Dynamic noisy signalling: Online appendix

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1 Uniqueness in one-shot signalling

The proof of uniqueness in a larger class of strategies closely follows McLennan et al. (2014) Theorem 1.2, which proves uniqueness in the one-shot Kyle (1985) model. Complex-variable functions are used, so more definitions are needed. A *real entire* function is smooth and coincides on $\mathbb{R}$ with its Taylor series centered at zero. A *region* is an open connected set $D \subseteq \mathbb{C}$. A function is *analytic* on D if it is complex-differentiable at every point in D. An *entire* function is analytic on $\mathbb{C}$. An analytic function is *single-valued* if it has an unambiguously defined maximal analytic continuation. If a real-valued function coincides on $\mathbb{C}$ with its Taylor series centered at zero and is smooth, then it is single-valued.

Proposition 1. *If $c(\cdot) = \exp(\cdot)$ or $(c')^{-1}$ is entire, then in the one-shot signalling game there is only one equilibrium that on some nonempty $(x_1, x_2) \subset \mathbb{R}$ coincides with a function that is single-valued on (x_1, x_2).*

Proof. Denote the mean of the market's posterior belief after signal s by $\mu_{\theta 1}(s)$. It depends on e^*. For any strategy e^* that the market expects, the posterior

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mean $\mathbb{E}[\mu_{\theta 1}(s)|e]$ that the sender expects is entire as a function of effort. This is proved in McLennan et al. (2014) Theorems 2.1 and 2.2.

Rearrange the FOC $\frac{\partial \mathbb{E}[\mu_{\theta 1}(s)|e]}{\partial e}|_{e=e^*(\theta)} - c'(e^*(\theta) - \alpha\theta) = 0$ as

$$\frac{1}{\alpha}\left[e^*(\theta) - (c')^{-1}\left(\frac{\partial \mathbb{E}[\mu_{\theta 1}(s)|e]}{\partial e}|_{e=e^*(\theta)}\right)\right] = \theta. \tag{1}$$

The left-hand side (LHS) is entire in $e^*(\theta)$ if $(c')^{-1}\left(\frac{\partial \mathbb{E}[\mu_{\theta 1}(s)|e]}{\partial e}|_{e=e^*(\theta)}\right)$ is. One sufficient condition for this is that $(c')^{-1}$ is entire, because the composition of entire functions is entire and the derivative of an entire function is entire. Another sufficient condition is that $c(\cdot) = \exp(\cdot)$ and $\frac{\partial \mathbb{E}[\mu_{\theta 1}(s)|e]}{\partial e}|_{e=e^*(\theta)} \neq 0$, because the logarithm of a nowhere zero entire function is entire. (The logarithm function itself is not entire.)

To prove $\frac{\partial \mathbb{E}[\mu_{\theta 1}(s)|e]}{\partial e}|_{e=e^*(\theta)} > 0 \ \forall\theta$ in any equilibrium, use the incentive constraints (ICs) that an equilibrium e^* must satisfy. Take any e^*, arbitrary θ_1, θ_2 and the effort levels $e_i = e^*(\theta_i)$, $i = 1,2$ that these types choose in equilibrium. The ICs are

$$\mathbb{E}[\mu_{\theta 1}(s)|e_i] - c(e_i - \theta_i) \geq \mathbb{E}[\mu_{\theta 1}(s)|e_j] - c(e_j - \theta_i), \quad i \neq j.$$

Adding the ICs,

$$c(e_1 - \theta_1) + c(e_2 - \theta_2) \leq c(e_2 - \theta_1) + c(e_1 - \theta_2).$$

If $\theta_1 < \theta_2$ and $e_1 \geq e_2$, then

$$e_1 - \theta_1 > \max\{e_1 - \theta_2, e_2 - \theta_1\} \geq \min\{e_1 - \theta_2, e_2 - \theta_1\} > e_2 - \theta_2,$$

so $\{e_1 - \theta_1, e_2 - \theta_2\}$ is a mean-preserving spread of $\{e_1 - \theta_2, e_2 - \theta_1\}$. The mean is $\frac{1}{2}(e_1 + e_2 - \theta_1 - \theta_2)$. Applying a strictly convex $c(\cdot)$ to a mean-preserving spread, one gets $c(e_1 - \theta_1) + c(e_2 - \theta_2) > c(e_2 - \theta_1) + c(e_1 - \theta_2)$, a contradiction. So $\theta_1 < \theta_2 \Rightarrow e_1 < e_2$. The strict monotone likelihood ratio property (MLRP) of normal distributions yields $e_1 < e_2 \Rightarrow \mathbb{E}[\mu_{\theta 1}(s)|e_1] < \mathbb{E}[\mu_{\theta 1}(s)|e_2]$. Smoothness of $\mathbb{E}[\mu_{\theta 1}(s)|e]$ implies that $\frac{\partial \mathbb{E}[\mu_{\theta 1}(s)|e]}{\partial e}|_{e=e^*(\theta)} > 0$.

Proposition 3.2 of McLennan et al. (2014) applies unchanged to (1), proving that if e^* coincides on some (x_1, x_2) with a function that is single-valued on (x_1, x_2), then e^* is affine. There is only one affine equilibrium, as shown above.

$\square$

Unfortunately the uniqueness proof of Proposition 1 does not extend to multiple periods, because if the initial period effort is nonlinear in type, then the starting belief in subsequent periods is not normal. With a non-normal belief at the start of the final period, the results of McLennan et al. (2014) cannot be used. Without knowing the solution in the final period, backward induction cannot start.

2 Additional comparative statics

In the baseline model (main paper section Equilibrium effort and payoff), the expected payoff in equilibrium is

$$b\frac{\tau_\theta \mu_\theta + \alpha^2 \tau_\epsilon \theta}{\tau_\theta + \alpha^2 \tau_\epsilon} - c\left((c')^{-1}\left(\frac{b\alpha\tau_\epsilon}{\tau_\theta + \alpha^2 \tau_\epsilon}\right)\right). \tag{2}$$

This is obtained by substituting the equilibrium strategy into the payoff, which is defined in the main paper.

Focus on the benefit side of (2). In equilibrium, the variance of the benefit across types at time t is

$$\mathbb{E}_\theta \left(\frac{\tau_\theta \mu_\theta + \sum_{j=1}^{t} \alpha_j^2 \tau_{\epsilon j} \theta}{\tau_\theta + \sum_{j=1}^{t} \alpha_j^2 \tau_{\epsilon j}} - \frac{\tau_\theta \mu_\theta + \sum_{j=1}^{t} \alpha_j^2 \tau_{\epsilon j} \mu_\theta}{\tau_\theta + \sum_{j=1}^{t} \alpha_j^2 \tau_{\epsilon j}}\right)^2$$
$$= \mathbb{E}_\theta \left[\left(\frac{\sum_{j=1}^{t} \alpha_j^2 \tau_{\epsilon j} (\theta - \mu_\theta)}{\tau_\theta + \sum_{j=1}^{t} \alpha_j^2 \tau_{\epsilon j}}\right)^2\right] = \frac{b_t^2}{\tau_\theta} \left(\frac{\sum_{j=1}^{t} \alpha_j^2 \tau_{\epsilon j}}{\tau_\theta + \sum_{j=1}^{t} \alpha_j^2 \tau_{\epsilon j}}\right)^2,$$

which rises over time (when b_t is constant) due to the market learning the type. This rise mirrors the increase in income inequality with age among the employed (Badel and Huggett, 2014). The variance of the benefit rises faster when the

signals are more informative ($\tau_{\epsilon t}$ larger for all t). For example, if computers enable easier verification of credentials, their introduction should lead to more inequality and more education effort. If α_j is constant, then the variance of effort across types is constant at $\frac{1}{\alpha^2 \tau_\theta}$ over time. This matches Badel and Huggett (2014): the variance of the number of hours worked per week does not change with age.

3 Derivations for the extensions

3.1 Exogenously changing type

The type at the end of period t is $\theta_t = \theta_{t-1} + \nu_t$, with $\nu_t \sim N(0, \tau_\nu)$ i.i.d. The distribution of θ_0 is $N(\mu_{\theta,0}, \tau_{\theta,0})$.

Belief remains normal under Bayesian updating when the effort expected by the market is linear in the type. With a linear expected strategy $e_t^*(\theta_{t-1}) = k + \alpha_t \theta_{t-1} \ \forall t$, the precision of the belief is updated deterministically and independently of type, realized signal or chosen effort (but depending on the expected effort) by the formula

$$\tau_{\theta,t} = \frac{1}{\frac{1}{\tau_{\theta,t-1}+\alpha_t^2\tau_{\epsilon t}} + \frac{1}{\tau_\nu}} = \frac{\tau_\nu\left(\tau_{\theta,t-1} + \alpha_t^2\tau_{\epsilon t}\right)}{\tau_\nu + \tau_{\theta,t-1} + \alpha_t^2\tau_{\epsilon t}}.$$

The marginal benefit of signalling in period t can be decomposed into the part m_t received in the current period and the part obtained in the future. The latter is the product of three components. The first is $\frac{\alpha_t \tau_{\epsilon t}}{\tau_{\theta,t-1}+\alpha_t^2\tau_{\epsilon t}}$, the influence of the sender's current effort on the mean of the market's belief at the end of the current period. The second is $\prod_{k=1}^{n-1} \frac{\tau_{\theta,t+k-1}}{\tau_{\theta,t+k-1}+\alpha_{t+k}^2\tau_{\epsilon,t+k}}$, the effect of the mean of the belief at the end of period t on the mean of the belief at the end of period $t + n - 1$ (this component is absent for $n = 1$). The third is $\frac{b_{t+n}\tau_{\theta,t+n-1}}{\tau_{\theta,t+n-1}+\alpha_{t+n}^2\tau_{\epsilon,t+n}}$, the change in period $t + n$ benefit in response to a change in the mean of the belief at the end of period $t + n - 1$. The total future marginal benefit of effort

is then

$$MB_t^{fut} = \frac{\alpha_t \tau_{\epsilon t}}{\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t}} \sum_{n=1}^{T} b_{t+n} \prod_{k=1}^{n} \frac{\tau_{\theta,t+k-1}}{\tau_{\theta,t+k-1} + \alpha_{t+k}^2 \tau_{\epsilon,t+k}}.$$

The total marginal benefit of effort adds the marginal flow benefit m_t to MB_t^{fut}. Conjecture that the marginal flow benefit is constant in effort and type. Define $\gamma_{T-t} = (c_t')^{-1} \left(m_t + MB_t^{fut} \right)$. From equating the marginal benefit and the marginal cost, the sender's best response is $e_t(\theta_{t-1}) = \gamma_{T-t} + \alpha_t \theta_{t-1}$.

The updated distribution of effort from the market's viewpoint conditional on signal s_t is

$$e_t | s_t \sim N \left(\frac{\tau_{\theta,t-1} \left[\gamma_{T-t} + \alpha_t \mu_{\theta,t-1} \right] + \alpha_t^2 \tau_{\epsilon t} s_t}{\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t}}, \quad \frac{\tau_{\theta,t-1}}{\alpha_t^2} + \tau_{\epsilon t} \right).$$

Using $\theta_{t-1} = \frac{1}{\alpha_t} e_t - \frac{\gamma_{T-t}}{\alpha_t}$, the updated distribution of θ_{t-1} is

$$\theta_{t-1} | s_t \sim N \left(\frac{\tau_{\theta,t-1} \mu_{\theta,t-1} + \alpha_t \tau_{\epsilon t} s_t - \gamma_{T-t} \alpha_t \tau_{\epsilon t}}{\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t}}, \quad \tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t} \right).$$

From the viewpoint of the market at the end of period t (after receiving s_t), the mean of θ_t is the same as for θ_{t-1}, so the derivative of the mean of the belief after period t w.r.t. the mean before t is $\frac{\tau_{\theta,t-1}}{\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t}}$. This is used in the derivation of MB_t^{fut} above.

The expected flow benefit in period t from effort $e_t(\theta_{t-1})$ is

$$b_t \left[\frac{\tau_{\theta,t-1} \left(\gamma_{T-t} + \alpha_t \mu_{\theta,t-1} \right) + \alpha_t^2 \tau_{\epsilon t} e_t(\theta)}{\alpha_t (\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t})} - \frac{\gamma_{T-t}}{\alpha_t} \right].$$

The marginal flow benefit is $m_t = \frac{b_t \alpha_t \tau_{\epsilon t}}{\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t}}$. The total marginal benefit is

$$m_t + MB_t^{fut} = \frac{\alpha_t \tau_{\epsilon t}}{\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t}} \sum_{n=0}^{T} b_{t+n} \prod_{k=1}^{n} \frac{\tau_{\theta,t+k-1}}{\tau_{\theta,t+k-1} + \alpha_{t+k}^2 \tau_{\epsilon,t+k}},$$

which is constant in effort and type, as conjectured. The optimal effort is

$$e_t^*(\theta_{t-1}) = (c_t')^{-1} \left(\frac{\alpha_t \tau_{\epsilon t}}{\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t}} \sum_{n=0}^{T} b_{t+n} \prod_{k=1}^{n} \frac{\tau_{\theta,t+k-1}}{\tau_{\theta,t+k-1} + \alpha_{t+k}^2 \tau_{\epsilon,t+k}} \right) + \alpha_t \theta_{t-1}.$$

3.2 Human capital accumulation

The type in period t is $\theta_t = \theta_{t-1} + h_t e_t$, with $h_t > 0$ and $\sum_{k=0}^{T} h_k < \infty$.

Conjecture that equilibrium effort is affine in type: $e_t(\theta_{t-1}) = \gamma_{T-t} + \alpha_t \theta_{t-1}$. Given that the prior on θ_{t-1} is normal and that in equilibrium, type evolves according to $\theta_t = (1 + \alpha_t h_t)\theta_{t-1} + h_t \gamma_{T-t}$, the belief about θ_t *before* receiving signal s_t is

$$\theta_t | s_{t-1} \sim N\left((1 + \alpha_t h_t)\mu_{\theta,t-1} + h_t \gamma_{T-t}, \quad \frac{\tau_{\theta,t-1}}{(1+\alpha_t h_t)^2} \right).$$

Expressing $\theta_{t-1} = \frac{\theta_t - h_t \gamma_{T-t}}{1 + \alpha_t h_t}$, effort can be written as $e_t = \gamma_{T-t} + \frac{\alpha_t(\theta_t - h_t \gamma_{T-t})}{1 + \alpha_t h_t}$, so the signal distribution conditional on θ_t is normal with precision $\tau_{\epsilon t}$ and mean $\gamma_{T-t} + \frac{\alpha_t(\theta_t - h_t \gamma_{T-t})}{1 + \alpha_t h_t} = \frac{\alpha_t \theta_t}{1 + \alpha_t h_t} + \frac{\gamma_{T-t}}{1 + \alpha_t h_t}$. Linearly transform the signal s_t to $z_t = \frac{(1 + \alpha_t h_t)s_t - \gamma_{T-t}}{\alpha_t}$, which has mean θ_t and precision $\frac{\alpha_t^2 \tau_{\epsilon t}}{(1+\alpha_t h_t)^2}$. Then the updated type at the end of period t is

$$\theta_t | z_t \sim N\left(\frac{\tau_{\theta,t-1}[(1 + \alpha_t h_t)\mu_{\theta,t-1} + h_t \gamma_{T-t}] + \alpha_t^2 \tau_{\epsilon t} z_t}{\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t}}, \quad \frac{\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t}}{(1+\alpha_t h_t)^2} \right).$$

The precision of the belief at the end of period t is $\tau_{\theta,t} = \frac{\tau_{\theta,0}}{\prod_{n=1}^{t}(1+\alpha_n h_n)^2} + \sum_{n=1}^{t} \frac{\alpha_n^2 \tau_{\epsilon n}}{\prod_{k=n}^{t}(1+\alpha_k h_k)^2}$, smaller than without human capital accumulation. Since higher types accumulate human capital faster, the type distribution becomes more dispersed over time. This counteracts learning by the market and may even make the precision of the posterior belief decrease in time.

Given effort e_t, the expected benefit in period t is

$$b_t \frac{\tau_{\theta,t-1}[(1 + \alpha_t h_t)\mu_{\theta,t-1} + h_t \gamma_{T-t}] + \alpha_t \tau_{\epsilon t}[(1 + \alpha_t h_t)e_t - \gamma_{T-t}]}{\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t}},$$

so the marginal flow benefit of effort is $\frac{b_t \alpha_t \tau_{\epsilon t}(1+\alpha_t h_t)}{\tau_{\theta,t-1}+\alpha_t^2 \tau_{\epsilon t}}$. The marginal benefit at time t of shifting μ_{t-1} is $\frac{b_t \tau_{\theta,t-1}(1+\alpha_t h_t)}{\tau_{\theta,t-1}+\alpha_t^2 \tau_{\epsilon t}}$. The response of μ_{t+n-1} ($n \geq 2$) to μ_t is

$$\frac{\partial \mu_{t+n-1}}{\partial \mu_t} = \frac{\tau_{\theta,t}}{\tau_{\theta,t+n}} \prod_{k=0}^{n-1} \frac{1}{1 + \alpha_{t+k+1} h_{t+k+1}}.$$

The effect of e_t on μ_t is $\frac{\alpha_t \tau_{\epsilon t}(1+\alpha_t h_t)}{\tau_{\theta,t-1}+\alpha_t^2 \tau_{\epsilon t}}$. The equilibrium effort is $e_t^*(\theta_{t-1}) = (c_t')^{-1}(m_t) + \alpha_t \theta_{t-1}$, where m_t is the total marginal benefit of effort

$$\alpha_t \tau_{\epsilon t} \sum_{n=0}^{T-t} \frac{b_{t+n}}{\frac{\tau_{\theta,0}}{\prod_{j=1}^{t-1}(1+\alpha_j h_j)^2 \prod_{j=t}^{t+n}(1+\alpha_j h_j)} + \sum_{j=1}^{t+n} \frac{\alpha_j^2 \tau_{\epsilon j}}{\prod_{k=j}^{t-1}(1+\alpha_k h_k)^2 \prod_{k=t}^{t+n}(1+\alpha_k h_k)}}. \quad (3)$$

3.3 Exogenous information revelation

An exogenous signal $x_t = \theta + \xi_t$ is added to the baseline model, with $\xi_t \sim N(0, \tau_\xi)$ i.i.d. Assume the return is obtained at the end of each period after both s_t and x_t have been observed.

Denote by $\mu_{\theta,t}^s$ and $\tau_{\theta,t}^s$ the mean and precision of the belief after observing s_t, but before x_t. The mean and precision after observing x_t are written $\mu_{\theta,t}, \tau_{\theta,t}$. The belief at the end of period t is

$$N\left(\frac{\mu_{\theta,t}^s \tau_{\theta,t}^s + x_t \tau_\xi}{\tau_{\theta,t}^s + \tau_\xi}, \quad \tau_{\theta,t}^s + \tau_\xi \right)$$
$$= N\left(\frac{\tau_{\theta,t-1}\mu_{\theta,t-1} + \alpha_t \tau_{\epsilon t}(s_t - \gamma_{T-t}) + x_t \tau_\xi}{\tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t} + \tau_\xi}, \quad \tau_{\theta,t-1} + \alpha_t^2 \tau_{\epsilon t} + \tau_\xi \right).$$

The derivative of the expected flow benefit w.r.t. effort is $\frac{b_t \alpha \tau_{\epsilon t}}{\tau_{\theta,t}}$, consisting of the derivative of the benefit w.r.t. the mean of the belief and the derivative of the mean w.r.t. the effort. The derivative of the mean of the belief at the end of period t w.r.t. the mean at the start of the period is $\frac{\tau_{\theta,t-1}}{\tau_{\theta,t}}$. The total expected marginal benefit is

$$\frac{\alpha \tau_{\epsilon t}}{\tau_{\theta,t}} \sum_{k=0}^{T-t} b_{t+k} \prod_{i=1}^{k} \frac{\tau_{\theta,t-1+i}}{\tau_{\theta,t+i}} = \alpha \tau_{\epsilon t} \sum_{k=0}^{T-t} b_{t+k} \frac{\tau_{\theta,t}}{\tau_{\theta,t+k}}$$

The optimal effort is

$$e_t^*(\theta) = (c_t')^{-1}\left(\alpha_t \tau_{\epsilon t} \sum_{n=t}^{T} \frac{b_n}{\tau_{\theta,t-1} + (n-t+1)\tau_\xi + \sum_{k=t}^{n} \tau_{\epsilon k}\alpha_k^2} \right) + \alpha_t \theta.$$

The derivative of effort w.r.t. τ_ξ is

$$\left[c_t''\left((c_t')^{-1}\left(\alpha_t \tau_{\epsilon t} \sum_{n=t}^{T} \frac{b_n}{\tau_{\theta,n}} \right) \right) \right]^{-1} \alpha_t \tau_{\epsilon t} \sum_{n=t}^{T} \frac{-(n-t+1)b_n}{(\tau_{\theta,t-1} + (n-t+1)\tau_\xi + \sum_{k=t}^{n} \tau_{\epsilon k}\alpha_k^2)^2}$$

3.3.1 One signal depending on both type and effort

The signal is a weighted sum of the type and the effort: $s_t = r\theta + e_t + \epsilon_t$, with $r > 0$. If an affine strategy $e_t^*(\theta) = k_{1t} + k_{2t}\theta$ is expected, then the belief updating formula is

$$\theta|s_t \sim N\left(\frac{\tau_{\theta,t-1}\mu_{\theta,t-1} + (r+k_{2t})\tau_{\epsilon t}(s_t - k_{1t})}{\tau_{\theta,t-1} + (r+k_{2t})^2\tau_{\epsilon t}}, \quad \tau_{\theta,t-1} + (r+k_{2t})^2\tau_{\epsilon t}\right).$$

The marginal flow benefit is therefore $\frac{b_t(r+k_{2t})\tau_{\epsilon t}}{\tau_{\theta,t-1}+(r+k_{2t})^2\tau_{\epsilon t}}$. The marginal future benefit is $\frac{(r+k_{2t})\tau_{\epsilon t}}{\tau_{\theta,t-1}+(r+k_{2t})^2\tau_{\epsilon t}} \sum_{n=t+1}^{T} \frac{b_n\tau_{\theta,t}}{\tau_{\theta,n}}$. The total marginal benefit is constant.

The equilibrium effort is

$$e_t^*(\theta) = (c_t')^{-1}\left((r+\alpha_t)\tau_{\epsilon t}\sum_{n=t}^{T}\frac{b_n}{\tau_{\theta,t-1}+\sum_{k=t}^{n}\tau_{\epsilon k}(r+\alpha_k)^2}\right) + \alpha_t\theta.$$

The comparative statics in r are given by

$$\frac{\partial e_t^*(\theta)}{\partial r} = \left[c_t''\left((c_t')^{-1}\left((r+\alpha_t)\tau_{\epsilon t}\sum_{j=t}^{T}\frac{b_j}{\tau_{\theta,j}}\right)\right)\right]^{-1}$$
$$\times\left(\sum_{n=t}^{T}\frac{b_n\tau_{\epsilon t}}{\tau_{\theta,n}} - (r+\alpha_t)\tau_{\epsilon t}\sum_{n=t}^{T}\frac{2b_n\sum_{k=t}^{n}\tau_{\epsilon k}(r+\alpha_k)}{\tau_{\theta,n}^2}\right).$$

3.3.2 Signal is a convex combination of type and effort

Assume the signal depends on type according to $s_t = \rho\theta + (1-\rho)e_t + \epsilon_t$, with $\rho \in (0,1)$. If a linear strategy $e_t^*(\theta) = k_{1t} + k_{2t}\theta$ is expected, with $k_{2t} > 0$ $\forall t$, then the updating formula is

$$\theta|s_t \sim N\left(\frac{\tau_{\theta,t-1}\mu_{\theta,t-1} + (\rho+(1-\rho)k_{2t})\tau_{\epsilon t}(s_t - (1-\rho)k_{1t})}{\tau_{\theta,t-1} + (\rho+(1-\rho)k_{2t})^2\tau_{\epsilon t}}, \quad \tau_{\theta,t-1} + (\rho+(1-\rho)k_{2t})^2\tau_{\epsilon t}\right).$$

The marginal flow benefit is therefore $\frac{b_t(\rho+(1-\rho)k_{2t})\tau_{\epsilon t}}{\tau_{\theta,t-1}+(\rho+(1-\rho)k_{2t})^2\tau_{\epsilon t}}$. The marginal future benefit is $\frac{(\rho+(1-\rho)k_{2t})\tau_{\epsilon t}}{\tau_{\theta,t-1}+(\rho+(1-\rho)k_{2t})^2\tau_{\epsilon t}} \sum_{n=t+1}^{T} \frac{b_n\tau_{\theta,t}}{\tau_{\theta,n}}$. The total marginal benefit is constant.

Equilibrium effort is

$$e_t^*(\theta) = (c_t')^{-1}\left((\alpha_t + \rho(1-\alpha_t))\tau_{\epsilon t}\sum_{n=t}^{T}\frac{b_n}{\tau_{\theta,t-1}+\sum_{k=t}^{n}\tau_{\epsilon k}(\alpha_k+\rho(1-\alpha_k))^2}\right) + \alpha_t\theta.$$

The comparative statics in ρ are given by

$$\frac{\partial e_t^*(\theta)}{\partial \rho} = \left[c_t'' \left((c_t')^{-1} \left((\alpha_t + \rho(1-\alpha_t)) \tau_\epsilon \sum_{j=t}^{T} \frac{b_j}{\tau_{\theta,j}} \right) \right) \right]^{-1}$$

$$\times \left((1-\alpha_t) \sum_{n=t}^{T} \frac{b_n \tau_{\epsilon n}}{\tau_{\theta,n}} - (\alpha_t + \rho(1-\alpha_t)) \sum_{n=t}^{T} \frac{2 b_n \tau_\epsilon^2 \sum_{k=t}^{n} (\alpha_k + \rho(1-\alpha_k)) (1-\alpha_k)}{\tau_{\theta,n}^2} \right).$$

Effort may increase or decrease in ρ.

3.4 Productive effort

The value of type θ and effort e_t to an employer is $q_t\theta + (1-q_t)e_t$ in any period t, where $q_t \in [0,1]$ is the relative weight of the type in the productivity. The signal is still $s_t = e_t + \epsilon_t$. Competition between risk-neutral employers will drive the wage of the worker to $w = \mathbb{E}\left[q_t\theta + (1-q_t)e_t^*(\theta)\right]$. The worker is risk-neutral and gets utility $b_t w$ from w.

Conjecture an affine strategy $e_t^*(\theta) = k_{1t} + k_{2t}\theta$, with $k_{2t} > 0 \ \forall t$. Updating based on the signal $s_t = e_t + \epsilon_t$ yields

$$\theta | s_t \sim N \left(\frac{\tau_{\theta,t-1}\mu_{\theta,t-1} + k_{2t}\tau_{\epsilon t}(s_t - k_{1t})}{\tau_{\theta,t-1} + k_{2t}^2 \tau_{\epsilon t}}, \quad \tau_{\theta,t-1} + k_{2t}^2 \tau_{\epsilon t} \right).$$

The expected wage given effort e is then

$$(1-q_t)k_{1t} + (q_t + (1-q_t)k_{2t}) \frac{\tau_{\theta,t-1}\mu_{\theta,t-1} + k_{2t}\tau_{\epsilon t}(e - k_{1t})}{\tau_{\theta,t-1} + k_{2t}^2 \tau_{\epsilon t}},$$

because the expected signal equals the effort. The marginal flow benefit of effort is $\frac{b_t(q_t + (1-q_t)k_{2t})k_{2t}\tau_{\epsilon t}}{\tau_{\theta,t-1} + k_{2t}^2 \tau_{\epsilon t}}$. Current effort shifts subsequent beliefs as well and this provides the marginal future benefit $\frac{k_{2t}\tau_{\epsilon t}}{\tau_{\theta,t-1} + k_{2t}^2 \tau_{\epsilon t}} \sum_{n=t+1}^{T} \frac{b_n(q_n + (1-q_n)k_{2n})\tau_{\theta,t}}{\tau_{\theta,n}}$. The total marginal benefit is constant, so the equilibrium effort is

$$e_t^*(\theta) = (c_t')^{-1} \left(\alpha_t \tau_{\epsilon t} \sum_{n=t}^{T} \frac{b_n(q_n + (1-q_n)\alpha_n)}{\tau_{\theta,t-1} + \sum_{k=t}^{n} \tau_{\epsilon k}\alpha_k^2} \right) + \alpha_t\theta,$$

affine in type as required, with $k_{2t} = \alpha_t \ \forall t$.

The comparative statics in q_t are discussed in the main paper. In the following, D_t is defined as in the main paper, and D_t^q is D_t modified to include

the $q_n + (1 - q_n)\alpha_n$ terms. The comparative statics in α_t are given by

$$\frac{\partial e_t^*(\theta)}{\partial \alpha_t} = \theta + D_t^q \tau_{\epsilon t} \sum_{n=t}^{T} \frac{b_n(q_n + (1 - q_n)\alpha_n)}{\tau_{\theta,t-1} + \sum_{k=t}^{n} \tau_{\epsilon k}\alpha_k^2}$$

$$+ D_t^q \alpha_t \tau_{\epsilon t} \sum_{n=t}^{T} \frac{\mathbf{1}_{n=t} b_n(1 - q_n)(\tau_{\theta,t-1} + \sum_{k=t}^{n} \tau_{\epsilon k}\alpha_k^2) - 2b_n(q_n + (1 - q_n)\alpha_n)\tau_{\epsilon t}\alpha_t}{(\tau_{\theta,t-1} + \sum_{k=t}^{n} \tau_{\epsilon k}\alpha_k^2)^2}.$$

The function $\mathbf{1}_{n=t}$ equals 1 if $n = t$ and zero otherwise. The effect of α_t on effort can be positive or negative, as in the baseline model.

For $j > t$,

$$\frac{\partial e_t^*(\theta)}{\partial \alpha_j} = D_t^q \alpha_t \tau_{\epsilon t} \sum_{n=j}^{T} \frac{\mathbf{1}_{n=j} b_n(1 - q_n)(\tau_{\theta,t-1} + \sum_{k=t}^{n} \tau_{\epsilon k}\alpha_k^2) - 2b_n(q_n + (1 - q_n)\alpha_n)\tau_{\epsilon j}\alpha_j}{(\tau_{\theta,t-1} + \sum_{k=t}^{n} \tau_{\epsilon k}\alpha_k^2)^2}.$$

The effect of α_j, $j > t$ on effort at time t can be positive or negative, unlike in the baseline model.

3.5 Multiple signallers

The cost $c\left(e_{it} - \alpha\theta_i - \sum_{k=1}^{t-1} \kappa_{ikt} s_{jk}\right)$ of sender i is influenced by the past signals s_{jk} of sender $j \neq i$. The influence κ_{ikt} of the other sender's success in period k on i's cost in period t may be positive or negative. The distribution of the noise is normal, i.i.d. across the senders.

As usual, conjecture a linear strategy $e_{it}^*(\theta) = k_{i1t} + k_{i2t}\theta$, with $k_{i2t} > 0 \ \forall t$. Updating based on the signal $s_{it} = e_{it} + \epsilon_{it}$ yields

$$\theta_i | s_{it} \sim N\left(\frac{\tau_{i\theta,t-1}\mu_{i\theta,t-1} + k_{i2t}\tau_{\epsilon t}(s_{it} - k_{i1t})}{\tau_{i\theta,t-1} + k_{i2t}^2\tau_{\epsilon t}}, \quad \tau_{i\theta,t-1} + k_{i2t}^2\tau_{\epsilon t}\right).$$

The marginal flow benefit of effort is $\frac{b_i k_{i2t}\tau_{\epsilon t}}{\tau_{i\theta,t-1} + k_{i2t}^2\tau_{\epsilon t}}$ as in the baseline model. Together with the marginal benefit of shifting the market's belief about oneself, this is $k_{i2t}\tau_{\epsilon t} \sum_{n=t}^{T} \frac{b_i}{\tau_{i\theta,t-1} + \sum_{l=t}^{n} \tau_{\epsilon l}k_{i2l}^2}$. There is an additional benefit or cost of shifting the other player's future effort and therefore signals.

The FOC yields

$$e_{it} = (c_i')^{-1}\left(k_{i2t}\tau_{\epsilon t} \sum_{n=t}^{T} \frac{b_i}{\tau_{i\theta,t-1} + \sum_{l=t}^{n} \tau_{\epsilon l}k_{i2l}^2}\right) + \alpha_i\theta_i + \kappa_i \sum_{k=1}^{t-1} s_{jk},$$

which is linear as conjectured. Based on this effort formula, $k_{i2t} = \alpha_i \ \forall t$.

References

BADEL, A. AND M. HUGGETT (2014): "Interpreting life-cycle inequality patterns as an efficient allocation: Mission impossible?" *Review of Economic Dynamics*, 17, 613–629.

KYLE, A. S. (1985): "Continuous auctions and insider trading," *Econometrica*, 1315–1335.

MCLENNAN, A., P. K. MONTEIRO, AND R. TOURKY (2014): "On the uniqueness of equilibrium in the Kyle model," CORE Discussion Papers 761.