t^i &= -D' (\hat{\tilde{\pi}}_{H,t}^i + \dot{\tilde{s}}_t^i) + D \dot{\hat{\tilde{\tau}}}_t^{i,r} \\
\dot{\hat{\tilde{\pi}}}_{H,t}^i &= \rho \hat{\tilde{\pi}}_{H,t}^i - \kappa_y \hat{\tilde{y}}_t^i - \kappa_\tau \hat{\tilde{\tau}}_t^{i,r} \\
\Lambda_y \hat{\tilde{y}}_0^i &= -\Lambda_s \tilde{s}_0^i + \Lambda_\tau \hat{\tilde{\tau}}_0^{i,r} \\
0 &= \int_0^\infty e^{-\rho t} \left[ \Gamma_y^{**} \hat{\tilde{y}}_t^i + \Gamma_\tau^{**} \hat{\tilde{\tau}}_t^{i,r} \right] dt
\end{aligned}$$

where $D' = \frac{\Lambda_s}{\Lambda_y}$ and $D = \frac{\Lambda_\tau}{\Lambda_y}$, and

$$\hat{U}_t^i |_{\hat{\theta}_t^i = 0, \hat{g}_t^i = 0} = \alpha_{yy} (\hat{\tilde{y}}_t^i)^2 + \alpha_{\tau\tau} (\hat{\tilde{\tau}}_t^{i,r})^2 + \alpha_{y\tau} \hat{\tilde{y}}_t^i \hat{\tilde{\tau}}_t^{i,r}$$

**Control problem.** To transform this planning problem into an optimal control problem that we can work with we replace the time derivative $\dot{\hat{\tilde{\tau}}}_t^{i,r}$ in the (IS) equation by a placeholder, $\hat{v}_t^i$, and introduce a new dynamic state equation. The Hamiltonian associated with this control problem can then be written as

$$\begin{aligned} H(x_t^i, u_t^i, \mu_t^i) =& \alpha_\pi (\hat{\tilde{\pi}}_{H,t}^i)^2 - \hat{U}_t^i|_{\hat{\theta}_t^i=0, \hat{g}_t^i=0} \\ &+ \Delta \left[ \Lambda_y \hat{\tilde{y}}_0^i + \Lambda_s \tilde{s}_0^i - \Lambda_\tau \hat{\tilde{\tau}}_0^{i,r} \right] \\ &+ \lambda \left[ \Gamma_y^{**} \hat{\tilde{y}}_t^i + \Gamma_\tau^{**} \hat{\tilde{\tau}}_t^{i,r} \right] \\ &+ \mu_{y,t}^i \left[ D \hat{\tilde{v}}_t^i - D' (\hat{\tilde{\pi}}_{H,t}^i + \dot{\tilde{s}}_t^i) \right] \\ &+ \mu_{\tau,t}^i \hat{\tilde{v}}_t^i \\ &+ \mu_{\pi,t}^i \left[ \rho \hat{\tilde{\pi}}_{H,t}^i - \kappa_y \hat{\tilde{y}}_t^i - \kappa_\tau \hat{\tilde{\tau}}_t^{i,r} \right], \end{aligned} \tag{263}$$

where $x_t^i = \{\hat{\tilde{\pi}}_{H,t}^i, \hat{\tilde{y}}_t^i, \hat{\tilde{\tau}}_t^{i,r}\}$ denotes the vector of state variables, $u_t^i = \{\hat{\tilde{v}}_t^i\}$ the vector of control variables, and $\mu_t^i = \{\mu_{\pi,t}^i, \mu_{y,t}^i, \mu_{\tau,t}^i\}$ the vector of costates.

### D.8.1 Optimality Conditions

The Hamiltonian yields the nine first-order conditions

**Optimality**:

$$D\mu_{y,t}^i + \mu_{\tau,t}^i = 0. \tag{264}$$

**Multiplier**:

$$\dot{\mu}_{\pi,t}^i = D' \mu_{y,t}^i - 2\alpha_\pi \hat{\tilde{\pi}}_{H,t}^i \tag{265}$$

$$\rho \mu_{y,t}^i - \dot{\mu}_{y,t}^i = \Gamma_y^{**} \lambda - \kappa_y \mu_{\pi,t}^i - 2\alpha_{yy} \hat{\tilde{y}}_t^i - \alpha_{y\tau} \hat{\tilde{\tau}}_t^{i,r} \tag{266}$$

$$\rho \mu_{\tau,t}^i - \dot{\mu}_{\tau,t}^i = \Gamma_\tau^{**} \lambda - \kappa_\tau \mu_{\pi,t}^i - 2\alpha_{\tau\tau} \hat{\tilde{\tau}}_t^{i,r} - \alpha_{y\tau} \hat{\tilde{y}}_t^i \tag{267}$$

**State**:

$$\begin{aligned} \dot{\hat{\tilde{y}}}_t^i &= -D'(\hat{\tilde{\pi}}_{H,t}^i + \dot{\tilde{s}}_t^i) + D\hat{\tilde{v}}_t^i \\ \dot{\hat{\tilde{\tau}}}_t^{i,r} &= \hat{\tilde{v}}_t^i \\ \dot{\hat{\tilde{\pi}}}_{H,t}^i &= \rho \hat{\tilde{\pi}}_{H,t}^i - \kappa_y \hat{\tilde{y}}_t^i - \kappa_\tau \hat{\tilde{\tau}}_t^{i,r} \end{aligned}$$

**Initial conditions**:

$$\begin{aligned} \mu_{\pi,0}^i &= 0 \\ \mu_{y,0}^i &= \Delta \Lambda_y \end{aligned}$$

$$\mu^i_{\tau,0} = -\Delta\Lambda_\tau.$$

We can now solve out for $\mu^i_{\pi,t}$ and $\mu^i_{\tau,t}$ to reduce the dimensionality of this dynamical system. Using equations (264), (266) and (267), we can write

$$(\kappa_\tau + D\kappa_y)\mu^i_{\pi,t} = (\Gamma^{**}_y D + \Gamma^{**}_\tau)\lambda - (2\alpha_{\tau\tau} + D\alpha_{y\tau})\hat{\tau}^{i,r}_t - (2D\alpha_{yy} + \alpha_{y\tau})\hat{y}^i_t. \tag{268}$$

Taking the derivative with respect to time and using (265), we have

$$(\kappa_\tau + D\kappa_y)(D'\mu^i_{y,t} - 2\alpha_\pi \hat{\pi}^i_{H,t}) + (2\alpha_{\tau\tau} + D\alpha_{y\tau})\dot{\hat{\tau}}^{i,r}_t + (2D\alpha_{yy} + \alpha_{y\tau})\dot{\hat{y}}^i_t = 0,$$

from which we can solve for the rate of change of transfers using the (IS) equation:

$$\dot{\hat{\tau}}^{i,r}_t = -\frac{(\kappa_\tau + D\kappa_y)(D'\mu^i_{y,t} - 2\alpha_\pi \hat{\pi}^i_{H,t}) - D'(2D\alpha_{yy} + \alpha_{y\tau})(\hat{\pi}^i_{H,t} + \dot{\hat{s}}^i_t)}{2\alpha_{\tau\tau} + D\alpha_{y\tau} + D(2D\alpha_{yy} + \alpha_{y\tau})} \tag{269}$$

Using the (IS) equation again, we have

$$\dot{\hat{y}}^i_t = -D'(\hat{\pi}^i_{H,t} + \dot{\hat{s}}^i_t) - D\frac{(\kappa_\tau + D\kappa_y)(D'\mu^i_{y,t} - 2\alpha_\pi \hat{\pi}^i_{H,t}) - D'(2D\alpha_{yy} + \alpha_{y\tau})(\hat{\pi}^i_{H,t} + \dot{\hat{s}}^i_t)}{2\alpha_{\tau\tau} + D\alpha_{y\tau} + D(2D\alpha_{yy} + \alpha_{y\tau})}. \tag{270}$$

Finally, using (266) and our solution for $\mu^i_{\pi,t}$ above, we find

$$\dot{\mu}^i_{y,t} = \rho\mu^i_{y,t} + \lambda\frac{\kappa_y\Gamma^{**}_\tau - \kappa_\tau\Gamma^{**}_y}{\kappa_\tau + D\kappa_y} - \hat{\tau}^{i,r}_t\frac{2\kappa_y\alpha_{\tau\tau} - \kappa_\tau\alpha_{y\tau}}{\kappa_\tau + D\kappa_y} - \hat{y}^i_t\frac{\alpha_{y\tau}\kappa_y - 2\kappa_\tau\alpha_{yy}}{\kappa_\tau + D\kappa_y}. \tag{271}$$

Therefore, we can write the reduced system of optimality conditions as

$$\dot{\mu}^i_{y,t} = \rho\mu^i_{y,t} + K_\tau\hat{\tau}^{i,r}_t + K_y\hat{y}^i_t + K_\lambda\lambda \tag{272}$$

$$\dot{\hat{y}}^i_t = J_{\mu_y}\mu^i_{y,t} + J_\pi\hat{\pi}^i_{H,t} + J_s\dot{\hat{s}}^i_t \tag{273}$$

$$\dot{\hat{\tau}}^{i,r}_t = H_{\mu_y}\mu^i_{y,t} + H_\pi\hat{\pi}^i_{H,t} + H_s\dot{\hat{s}}^i_t \tag{274}$$

$$\dot{\hat{\pi}}^i_{H,t} = \rho\hat{\pi}^i_{H,t} - \kappa_y\hat{y}^i_t - \kappa_\tau\hat{\tau}^{i,r}_t. \tag{275}$$

### D.8.2 Solving the Dynamical System

We can thus express the dynamical system of first-order conditions as a system of linear ODEs,

$$\dot{X}^i_t = AX^i_t + B^i_t \tag{276}$$

for $X_t^i = \{\hat{\hat{\pi}}_{H,t}^i, \hat{y}_t^i, \hat{\hat{\tau}}_t^{i,r}, \mu_{y,t}^i\}$. We have

$$A = \begin{pmatrix} \rho & -\kappa_y & -\kappa_\tau & 0 \\ J_\pi & 0 & 0 & J_{\mu_y} \\ H_\pi & 0 & 0 & H_{\mu_y} \\ 0 & K_y & K_\tau & \rho \end{pmatrix}, \quad B_t^i = \begin{pmatrix} 0 \\ \bar{J}_t^i \\ \bar{H}_t^i \\ \bar{K}_t^i \end{pmatrix}, \quad X_0^i = \begin{pmatrix} \hat{\hat{\pi}}_{H,0}^i \\ \hat{y}_0^i \\ \hat{\hat{\tau}}_0^{i,r} \\ \Delta\Lambda_y \end{pmatrix},$$

and $\bar{J}_t^i = J_s \dot{\tilde{s}}_t^i$, $\bar{H}_t^i = H_s \dot{\tilde{s}}_t^i$, and $\bar{K}_t^i = K_\lambda \lambda$.

Following the same steps as in Section 4, we can write the solution as

$$X_t^i = e^{At}\left[X_0^i + K_\lambda \lambda \int_0^t e^{-As} E_4 ds + W_2 \int_0^t e^{-As-\psi s} E_2 ds + W_3 \int_0^t e^{-As-\psi s} E_3 ds\right],$$

where $W_2 = -\psi a_0^i J_s$ and $W_3 = -\psi a_0^i H_s$. Assuming that the economy is in the parameter subspace where $A$ is nonsingular, we can solve out the integrals to arrive at

$$\begin{aligned} X_t^i =& e^{At}\left[X_0^i + K_\lambda \lambda A^{-1} E_4 + (A+\psi I)^{-1}\big(W_2 E_2 + W_3 E_3\big)\right] \\ & - K_\lambda \lambda A^{-1} E_4 - (A+\psi I)^{-1} e^{-\psi t}\big(W_2 E_2 + W_3 E_3\big). \end{aligned} \tag{277}$$

### D.8.3 Stability

Our main departure from the solution in Section 4 is the following stability analysis. To guarantee uniqueness, we still require

$$X_0^i + K_\lambda \lambda A^{-1} E_4 + (A+\psi I)^{-1}\big(W_2 E_2 + W_3 E_3\big)$$

to be in the stable manifold. Let $V_j$, $j \in \{1,2\}$, denote the eigenvector of $A$ associated with the negative eigenvalue $\lambda_j$. We can express this stability condition as

$$0 = \alpha_1 V_1 + \alpha_2 V_2 + X_0^i + K_\lambda \lambda A^{-1} E_4 + (A+\psi I)^{-1}\big(W_2 E_2 + W_3 E_3\big), \tag{278}$$

for some $\alpha_j \in \mathbb{C}, j \in \{1,2\}$.

Under deficit spending, however, we now have an extra parameter that we need to solve for. That is, we now need to solve the vector $Z = (\alpha_1, \alpha_2, \lambda, \Delta, \hat{\hat{\pi}}_{H,0}^i, \hat{y}_0^i, \hat{\hat{\tau}}_0^{i,r})' \in \mathbb{C}^7$ to satisfy the stability requirement and initial conditions. Equation (278) yields four of the desired eight conditions. As before, we left-multiply (278) by $E_i'$ for each $i \in \{1,2,3,4\}$.

The first initial condition we can use is $\Lambda_y \hat{y}_0^i = -\Lambda_s \tilde{s}_0^i + \Lambda_\tau \hat{\hat{\tau}}_0^{i,r}$. Another one we obtain by writing

$$(\kappa_\tau + D\kappa_y)\mu_{\pi,0}^i = 0 = (\Gamma_y^{**} D + \Gamma_\tau^{**})\lambda - (2\alpha_{\tau\tau} + D\alpha_{y\tau})\hat{\hat{\tau}}_0^{i,r} - (2D\alpha_{yy} + \alpha_{y\tau})\hat{y}_0^i. \tag{279}$$

The remaining two conditions we obtain from the two constraints associated with $\lambda_1$ and $\lambda_2$, respectively. Note that we can write

$$
\begin{aligned}
X_t^i =& e^{At}\big(-\alpha_1 V_1 - \alpha_2 V_2\big) - K_\lambda \lambda A^{-1}E_4 - (A+\psi I)^{-1}e^{-\psi t}\big(W_2E_2 + W_3E_3\big) \\
=& -\alpha_1 e^{\lambda_1 t}V_1 - \alpha_2 e^{\lambda_2 t}V_2 - K_\lambda \lambda A^{-1}E_4 - (A+\psi I)^{-1}e^{-\psi t}\big(W_2E_2 + W_3E_3\big).
\end{aligned}
$$

The budget constraint then becomes

$$
\begin{aligned}
0 =& \int_0^\infty e^{-\rho t}\left[\Gamma_y^{**}\hat{\tilde{y}}_t^i + \Gamma_\tau^{**}\hat{\tilde{\tau}}_t^{i,r}\right]dt \\
=& \int_0^\infty e^{-\rho t}(\Gamma_y^{**}E_2 + \Gamma_\tau^{**}E_3)' X_t^i dt,
\end{aligned}
$$

where we can plug in the solution for $X_t^i$ and solve out the integrals to obtain

$$
\begin{aligned}
0 =& \frac{\alpha_1}{\lambda_1-\rho}(\Gamma_y^{**}E_2 + \Gamma_\tau^{**}E_3)'V_1 + \frac{\alpha_2}{\lambda_2-\rho}(\Gamma_y^{**}E_2 + \Gamma_\tau^{**}E_3)'V_2 \\
& - \frac{K_\lambda \lambda}{\rho}(\Gamma_y^{**}E_2 + \Gamma_\tau^{**}E_3)'A^{-1}E_4 \\
& - \frac{1}{\rho+\psi}(\Gamma_y^{**}E_2 + \Gamma_\tau^{**}E_3)'(A+\psi I)^{-1}[W_2E_2 + W_3E_3].
\end{aligned} \tag{280}
$$

Putting everything together, we can write $MZ = N$. Letting $\tilde{E} = (\Gamma_y^{**}E_2 + \Gamma_\tau^{**}E_3)$ for ease of notation, we have

$$
M = \begin{pmatrix}
E_1'V_1 & E_1'V_2 & K_\lambda E_1'A^{-1}E_4 & 0 & 1 & 0 & 0 \\
E_2'V_1 & E_2'V_2 & K_\lambda E_2'A^{-1}E_4 & 0 & 0 & 1 & 0 \\
E_3'V_1 & E_3'V_2 & K_\lambda E_3'A^{-1}E_4 & 0 & 0 & 0 & 1 \\
E_4'V_1 & E_4'V_2 & K_\lambda E_4'A^{-1}E_4 & \Lambda_y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Lambda_y & -\Lambda_\tau \\
0 & 0 & \Gamma_\tau^{**} + D\Gamma_y^{**} & 0 & 0 & -(2\alpha_{yy}D + \alpha_{y\tau}) & -(2\alpha_{\tau\tau} + D\alpha_{y\tau}) \\
\frac{\tilde{E}'}{\lambda_1-\rho}V_1 & \frac{\tilde{E}'}{\lambda_2-\rho}V_2 & -\frac{K_\lambda}{\rho}\tilde{E}'A^{-1}E_4 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

and

$$
N = \begin{pmatrix}
-E_1'W \\
-E_2'W \\
-E_3'W \\
-E_4'W \\
-\Lambda_s \tilde{s}_0^i \\
0 \\
\frac{\tilde{E}'}{\rho+\psi}(A+\psi I)^{-1}[W_2E_2 + W_3E_3]
\end{pmatrix}
$$

where $W = (A+\psi I)^{-1}[W_2E_2 + W_3E_3]$. Given these matrices, we can compute the vector $Z$ numerically by setting

$$Z = M^{-1}N \tag{281}$$

over the parameter subspace on which $M$ is invertible.

## D.9 Jointly Optimal Fiscal Policy

Finally, we consider optimal fiscal policy, allowing for both government spending and redistribution. The allocation derived under jointly optimal government and deficit spending would be identical.

In the absence of international transfers, we have $NFA_0^i = N\hat{\hat{F}}A_0^i = 0$. We do not allow for capital controls, $\hat{\hat{\theta}}_t^i = \hat{\hat{\theta}}^i$, and impose the per-period government budget constraint, $\chi\hat{\hat{\tau}}_t^{i,r} + (1-\chi)\hat{\hat{\tau}}_t^{i,o} + v\hat{\hat{g}}_t^i - \alpha v\hat{\hat{s}}_t^i = 0$.

Since the government's budget is balanced in every period, the original external budget constraint for country $i$ holds,

$$0 = \int_0^\infty e^{\rho t}\left[\hat{\hat{y}}_t^i - \alpha(1-v)\hat{\hat{s}}_t^i - (1-v)\hat{\hat{c}}_t^i - v\hat{\hat{g}}_t^i\right]dt \equiv \int_0^\infty e^{-\rho t}N\hat{\hat{X}}_t^i dt \tag{282}$$

where as before $\hat{\hat{s}}_t^i = \nu_y\hat{\hat{y}}_t^i + \nu_\tau\hat{\hat{\tau}}_t^{i,r} + \nu_\theta\hat{\hat{\theta}}^i + \nu_g\hat{\hat{g}}_t^i$ and $\hat{\hat{c}}_t^i = \zeta_y\hat{\hat{y}}_t^i + \zeta_g\hat{\hat{g}}_t^i + \zeta_\tau\hat{\hat{\tau}}_t^{i,r} + \zeta_\theta\hat{\hat{\theta}}^i$. We can simplify to write the NFA condition as

$$0 = \int_0^\infty e^{-\rho t}\left[\Gamma_y^*\hat{\hat{y}}_t^i + \Gamma_\theta^*\hat{\hat{\theta}}^i + \Gamma_\tau^*\hat{\hat{\tau}}_t^{i,r} + \Gamma_g^*\hat{\hat{g}}_t^i\right]dt, \tag{283}$$

where

$$\begin{aligned}
\Gamma_y^* &= 1 - \alpha(1-v)\nu_y - (1-v)\zeta_y \\
\Gamma_\theta^* &= -\alpha(1-v)\nu_\tau - (1-v)\zeta_\tau \\
\Gamma_\tau^* &= -\alpha(1-v)\nu_\theta - (1-v)\zeta_\theta \\
\Gamma_g^* &= -v - \alpha(1-v)\nu_g - (1-v)\zeta_g.
\end{aligned}$$

**Planning problem.** We can again adopt the (IS), (NKPC) and initial condition as they were specified in Section 4. Therefore, we can write country $i$'s disaggregated planning problem as

$$\min_{\hat{\hat{\theta}}^i,\{\hat{\hat{\tau}}_t^{i,r},\hat{\hat{g}}_t^i\}_{t\geq 0}} \int_0^\infty e^{-\rho t}\left[\alpha_\pi(\hat{\hat{\pi}}_{H,t}^i)^2 - \hat{U}_t^i\big|_{\hat{\hat{\theta}}_t^i=\hat{\hat{\theta}}^i}\right]dt, \tag{284}$$

subject to

$$\begin{aligned}
\dot{\hat{\hat{y}}}_t^i &= -D'(\hat{\hat{\pi}}_{H,t}^i + \dot{\hat{\hat{s}}}_t^i) + \bar{D}\dot{\hat{\hat{g}}}_t^i + D\dot{\hat{\hat{\tau}}}_t^{i,r} \\
\dot{\hat{\hat{\pi}}}_{H,t}^i &= \rho\hat{\hat{\pi}}_{H,t}^i - \kappa_y\hat{\hat{y}}_t^i - \kappa_\theta\hat{\hat{\theta}}^i - \kappa_\tau\hat{\hat{\tau}}_t^{i,r} - \kappa_g\hat{\hat{g}}_t^i
\end{aligned}$$

$$\Lambda_y \hat{\tilde{y}}_0^i = -\Lambda_s \tilde{s}_0^i + \Lambda_\theta \hat{\tilde{\theta}}^i + \Lambda_\tau \hat{\tilde{\tau}}_0^{i,r} + \Lambda_g \hat{\tilde{g}}_0^i$$
$$0 = \int_0^\infty e^{-\rho t} \left[ \Gamma_y^* \hat{\tilde{y}}_t^i + \Gamma_\theta^* \hat{\tilde{\theta}}^i + \Gamma_\tau^* \hat{\tilde{\tau}}_t^{i,r} + \Gamma_g^* \hat{\tilde{g}}_t^i \right] dt$$

where $D' = \frac{\Lambda_s}{\Lambda_y}$, $\bar{D} = \frac{\Lambda_g}{\Lambda_y}$ and $D = \frac{\Lambda_\tau}{\Lambda_y}$, and

$$\begin{aligned} \hat{U}_t^i \big|_{\hat{\tilde{\theta}}_t^i = \hat{\tilde{\theta}}^i} =& \alpha_{yy} (\hat{\tilde{y}}_t^i)^2 + \alpha_{\theta\theta} (\hat{\tilde{\theta}}^i)^2 + \alpha_{\tau\tau} (\hat{\tilde{\tau}}_t^{i,r})^2 + \alpha_{gg} (\hat{\tilde{g}}_t^i)^2 \\ &+ \alpha_{y\theta} \hat{\tilde{y}}_t^i \hat{\tilde{\theta}}^i + \alpha_{yg} \hat{\tilde{y}}_t^i \hat{\tilde{g}}_t^i + \alpha_{y\tau} \hat{\tilde{y}}_t^i \hat{\tilde{\tau}}_t^{i,r} + \alpha_{g\theta} \hat{\tilde{g}}_t^i \hat{\tilde{\theta}}^i + \alpha_{g\tau} \hat{\tilde{g}}_t^i \hat{\tilde{\tau}}_t^{i,r} + \alpha_{\tau\theta} \hat{\tilde{\tau}}_t^{i,r} \hat{\tilde{\theta}}^i. \end{aligned}$$

**Control problem.** As before, we incorporate the initial condition and the country budget constraint into the objective function using Lagrange multipliers $\Delta$ and $\lambda$, respectively. Furthermore, we replace the time derivatives $\dot{\hat{\tilde{g}}}_t^i$ and $\dot{\hat{\tilde{\tau}}}_t^{i,r}$ in the (IS) equation with $\hat{\omega}_t^i$ and $\hat{v}_t^i$, respectively, to recover the standard optimal control problem structure. We add these equations as dynamic state equations. We can now write down the Hamiltonian associated with this control problem. We have

$$\begin{aligned} H(x_t^i, u_t^i, \mu_t^i) =& \alpha_\pi (\hat{\tilde{\pi}}_{H,t}^i)^2 - \hat{U}_t^i \big|_{\hat{\tilde{\theta}}_t^i = \hat{\tilde{\theta}}^i} \\ &+ \Delta \left[ \Lambda_y \hat{\tilde{y}}_0^i + \Lambda_s \tilde{s}_0^i - \Lambda_g \hat{\tilde{g}}_0^i - \Lambda_\theta \hat{\tilde{\theta}}^i - \Lambda_\tau \hat{\tilde{\tau}}_0^{i,r} \right] \\ &+ \lambda \left[ \Gamma_y^* \hat{\tilde{y}}_t^i + \Gamma_g^* \hat{\tilde{g}}_t^i + \Gamma_\tau^* \hat{\tilde{\tau}}_t^{i,r} + \Gamma_\theta^* \hat{\tilde{\theta}}^i \right] \\ &+ \mu_{y,t}^i \left[ D \hat{v}_t^i + \bar{D} \hat{\omega}_t^i - D' (\hat{\tilde{\pi}}_{H,t}^i + \dot{s}_t^i) \right] \\ &+ \mu_{\tau,t}^i \hat{v}_t^i \\ &+ \mu_{g,t}^i \hat{\omega}_t^i \\ &+ \mu_{\pi,t}^i \left[ \rho \hat{\tilde{\pi}}_{H,t}^i - \kappa_y \hat{\tilde{y}}_t^i - \kappa_\theta \hat{\tilde{\theta}}^i - \kappa_\tau \hat{\tilde{\tau}}_t^{i,r} - \kappa_g \hat{\tilde{g}}_t^i \right], \end{aligned} \tag{285}$$

where $x_t^i = \{\hat{\tilde{\pi}}_{H,t}^i, \hat{\tilde{y}}_t^i, \hat{\tilde{\tau}}_t^{i,r}, \hat{\tilde{g}}_t^i\}$ denotes the vector of state variables, $u_t^i = \{\hat{v}_t^i, \hat{\omega}_t^i\}$ the vector of control variables, and $\mu_t^i = \{\mu_{\pi,t}^i, \mu_{y,t}^i, \mu_{g,t}^i, \mu_{\tau,t}^i\}$ the vector of costates.

### D.9.1 Optimality Conditions

The first-order optimality conditions associated with this problem are given by

**Optimality**:

$$D \mu_{y,t}^i + \mu_{\tau,t}^i = 0 \tag{286}$$
$$\bar{D} \mu_{y,t}^i + \mu_{g,t}^i = 0 \tag{287}$$

**Multiplier**:

$$\dot{\mu}^i_{\pi,t} = D'\mu^i_{y,t} - 2\alpha_\pi \hat{\tilde{\pi}}^i_{H,t} \tag{288}$$

$$\rho\mu^i_{y,t} - \dot{\mu}^i_{y,t} = \Gamma^*_y\lambda - \kappa_y\mu^i_{\pi,t} - 2\alpha_{yy}\hat{\tilde{y}}^i_t - \alpha_{y\theta}\hat{\tilde{\theta}}^i - \alpha_{yg}\hat{\tilde{g}}^i_t - \alpha_{y\tau}\hat{\tilde{\tau}}^{i,r}_t \tag{289}$$

$$\rho\mu^i_{\tau,t} - \dot{\mu}^i_{\tau,t} = \Gamma^*_\tau\lambda - \kappa_\tau\mu^i_{\pi,t} - 2\alpha_{\tau\tau}\hat{\tilde{\tau}}^{i,r}_t - \alpha_{\tau\theta}\hat{\tilde{\theta}}^i - \alpha_{y\tau}\hat{\tilde{y}}^i_t - \alpha_{g\tau}\hat{\tilde{g}}^i_t \tag{290}$$

$$\rho\mu^i_{g,t} - \dot{\mu}^i_{g,t} = \Gamma^*_g\lambda - \kappa_g\mu^i_{\pi,t} - 2\alpha_{gg}\hat{\tilde{g}}^i_t - \alpha_{yg}\hat{\tilde{g}}^i - \alpha_{g\theta}\hat{\tilde{\theta}}^i - \alpha_{g\tau}\hat{\tilde{\tau}}^{i,r}_t \tag{291}$$

**State**:

$$\begin{aligned}
\dot{\hat{\tilde{y}}}^i_t &= -D'(\hat{\tilde{\pi}}^i_{H,t} + \dot{\tilde{s}}^i_t) + D\hat{\tilde{v}}^i_t + \bar{D}\hat{\tilde{\omega}}^i_t \\
\dot{\hat{\tilde{\tau}}}^{i,r}_t &= \hat{\tilde{v}}^i_t \\
\dot{\hat{\tilde{g}}}^{i,r}_t &= \hat{\tilde{\omega}}^i_t \\
\dot{\hat{\tilde{\pi}}}^i_{H,t} &= \rho\hat{\tilde{\pi}}^i_{H,t} - \kappa_y\hat{\tilde{y}}^i_t - \kappa_g\hat{\tilde{g}}^i_t - \kappa_\theta\hat{\tilde{\theta}}^i - \kappa_\tau\hat{\tilde{\tau}}^{i,r}_t
\end{aligned}$$

**Initial conditions**:

$$\begin{aligned}
\mu^i_{\pi,0} &= 0 \\
\mu^i_{y,0} &= \Delta\Lambda_y \\
\mu^i_{\tau,0} &= -\Delta\Lambda_\tau \\
\mu^i_{g,0} &= -\Delta\Lambda_g.
\end{aligned}$$

We can conveniently reduce the dimensionality of this system of differential equations. Following steps analogous to those in earlier sections, we can write

$$\begin{aligned}
\dot{\hat{\tilde{g}}}^i_t =& -\frac{1}{\alpha_{g\tau} + D\alpha_{yg} + \bar{D}(2\alpha_{yy}D + \alpha_{y\tau})}\Bigg[(\kappa_\tau + D\kappa_y)D'\mu^i_{y,t} \\
& - \Big(2\alpha_\pi(\kappa_\tau + D\kappa_y) + D'(2\alpha_{yy}D + \alpha_{y\tau})\Big)\hat{\tilde{\pi}}^i_{H,t} - D'(2\alpha_{yy}D + \alpha_{y\tau})\dot{\tilde{s}}^i_t \\
& + \Big(D(2\alpha_{yy}D + \alpha_{y\tau}) + 2\alpha_{\tau\tau} + D\alpha_{y\tau}\Big)\dot{\hat{\tilde{\tau}}}^{i,r}_t\Bigg] \\
\equiv & l_{\mu_y}\mu^i_{y,t} + l_\pi\hat{\tilde{\pi}}^i_{H,t} + l_s\dot{\tilde{s}}^i_t + l_\tau\dot{\hat{\tilde{\tau}}}^{i,r}_t.
\end{aligned}$$

We then find that

$$\begin{aligned}
\dot{\hat{\tilde{\tau}}}^{i,r}_t =& -\frac{1}{2\alpha_{yy}D\bar{D} + \alpha_{yg}D + \alpha_{g\tau} + \bar{D}\alpha_{y\tau} + l_\tau(2\alpha_{yy}\bar{D}^2 + \alpha_{yg}\bar{D} + 2\alpha_{gg} + \bar{D}\alpha_{yg})}\cdot \\
& \Bigg[\Big(D'(\kappa_g + \bar{D}\kappa_y) + l_{\mu_y}(2\alpha_{yy}\bar{D}^2 + \alpha_{yg}\bar{D} + 2\alpha_{gg} + \bar{D}\alpha_{yg})\Big)\mu^i_{y,t} \\
& + \Big(l_\pi(2\alpha_{yy}\bar{D}^2 + \alpha_{yg}\bar{D} + 2\alpha_{gg} + \bar{D}\alpha_{yg}) - 2\alpha_\pi(\kappa_g + \bar{D}\kappa_y) - D'(2\alpha_{yy}\bar{D} + \alpha_{yg})\Big)\hat{\tilde{\pi}}^i_{H,t}
\end{aligned}$$

$$-\left(D'(2\alpha_{yy}\bar{D}+\alpha_{yg})+l_s(2\alpha_{yy}\bar{D}^2+\alpha_{yg}\bar{D}+2\alpha_{gg}+\bar{D}\alpha_{yg})\right)\dot{\hat{s}}_t^i\Big]$$
$$\equiv H_{\mu_y}\mu_{y,t}^i+H_\pi\hat{\pi}_{H,t}^i+H_s\hat{s}_t^i$$

and

$$\dot{\hat{g}}_t^i = l_{\mu_y}\mu_{y,t}^i+l_\pi\hat{\pi}_{H,t}^i+l_s\dot{\hat{s}}_t^i+l_\tau(H_{\mu_y}\mu_{y,t}^i+H_\pi\hat{\pi}_{H,t}^i+H_s\dot{\hat{s}}_t^i).$$

Using the (IS) equation, we have

$$\dot{\hat{y}}_t^i = (DH_{\mu_y}+\bar{D}L_{\mu_y})\mu_{y,t}^i+(DH_\pi+\bar{D}L_\pi-D')\hat{\pi}_{H,t}^i+(DH_s+\bar{D}L_s+D')\dot{\hat{s}}_t^i.$$

Finally, we can write

$$\dot{\mu}_{y,t}^i = \rho\mu_{y,t}^i+\lambda\frac{\kappa_y\Gamma_g^*-\kappa_g\Gamma_y^*}{\kappa_g+\bar{D}\kappa_y}+\hat{y}_t^i\frac{2\alpha_{yy}\kappa_g-\alpha_{yg}\kappa_y}{\kappa_g+\bar{D}\kappa_y}+\hat{g}_t^i\frac{\alpha_{yg}\kappa_g-2\alpha_{gg}\kappa_y}{\kappa_g+\bar{D}\kappa_y}$$
$$+\hat{\tau}_t^{i,r}\frac{\alpha_{y\tau}\kappa_g-\alpha_{g\tau}\kappa_y}{\kappa_g+\bar{D}\kappa_y}+\hat{\theta}^i\frac{\alpha_{y\theta}\kappa_g-\alpha_{g\theta}\kappa_y}{\kappa_g+\bar{D}\kappa_y}.$$

Summarizing, we obtain the system

$$\dot{\hat{\pi}}_{H,t}^i = \rho\hat{\pi}_{H,t}^i-\kappa_y\hat{y}_t^i-\kappa_g\hat{g}_t^i-\kappa_\theta\hat{\theta}^i-\kappa_\tau\hat{\tau}_t^{i,r} \tag{292}$$
$$\dot{\hat{y}}_t^i \equiv J_{\mu_y}\mu_{y,t}^i+J_\pi\hat{\pi}_{H,t}^i+J_s\dot{\hat{s}}_t^i \tag{293}$$
$$\dot{\hat{\tau}}_t^{i,r} = H_{\mu_y}\mu_{y,t}^i+H_\pi\hat{\pi}_{H,t}^i+H_s\dot{\hat{s}}_t^i \tag{294}$$
$$\dot{\hat{g}}_t^i \equiv L_{\mu_y}\mu_{y,t}^i+L_\pi\hat{\pi}_{H,t}^i+L_s\dot{\hat{s}}_t^i \tag{295}$$
$$\dot{\mu}_{y,t}^i \equiv \rho\mu_{y,t}^i+K_\lambda\lambda+K_y\hat{y}_t^i+K_g\hat{g}_t^i+K_\tau\hat{\tau}_t^{i,r}+K_\theta\hat{\theta}^i. \tag{296}$$

### D.9.2 A Targeting Rule for Government Spending

If we let $X_t^i = \{\hat{\pi}_{H,t}^i, \hat{y}_t^i, \hat{\tau}_t^{i,r}, \hat{g}_t^i, \mu_{y,t}^i\}$, then the dynamical system of optimality conditions can be written as

$$\dot{X}_t^i = AX_t^i+B_t^i, \tag{297}$$

where

$$A=\begin{pmatrix}\rho & -\kappa_y & -\kappa_\tau & -\kappa_g & 0\\ J_\pi & 0 & 0 & 0 & J_{\mu_y}\\ H_\pi & 0 & 0 & 0 & H_{\mu_y}\\ L_\pi & 0 & 0 & 0 & L_{\mu_y}\\ 0 & K_y & K_\tau & K_g & \rho\end{pmatrix},\quad B_t^i=\begin{pmatrix}-\kappa_\theta\hat{\theta}^i\\ \bar{J}_t^i\\ \bar{H}_t^i\\ \bar{L}_t^i\\ \bar{K}_t^i\end{pmatrix},\quad X_0^i=\begin{pmatrix}\hat{\pi}_{H,0}^i\\ \hat{y}_0^i\\ \hat{\tau}_0^{i,r}\\ \hat{g}_0^i\\ \Delta\Lambda_y\end{pmatrix},$$

and

$$\bar{J}_t^i = J_s\dot{\hat{s}}_t^i$$

$$\bar{H}_t^i = H_s \hat{\dot{s}}_t^i$$
$$\bar{L}_t^i = L_s \hat{\dot{s}}_t^i$$
$$\bar{K}_t^i = K_\theta \hat{\dot{\theta}}^i + K_\lambda \lambda.$$

The coefficient matrix $A$ has one 0-eigenvalue. This implies that one of the differential equations can be expressed as a function of the other variables in $X_t^i$. We will exploit this to derive an explicit targeting rule for government spending using tools from linear algebra. We could also prove this result directly starting from the system (292) - (296).

If $A$ has a 0-eigenvalue, then so does its transpose $A'$. Hence, there exists a vector $E$ such that $A'E = 0$, which gives us $E'A = 0$. That is, $E$ is the eigenvector associated with the 0-eigenvalue of the transpose matrix $A'$. We have

$$E'\dot{X}_t^i = E'AX_t^i + E'B_t^i = E'B_t^i.$$

Integrating, we find

$$E'X_t^i = E'X_0^i + \int_0^t E'B_s^i ds$$

It is easy to verify that $E(1) = E(5) = 0$. Therefore, we can write

$$E_g \hat{\dot{g}}_t^i = -E_y \hat{\dot{y}}_t^i - E_\tau \hat{\dot{\tau}}_t^{i,r} + E'X_0^i + \int_0^t E'B_s^i ds, \tag{298}$$

where $E_y = E(2)$, $E_\tau = E(3)$ and $E_g = E(4)$. Let

$$W = -\psi E_y J_s a_0^i - \psi E_\tau H_s a_0^i - \psi E_g L_s a_0^i.$$

It then follows from (298) that government spending follows the policy targeting rule

$$\hat{\dot{g}}_t^i = -\frac{E_y}{E_g}(\hat{\dot{y}}_t^i - \hat{\dot{y}}_0^i) - \frac{E_\tau}{E_g}(\hat{\dot{\tau}}_t^{i,r} - \hat{\dot{\tau}}_0^{i,r}) + \hat{\dot{g}}_0^i + \frac{W}{\psi E_g}(1 - e^{-\psi t}). \tag{299}$$

So far, we have only shown that the targeting rule holds for some family of eigenvectors $E$. However, we can even solve for $E_y, E_\tau$ and $E_g$ in closed form. Differentiating the targeting rule with respect to time, we have

$$\dot{\hat{\dot{g}}}_t^i = -\frac{E_y}{E_g}\dot{\hat{\dot{y}}}_t^i - \frac{E_\tau}{E_g}\dot{\hat{\dot{\tau}}}_t^{i,r} + \frac{W}{E_g}e^{-\psi t}. \tag{300}$$

Using the original system of equations (292) - (296) to substitute in for $\dot{\hat{\dot{y}}}_t^i$ and $\dot{\hat{\dot{\tau}}}_t^{i,r}$ yields, after some algebra, the two equations

$$E_g L_{\mu_y} = -E_y J_{\mu_y} - E_\tau H_{\mu_y} \tag{301}$$
$$E_g L_\pi = -E_y J_\pi - E_\tau H_\pi, \tag{302}$$

in the three unknowns $E_y, E_\tau$ and $E_g$. By construction, an uncountably large class of linearly dependent eigenvectors $E$ can give rise to the targeting rule (note that the coefficients in the policy rule are ratios). Therefore, we can normalize $E_g = 1$, and solve for $E_y$ and $E_\tau$:

$$E_y = -\frac{H_\pi L_{\mu_y} + H_{\mu_y} L_\pi}{H_\pi J_{\mu_y} + H_{\mu_y} J_\pi} \tag{303}$$

$$E_\tau = \frac{J_\pi}{H_\pi} \frac{H_\pi L_{\mu_y} + H_{\mu_y} L_\pi}{H_\pi J_{\mu_y} + H_{\mu_y} - \frac{L_\pi}{H_\pi}}. \tag{304}$$

### D.9.3 Solving the Dynamic System

We can use the targeting rule to simplify the dynamical system of optimality conditions. We have

$$\dot{\hat{\pi}}^i_{H,t} = \rho \hat{\pi}^i_{H,t} - \tilde{\kappa}_y \hat{y}^i_t - \tilde{\kappa}_\tau \hat{\tau}^{i,r}_t - \kappa_\theta \hat{\theta}^i - \frac{\kappa_g}{E_g}(E_y \hat{y}^i_0 + E_\tau \hat{\tau}^{i,r}_0) - \kappa_g \hat{g}^i_0 - \kappa_g \frac{W}{\psi E_g}(1 - e^{-\psi t}) \tag{305}$$

$$\dot{\hat{y}}^i_t \equiv J_{\mu_y} \mu^i_{y,t} + J_\pi \hat{\pi}^i_{H,t} + \bar{J}^i_t \tag{306}$$

$$\dot{\hat{\tau}}^{i,r}_t = H_{\mu_y} \mu^i_{y,t} + H_\pi \hat{\pi}^i_{H,t} + \bar{H}^i_t \tag{307}$$

$$\dot{\mu}^i_{y,t} = \rho \mu^i_{y,t} + \tilde{K}_y \hat{y}^i_t + \tilde{K}_\tau \hat{\tau}^{i,r}_t + \bar{K}^i_t + \frac{K_g}{E_g}(E_y \hat{y}^i_0 + E_\tau \hat{\tau}^{i,r}_0) + K_g \hat{g}^i_0 + K_g \frac{W}{\psi E_g}(1 - e^{-\psi t}), \tag{308}$$

where

$$\tilde{\kappa}_y = \kappa_y - \kappa_g \frac{E_y}{E_g}$$
$$\tilde{\kappa}_\tau = \kappa_\tau - \kappa_g \frac{E_\tau}{E_g}$$
$$\tilde{K}_y = K_y - K_g \frac{E_y}{E_g}$$
$$\tilde{K}_\tau = K_\tau - K_g \frac{E_\tau}{E_g}.$$

Next, we need to solve for $\hat{y}^i_0$ and $\hat{\tau}^{i,r}_0$ using the initial conditions

$$\Lambda_y \hat{y}^i_0 = -\Lambda_s \tilde{s}^i_0 + \Lambda_\theta \hat{\theta}^i + \Lambda_\tau \hat{\tau}^{i,r}_0 + \Lambda_g \hat{g}^i_0 \tag{309}$$

$$\begin{aligned} 0 =& \lambda(\Gamma^*_g + \bar{D}\Gamma^*_y) - (2\alpha_{gg} + \bar{D}\alpha_{yg})\hat{g}^i_0 - (2\alpha_{yy}\bar{D} + \alpha_{yg})\hat{y}^i_0 \\ & - (\alpha_{g\tau} + \bar{D}\alpha_{y\tau})\hat{\tau}^{i,r}_0 - (\alpha_{g\theta} + \bar{D}\alpha_{y\theta})\hat{\theta}^i. \end{aligned} \tag{310}$$

Rearranging and solving the two equations in the two unknowns, we can write

$$\hat{\tau}^{i,r}_0 \equiv Q_\lambda \lambda + Q_g \hat{g}^i_0 + Q_\theta \hat{\theta}^i + Q_s \tilde{s}^i_0 \tag{311}$$

$$\hat{y}^i_0 \equiv P_\lambda \lambda + P_g \hat{g}^i_0 + P_\theta \hat{\theta}^i + P_s \tilde{s}^i_0. \tag{312}$$

Hence, we have solved for $\hat{\tilde{y}}_0^i$ and $\hat{\tilde{\tau}}_0^{i,r}$ as functions of the Lagrange multiplier $\lambda$, $\hat{\tilde{g}}_0^i$ and $\hat{\tilde{\theta}}^i$. Recall that in previous sections, we have solved for the optimal allocation as a function of $\hat{\tilde{\theta}}^i$. Now, even though the path of government spending is tied down by the targeting rule, $\hat{\tilde{g}}_0^i$ is a free parameter that the social planner can choose optimally. Therefore, we will now go on to solve the optimal allocation as a function of $(\hat{\tilde{g}}_0^i, \hat{\tilde{\theta}}^i)$ and then numerically pin down these two free constants.

The transformed dynamical system can now be written in terms of $\tilde{X}_t^i = \{\hat{\tilde{\pi}}_{H,t}^i, \hat{\tilde{y}}_t^i, \hat{\tilde{\tau}}_t^{i,r}, \mu_{y,t}^i\}$ as

$$\dot{\tilde{X}}_t^i = \tilde{A}\tilde{X}_t^i + \tilde{B}_t^i, \tag{313}$$

where

$$\tilde{A} = \begin{pmatrix} \rho & -\tilde{\tilde{\kappa}}_y & -\tilde{\tilde{\kappa}}_\tau & 0 \\ J_\pi & 0 & 0 & J_{\mu_y} \\ H_\pi & 0 & 0 & H_{\mu_y} \\ 0 & \tilde{\tilde{K}}_y & \tilde{\tilde{K}}_\tau & \rho \end{pmatrix}, \qquad \tilde{B}_t^i = \begin{pmatrix} \kappa_t^i \\ \bar{J}_t^i \\ \bar{H}_t^i \\ K_t^i \end{pmatrix},$$

and

$$\tilde{X}_0^i = \begin{pmatrix} \hat{\tilde{\pi}}_{H,0}^i \\ P_\lambda \lambda + P_g \hat{\tilde{g}}_0^i + P_\theta \hat{\tilde{\theta}}^i + P_s \tilde{s}_0^i \\ Q_\lambda \lambda + Q_g \hat{\tilde{g}}_0^i + Q_\theta \hat{\tilde{\theta}}^i + Q_s \tilde{s}_0^i \\ \Delta\Lambda_y \end{pmatrix},$$

where

$$\begin{aligned} \kappa_t^i = & -\kappa_\theta \hat{\tilde{\theta}}^i - \kappa_g \frac{W}{\psi E_g}(1 - e^{-\psi t}) - \lambda \frac{\kappa_g}{E_g}(E_y P_\lambda + E_\tau Q_\lambda) \\ & - \hat{\tilde{g}}_0^i \frac{\kappa_g}{E_g}(E_y P_g + E_\tau Q_g + E_g) - \hat{\tilde{\theta}}^i \frac{\kappa_g}{E_g}(E_y P_\theta + E_\tau Q_\theta) \\ & - \tilde{s}_0^i \frac{\kappa_g}{E_g}(E_y P_s + E_\tau Q_s) \\ \equiv & -\tilde{\kappa}_\theta \hat{\tilde{\theta}}^i - \tilde{\kappa}_\lambda \lambda - \tilde{\kappa}_g \hat{\tilde{g}}_0^i - \tilde{\kappa} + \kappa_g \frac{W}{\psi E_g} e^{-\psi t} \end{aligned}$$

with

$$\tilde{\kappa} = \tilde{s}_0^i \frac{\kappa_g}{E_g}(E_y P_s + E_\tau Q_s) + \kappa_g \frac{W}{\psi E_g},$$

and

$$\begin{aligned} K_t^i = & K_g \frac{W}{\psi E_g}(1 - e^{-\psi t}) + \lambda \frac{K_g}{E_g}(E_y P_\lambda + E_\tau Q_\lambda) \\ & + \hat{\tilde{g}}_0^i \frac{K_g}{E_g}(E_y P_g + E_\tau Q_g) + \hat{\tilde{\theta}}^i \frac{K_g}{E_g}(E_y P_\theta + E_\tau Q_\theta) \\ & + \tilde{s}_0^i \frac{K_g}{E_g}(E_y P_s + E_\tau Q_s) + K_\lambda \lambda + K_\theta \hat{\tilde{\theta}}^i \\ \equiv & \tilde{\tilde{K}}_\lambda \lambda + \tilde{\tilde{K}}_g \hat{\tilde{g}}_0^i + \tilde{\tilde{K}}_\theta \hat{\tilde{\theta}}^i + \tilde{\tilde{K}} - K_g \frac{W}{\psi E_g} e^{-\psi t} \end{aligned}$$

with

$$\tilde{\tilde{K}} = \tilde{s}_0^i \frac{K_g}{E_g}(E_y P_s + E_\tau Q_s) + K_g \frac{W}{\psi E_g}.$$

Therefore, we can write the forcing term vector as

$$\tilde{B}_t^i = \begin{pmatrix} -\tilde{\kappa}_\theta \hat{\hat{\theta}}^i - \tilde{\kappa}_\lambda \lambda - \tilde{\kappa}_g \hat{g}_0^i - \tilde{\kappa} + \kappa_g \frac{W}{\psi E_g} e^{-\psi t} \\ \bar{J}_t^i \\ \bar{H}_t^i \\ \tilde{\tilde{K}}_\lambda \lambda + \tilde{\tilde{K}}_g \hat{g}_0^i + \tilde{\tilde{K}}_\theta \hat{\hat{\theta}}^i + \tilde{\tilde{K}} - K_g \frac{W}{\psi E_g} e^{-\psi t} \end{pmatrix}. \tag{314}$$

The solution is then given by

$$\tilde{X}_t^i = e^{\tilde{A}t} \left[ \tilde{X}_0^i + \int_0^t e^{-\tilde{A}s} \tilde{B}_s^i ds \right]. \tag{315}$$

Using the same steps as in previous sections, we can manipulate this solution to obtain

$$\begin{aligned} X_t^i =& e^{\tilde{A}t} \left[ \tilde{X}_0^i - (\tilde{\kappa}_\theta \hat{\hat{\theta}}^i + \tilde{\kappa}_\lambda \lambda + \tilde{\kappa}_g \hat{g}_0^i + \tilde{\kappa}) \tilde{A}^{-1} E_1 + (\tilde{\tilde{K}}_\theta \hat{\hat{\theta}}^i + \tilde{\tilde{K}}_\lambda \lambda + \tilde{\tilde{K}}_g \hat{g}_0^i + \tilde{\tilde{K}}) \tilde{A}^{-1} E_4 + (\tilde{A} + \psi I)^{-1} \hat{\hat{E}} \right] \\ &+ (\tilde{\kappa}_\theta \hat{\hat{\theta}}^i + \tilde{\kappa}_\lambda \lambda + \tilde{\kappa}_g \hat{g}_0^i + \tilde{\kappa}) \tilde{A}^{-1} E_1 - (\tilde{\tilde{K}}_\theta \hat{\hat{\theta}}^i + \tilde{\tilde{K}}_\lambda \lambda + \tilde{\tilde{K}}_g \hat{g}_0^i + \tilde{\tilde{K}}) \tilde{A}^{-1} E_4 - (\tilde{A} + \psi I)^{-1} e^{-\psi t} \hat{\hat{E}}, \end{aligned} \tag{316}$$

where

$$\hat{\hat{E}} = W_2 E_2 + W_3 E_3 - K_g \frac{W}{\psi E_g} E_4 + \kappa_g \frac{W}{\psi E_g} E_1$$

and $W_2$ and $W_3$ are defined as in previous sections.

### D.9.4 Stability

As before, we require the term in square brackets to be spanned in the stable manifold to guarantee the stability of the system. $\tilde{A}$ again has two negative eigenvalues. Let $V_j, j \in \{1,2\}$ denote the eigenvectors of $A$ associated with these negative eigenvalues, which we denote $\lambda_1$ and $\lambda_2$. The spanning condition can then be written as

$$0 = \alpha_1 V_1 + \alpha_2 V_2 \tilde{X}_0^i - (\tilde{\kappa}_\theta \hat{\hat{\theta}}^i + \tilde{\kappa}_\lambda \lambda + \tilde{\kappa}_g \hat{g}_0^i + \tilde{\kappa}) \tilde{A}^{-1} E_1 + (\tilde{\tilde{K}}_\theta \hat{\hat{\theta}}^i + \tilde{\tilde{K}}_\lambda \lambda + \tilde{\tilde{K}}_g \hat{g}_0^i + \tilde{\tilde{K}}) \tilde{A}^{-1} E_4 + (\tilde{A} + \psi I)^{-1} \hat{\hat{E}} \tag{317}$$

for some $\alpha_j \in \mathbb{C}, j \in \{1,2\}$.

The solution, including the above stability condition, is now given as a function of the free parameters $(\hat{\hat{g}}_0^i, \hat{\hat{\theta}}^i)$ and $Z = (\alpha_1, \alpha_2, \lambda, \Delta, \hat{\hat{\pi}}_{H,0}^i)$. We can pin down the latter set of free parameters by imposing the stability condition as well as the country budget constraint. The stability condition yields a system of four equations if we premultiply both sides by the projection matrices $E_i, i \in \{1,2,3,4\}$. Unlike in previous sections, we have already solved for and substituted in $\hat{\hat{y}}_0^i$ and $\hat{\hat{\tau}}_0^{i,r}$ using the two initial conditions.

We now substitute the target rule for government spending into the country budget constraint

$$
\begin{aligned}
0 =& \int_0^\infty e^{-\rho t}\left[\Gamma_y^* \hat{\tilde{y}}_t^i + \Gamma_\theta^* \hat{\tilde{\theta}}^i + \Gamma_\tau^* \hat{\tilde{\tau}}_t^{i,r} + \Gamma_g^* \hat{\tilde{g}}_t^i\right] dt \\
=& \frac{\Gamma_\theta^*}{\rho}\hat{\tilde{\theta}}^i + \frac{\Gamma_g^*}{\rho}\hat{\tilde{g}}_0^i + \frac{\Gamma_g^*}{\rho E_g}(E_y \hat{\tilde{y}}_0^i + E_\tau \hat{\tilde{\tau}}_0^{i,r}) + \Gamma_g^* \frac{W}{\rho \psi E_g} \\
&+ \int_0^\infty e^{-\rho t}\left[(\Gamma_y^* - \Gamma_g^* \frac{E_y}{E_g})\hat{\tilde{y}}_t^i + (\Gamma_\tau^* - \Gamma_g^* \frac{E_\tau}{E_g})\hat{\tilde{\tau}}_t^{i,r} - \Gamma_g^* \frac{W}{\psi E_g} e^{-\psi t}\Big)\right] dt \\
=& \frac{\Gamma_\theta^*}{\rho}\hat{\tilde{\theta}}^i + \frac{\Gamma_g^*}{\rho}\hat{\tilde{g}}_0^i + \frac{\Gamma_g^*}{\rho E_g} E_y (P_\lambda \lambda + P_g \hat{\tilde{g}}_0^i + P_\theta \hat{\tilde{\theta}}^i + P_s \tilde{s}_0^i) \\
&+ \frac{\Gamma_g^*}{\rho E_g} E_\tau (Q_\lambda \lambda + Q_g \hat{\tilde{g}}_0^i + Q_\theta \hat{\tilde{\theta}}^i + Q_s \tilde{s}_0^i) + \Gamma_g^* \frac{W}{\rho \psi E_g} - \Gamma_g^* \frac{W}{(\rho+\psi)\psi E_g} + \int_0^\infty e^{-\rho t} \tilde{\tilde{E}}' \tilde{X}_t^i dt
\end{aligned}
$$

where

$$
\tilde{\tilde{E}} = (\Gamma_y^* - \Gamma_g^* \frac{E_y}{E_g}) E_2 + (\Gamma_\tau^* - \Gamma_g^* \frac{E_\tau}{E_g}) E_3.
$$

We have

$$
\begin{aligned}
\int_0^\infty e^{-\rho t} \tilde{\tilde{E}}' \tilde{X}_t^i dt =& \frac{\alpha_1}{\lambda_1 - \rho} \tilde{\tilde{E}}' V_1 + \frac{\alpha_2}{\lambda_2 - \rho} \tilde{\tilde{E}}' V_2 + \frac{1}{\rho}(\tilde{\tilde{\kappa}}_\theta \hat{\tilde{\theta}}^i + \tilde{\tilde{\kappa}}_\lambda \lambda + \tilde{\tilde{\kappa}}_g \hat{\tilde{g}}_0^i + \tilde{\tilde{\kappa}}) \tilde{\tilde{E}}' \tilde{A}^{-1} E_1 \\
&- \frac{1}{\rho}(\tilde{\tilde{K}}_\theta \hat{\tilde{\theta}}^i + \tilde{\tilde{K}}_\lambda \lambda + \tilde{\tilde{K}}_g \hat{\tilde{g}}_0^i + \tilde{\tilde{K}}) \tilde{\tilde{E}}' \tilde{A}^{-1} E_4 - \frac{1}{\rho+\psi} \tilde{\tilde{E}}' (\tilde{A} + \psi I)^{-1} \hat{\tilde{E}}.
\end{aligned}
$$

Therefore, putting everything together and grouping terms, the country budget constraint becomes

$$
\begin{aligned}
0 =& \hat{\tilde{\theta}}^i \left(\frac{\Gamma_\theta^*}{\rho} + \frac{\Gamma_g^*}{\rho E_g}(E_y P_\theta + E_\tau Q_\theta)\right) + \Gamma_g^* \frac{W}{\rho(\rho+\psi)E_g} + \lambda \frac{\Gamma_g^*}{\rho E_g}(E_y P_\lambda + E_\tau Q_\lambda) \\
&+ \hat{\tilde{g}}_0^i \left(\frac{\Gamma_g^*}{\rho} + \frac{\Gamma_g^*}{\rho E_g}(E_y P_g + E_\tau Q_g)\right) + \tilde{s}_0^i \frac{\Gamma_g^*}{\rho E_g}(E_y P_s + E_\tau Q_s) \\
&+ \frac{\alpha_1}{\lambda_1 - \rho} \tilde{\tilde{E}}' V_1 + \frac{\alpha_2}{\lambda_2 - \rho} \tilde{\tilde{E}}' V_2 + \frac{1}{\rho}(\tilde{\tilde{\kappa}}_\theta \hat{\tilde{\theta}}^i + \tilde{\tilde{\kappa}}_\lambda \lambda + \tilde{\tilde{\kappa}}_g \hat{\tilde{g}}_0^i + \tilde{\tilde{\kappa}}) \tilde{\tilde{E}}' \tilde{A}^{-1} E_1 \\
&- \frac{1}{\rho}(\tilde{\tilde{K}}_\theta \hat{\tilde{\theta}}^i + \tilde{\tilde{K}}_\lambda \lambda + \tilde{\tilde{K}}_g \hat{\tilde{g}}_0^i + \tilde{\tilde{K}}) \tilde{\tilde{E}}' \tilde{A}^{-1} E_4 - \frac{1}{\rho+\psi} \tilde{\tilde{E}}' (\tilde{A} + \psi I)^{-1} \hat{\tilde{E}}.
\end{aligned}
\tag{318}
$$

Together, the stability condition and the country budget constraint imply a system of equations $MZ = N$, with

$$
M = \begin{pmatrix}
E_1' V_1 & E_1' V_2 & -\tilde{\tilde{\kappa}}_\lambda E_1' A^{-1} E_1 + \tilde{\tilde{K}}_\lambda E_1' A^{-1} E_4 & 0 & 1 \\
E_2' V_1 & E_2' V_2 & P_\lambda - \tilde{\tilde{\kappa}}_\lambda E_2' A^{-1} E_1 + \tilde{\tilde{K}}_\lambda E_2' A^{-1} E_4 & 0 & 0 \\
E_3' V_1 & E_3' V_2 & Q_\lambda - \tilde{\tilde{\kappa}}_\lambda E_3' A^{-1} E_1 + \tilde{\tilde{K}}_\lambda E_3' A^{-1} E_4 & 0 & 0 \\
E_4' V_1 & E_4' V_2 & -\tilde{\tilde{\kappa}}_\lambda E_4' A^{-1} E_1 + \tilde{\tilde{K}}_\lambda E_4' A^{-1} E_4 & \Lambda_y & 0 \\
\frac{\tilde{\tilde{E}}'}{\lambda_1 - \rho} V_1 & \frac{\tilde{\tilde{E}}'}{\lambda_2 - \rho} V_2 & \frac{\Gamma_g^*}{\rho E_g}(E_y P_\lambda + E_\tau Q_\lambda) + \frac{\tilde{\tilde{\kappa}}_\lambda}{\rho} \tilde{\tilde{E}}' \tilde{A}^{-1} E_1 - \frac{\tilde{\tilde{K}}_\lambda}{\rho} \tilde{\tilde{E}}' \tilde{A}^{-1} E_4 & 0 & 0
\end{pmatrix}
$$

and

$$N = \begin{pmatrix} -E_1'\tilde{\tilde{W}} \\ -E_2'\tilde{\tilde{W}} - P_g\hat{\tilde{g}}_0^i - P_\theta\hat{\tilde{\theta}}^i - P_s\tilde{s}_0^i \\ -E_3'\tilde{\tilde{W}} - Q_g\hat{\tilde{g}}_0^i - Q_\theta\hat{\tilde{\theta}}^i - Q_s\tilde{s}_0^i \\ -E_4'\tilde{\tilde{W}} \\ -\hat{\tilde{\theta}}^i\big(\frac{\Gamma_\theta^*}{\rho} + \frac{\Gamma_g^*}{\rho E_g}(E_yP_\theta + E_\tau Q_\theta)\big) - \Gamma_g^*\frac{W}{\rho(\rho+\psi)E_g} - \hat{\tilde{g}}_0^i\big(\frac{\Gamma_g^*}{\rho} + \frac{\Gamma_g^*}{\rho E_g}(E_yP_g + E_\tau Q_g)\big)\dots \\ \dots - \tilde{s}_0^i\frac{\Gamma_g^*}{\rho E_g}(E_yP_s + E_\tau Q_s) - \frac{1}{\rho}(\tilde{\kappa}_\theta\hat{\tilde{\theta}}^i + \tilde{\kappa}_g\hat{\tilde{g}}_0^i + \tilde{\kappa})\tilde{\tilde{E}}'\tilde{A}^{-1}E_1\dots \\ \dots + \frac{1}{\rho}(\tilde{\tilde{K}}_\theta\hat{\tilde{\theta}}^i + \tilde{\tilde{K}}_g\hat{\tilde{g}}_0^i + \tilde{\tilde{K}})\tilde{\tilde{E}}'\tilde{A}^{-1}E_4 + \frac{1}{\rho+\psi}\tilde{\tilde{E}}'(\tilde{A}+\psi I)^{-1}\hat{\tilde{E}} \end{pmatrix},$$

where $\tilde{\tilde{W}} = -(\tilde{\kappa}_\theta\hat{\tilde{\theta}}^i + \tilde{\kappa}_g\hat{\tilde{g}}_0^i + \tilde{\kappa})\tilde{A}^{-1}E_1 + (\tilde{\tilde{K}}_\theta\hat{\tilde{\theta}}^i + \tilde{\tilde{K}}_g\hat{\tilde{g}}_0^i + \tilde{\tilde{K}})\tilde{A}^{-1}E_4 + (\tilde{A}+\psi I)^{-1}\hat{\tilde{E}}$.

Inverting $M$, which is non-singular in all of our calibrations, we can solve for $Z$ numerically. Given $Z$, we have the full solution of the optimal allocation as a function of the two remaining free parameters $(\hat{\tilde{g}}_0^i, \hat{\tilde{\theta}}^i)$, over which we optimize numerically.

## D.10 The Special Case $\chi = 0$

We now provide a general formulation the allocation and planning problem without hand-to-mouth agents as an important special case.

We start by recording several important parameters. For $\chi = 0$, we have $\Lambda_y = \frac{1}{1-v}$, $\Lambda_\tau = 0$, $\Lambda_s = 1$, $\Lambda_g = \frac{v}{1-v}$ and $\Lambda_\theta = 1-\alpha$. Furthermore, we have $\nu_\tau = 0$, $\nu_y = \frac{1}{1-v}$, $\nu_\theta = -(1-\alpha)$, $\nu_g = -\frac{v}{1-v}$. And $\zeta_y = \frac{1-\alpha}{1-v}$, $\zeta_g = -(1-\alpha)\frac{v}{1-v}$, $\zeta_\tau = 0$, and $\zeta_\theta = 1-(1-\alpha)^2$.

### D.10.1 General Allocation for $\chi = 0$

Give these parameters, we can write the (IS) equation for $\chi = 0$ as

$$\frac{1}{1-v}\dot{\hat{\tilde{y}}}_t^i = -(\hat{\tilde{\pi}}_{H,t}^i + \dot{\tilde{s}}_t^i) + \frac{v}{1-v}\dot{\hat{\tilde{g}}}_t^i + (1-\alpha)\dot{\hat{\tilde{\theta}}}_t^i. \tag{319}$$

For the Phillips Curve, we record the following parameters: $\kappa_y = \lambda(\phi + \frac{1}{1-v})$, $\kappa_g = -\lambda\frac{v}{1-v}$, $\kappa_\tau = 0$, and $\kappa_\theta = \alpha\lambda$. Therefore, we can write the (NKPC) as

$$\dot{\hat{\tilde{\pi}}}_{H,t}^i = \rho\hat{\tilde{\pi}}_{H,t}^i - \lambda(\phi + \frac{1}{1-v})\hat{\tilde{y}}_t^i - \alpha\lambda\hat{\tilde{\theta}}_t^i + \lambda\frac{v}{1-v}\hat{\tilde{g}}_t^i \tag{320}$$

The initial condition is the given by

$$\frac{1}{1-v}\hat{\tilde{y}}_0^i = -\tilde{s}_0^i + \frac{v}{1-v}\hat{\tilde{g}}_0^i + (1-\alpha)\hat{\tilde{\theta}}_0^i. \tag{321}$$

Finally, for country $i$'s budget constraint, we note that for $\chi = 0$, $\Gamma_y = -\frac{\alpha v}{1-v}$, $\Gamma_\tau = 1$, $\Gamma_\theta = 1 - \alpha - (1-v) - v(1-\alpha)^2$, and $\Gamma_g = v + \alpha\frac{v^2}{1-v}$. We can then use the government budget constraint to substitute in for $\hat{\hat{\tau}}_t^{i,o}$, which yields that in the absence of transfers we have the following NFA condition

$$0 = -\alpha(1-v)\int_0^\infty e^{-\rho t}\hat{\hat{\theta}}_t^i dt. \tag{322}$$

### D.10.2 Loss Function and Planning Problem for $\chi = 0$

For $\chi = 0$, the loss function coefficients are as follows: $\alpha_{\tau\tau} = \alpha_{g\tau} = \alpha_{y\tau} = \alpha_{\tau\theta} = \alpha_{y\theta} = \alpha_{g\theta} = 0$. This leaves us with

$$\begin{aligned}
\alpha_\pi &= \frac{\epsilon}{2\lambda} \\
\alpha_{yy} &= -\frac{1+(1-v)\phi}{2(1-v)} \\
\alpha_{gg} &= -\frac{v}{2(1-v)} \\
\alpha_{\theta\theta} &= -\frac{1-v}{2}\left(1-(1-\alpha)^2\right) \\
\alpha_{yg} &= \frac{v}{1-v}.
\end{aligned}$$

Hence, the loss function can be written as

$$\begin{aligned}
\mathbb{L} = \int_0^\infty \int_0^1 e^{-\rho t}\Bigg[&\frac{\epsilon}{2\lambda}(\hat{\hat{\pi}}_{H,t}^i)^2 + \frac{1+(1-v)\phi}{2(1-v)}(\hat{\hat{y}}_t^i)^2 + \frac{1-v}{2}\left(1-(1-\alpha)^2\right)(\hat{\hat{\theta}}_t^i)^2 \\
&+ \frac{v}{2(1-v)}(\hat{\hat{g}}_t^i)^2 - \frac{v}{1-v}\hat{\hat{y}}_t^i\hat{\hat{g}}_t^i\Bigg] di dt.
\end{aligned} \tag{323}$$

where of course $a_t^i = e^{-\psi t}a_0^i$ is the initial productivity shock.

### D.10.3 Planning Problem for Transfers when $\chi = 0$

When the social planner can only use transfers, we have $\hat{\hat{g}}_t^i = 0$ and $\hat{\hat{\theta}}_t^i = \hat{\hat{\theta}}^i$. Therefore, we can write the planning problem as

$$\min_{\hat{\hat{\theta}}^i} \int_0^\infty \int_0^1 e^{-\rho t}\left[\frac{\epsilon}{2\lambda}(\hat{\hat{\pi}}_{H,t}^i)^2 + \frac{1+(1-v)\phi}{2(1-v)}(\hat{\hat{y}}_t^i)^2 + \frac{1-v}{2}\left(1-(1-\alpha)^2\right)(\hat{\hat{\theta}}^i)^2\right] di dt,$$

subject to

$$\begin{aligned}
\frac{1}{1-v}\dot{\hat{\hat{y}}}_t^i &= -(\hat{\hat{\pi}}_{H,t}^i + \hat{\hat{s}}_t^i) \\
\dot{\hat{\hat{\pi}}}_{H,t}^i &= \rho\hat{\hat{\pi}}_{H,t}^i - \lambda(\phi + \frac{1}{1-v})\hat{\hat{y}}_t^i - \alpha\lambda\hat{\hat{\theta}}^i
\end{aligned}$$

$$\frac{1}{1-v}\hat{\tilde{y}}_0^i = -\tilde{s}_0^i + (1-\alpha)\hat{\tilde{\theta}}^i.$$

### D.10.4 Planning Problem for Capital Controls when $\chi = 0$

When the social planner can only use transfers, we have $\hat{\tilde{g}}_t^i = 0$ and $\hat{\tilde{\tau}}_t^{i,r} = v\alpha\hat{\tilde{s}}_t^i$. Therefore, we can write the planning problem as

$$\min_{\{\hat{\tilde{\theta}}_t^i\}_{t\geq 0}} \int_0^\infty \int_0^1 e^{-\rho t}\left[\frac{\epsilon}{2\lambda}(\hat{\tilde{\pi}}_{H,t}^i)^2 + \frac{1+(1-v)\phi}{2(1-v)}(\hat{\tilde{y}}_t^i)^2 + \frac{1-v}{2}\big(1-(1-\alpha)^2\big)(\hat{\tilde{\theta}}_t^i)^2\right] didt,$$

subject to

$$\begin{aligned}
\frac{1}{1-v}\dot{\hat{\tilde{y}}}_t^i &= -(\hat{\tilde{\pi}}_{H,t}^i + \dot{\tilde{s}}_t^i) + (1-\alpha)\dot{\hat{\tilde{\theta}}}_t^i \\
\dot{\hat{\tilde{\pi}}}_{H,t}^i &= \rho\hat{\tilde{\pi}}_{H,t}^i - \lambda(\phi + \frac{1}{1-v})\hat{\tilde{y}}_t^i - \alpha\lambda\hat{\tilde{\theta}}_t^i \\
\frac{1}{1-v}\hat{\tilde{y}}_0^i &= -\tilde{s}_0^i + (1-\alpha)\hat{\tilde{\theta}}_0^i \\
0 &= -\alpha(1-v)\int_0^\infty e^{-\rho t}\hat{\tilde{\theta}}_t^i dt.
\end{aligned}$$

### D.10.5 Planning Problem for Government Spending when $\chi = 0$

When the social planner can only use government spending, we have $\hat{\tilde{\theta}}_t^i = 0$ and $\hat{\tilde{\tau}}_t^{i,r} = v\alpha\hat{\tilde{s}}_t^i$. Therefore, we can write the planning problem as

$$\min_{\{\hat{\tilde{\theta}}_t^i\}_{t\geq 0}} \int_0^\infty \int_0^1 e^{-\rho t}\left[\frac{\epsilon}{2\lambda}(\hat{\tilde{\pi}}_{H,t}^i)^2 + \frac{1+(1-v)\phi}{2(1-v)}(\hat{\tilde{y}}_t^i)^2 + \frac{v}{2(1-v)}(\hat{\tilde{g}}_t^i)^2 - \frac{v}{1-v}\hat{\tilde{y}}_t^i\hat{\tilde{g}}_t^i\right] didt,$$

subject to

$$\begin{aligned}
\frac{1}{1-v}\dot{\hat{\tilde{y}}}_t^i &= -(\hat{\tilde{\pi}}_{H,t}^i + \dot{\tilde{s}}_t^i) + \frac{v}{1-v}\dot{\hat{\tilde{g}}}_t^i \\
\dot{\hat{\tilde{\pi}}}_{H,t}^i &= \rho\hat{\tilde{\pi}}_{H,t}^i - \lambda(\phi + \frac{1}{1-v})\hat{\tilde{y}}_t^i + \lambda\frac{v}{1-v}\hat{\tilde{g}}_t^i \\
\frac{1}{1-v}\hat{\tilde{y}}_0^i &= -\tilde{s}_0^i + \frac{v}{1-v}\hat{\tilde{g}}_0^i.
\end{aligned}$$