

Online Appendix for “Electoral Campaign Attacks: Theory and Evidence”

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This Online Appendix contains supplemental material for “*Electoral Campaign Attacks: Theory and Evidence*” and is organized into six sections, Appendices A–F.

A Additional Figures and Tables

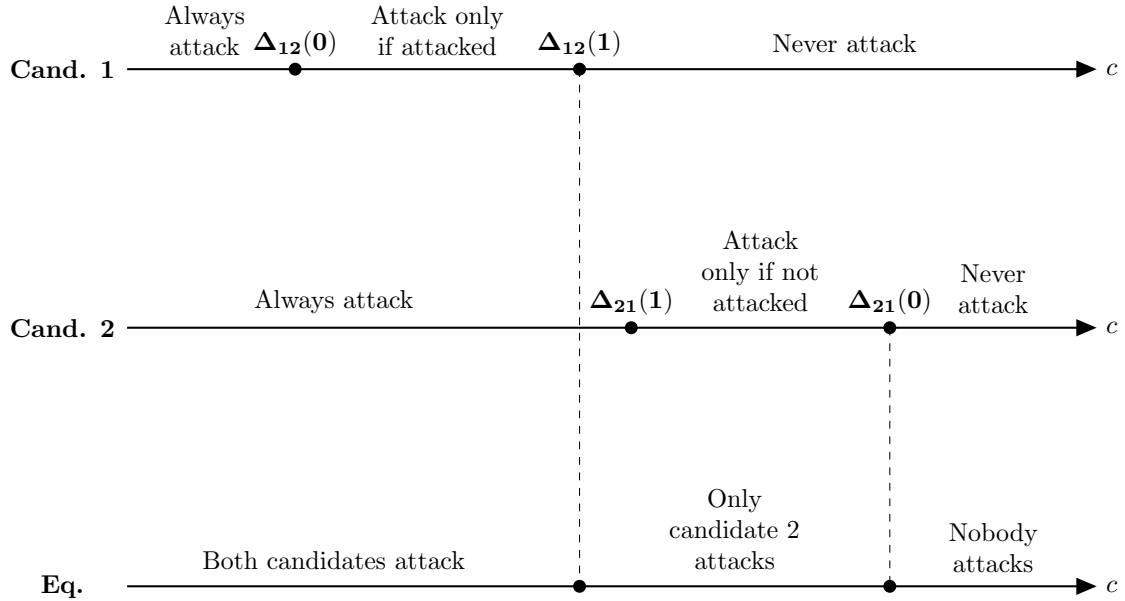


Figure A.1: Two-Candidate Races: Equilibria

Notes: This figure depicts (i) the best-response function of candidate 1, (ii) the best-response function of candidate 2, and (iii) the unique equilibrium that exists in each region of the parameters.

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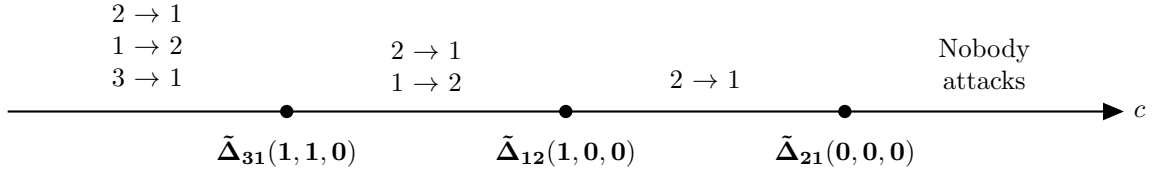


Figure A.2: Three-Candidate Races: Equilibria

Notes: This figure depicts the unique equilibrium that exists in each region of the parameters for the case where $\tilde{\Delta}_{31}(n_3) < \tilde{\Delta}_{12}(n_1) < \tilde{\Delta}_{21}(n_2)$ for any $n_i \in \mathcal{N}_i$. Here, $i \rightarrow j$ denotes an attack from candidate i against j .

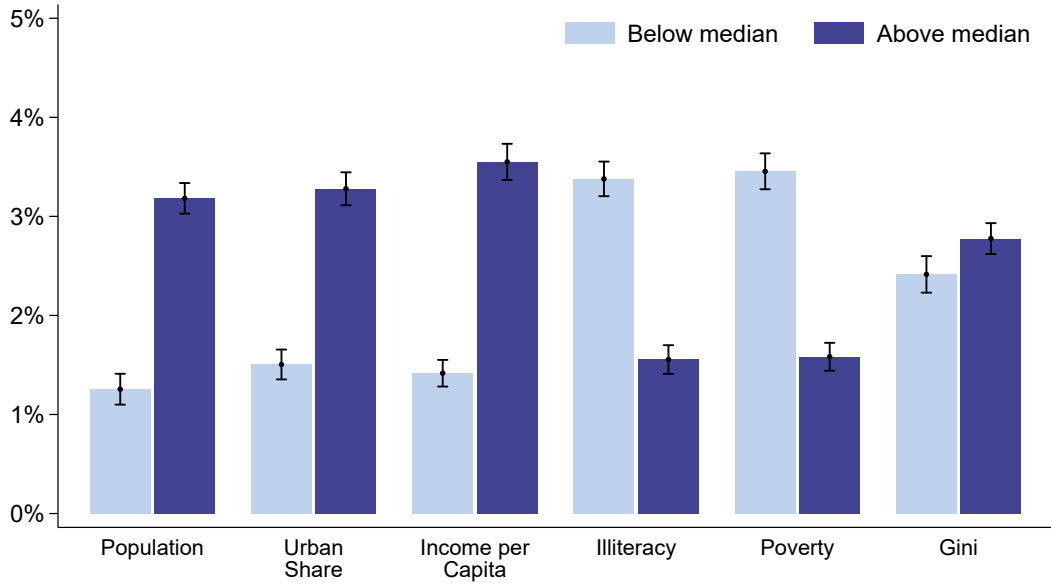
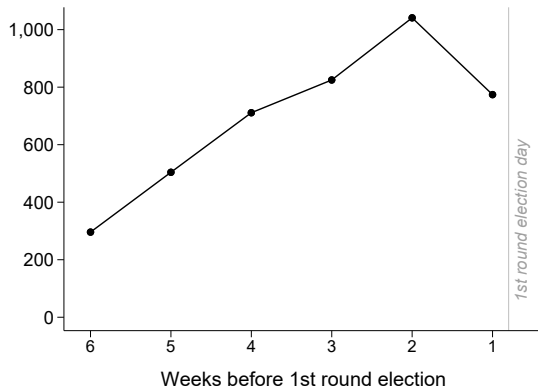


Figure A.3: Frequency of Campaign Attacks and Municipality Characteristics

Notes: This figure reports the frequency of campaign attacks for municipalities below and above the median of the following socioeconomic characteristics (medians in parenthesis): population (10,761), share of urban population (0.65), monthly income per capita (R\$ 472.28), illiteracy rate (0.13), poverty rate (0.06) and Gini index (0.49). Frequencies are computed based on the full sample of ordered pairs of candidates, with 95% confidence intervals reported.

Panel A: Full sample



Panel B: By election year

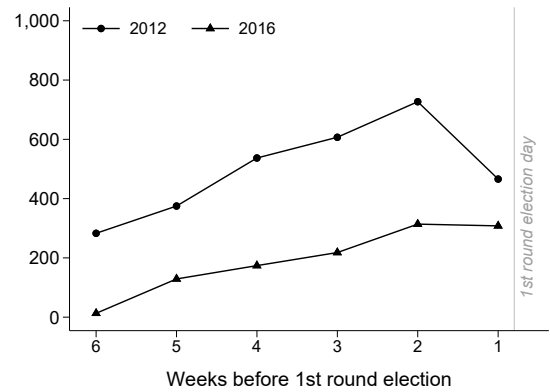


Figure A.4: Timing of Campaign Attacks

Notes: This figure plots the number of right of reply lawsuits filed in the weeks preceding the Election Day, for the full sample (Panel A) and separately for the 2012 and 2016 elections (Panel B).

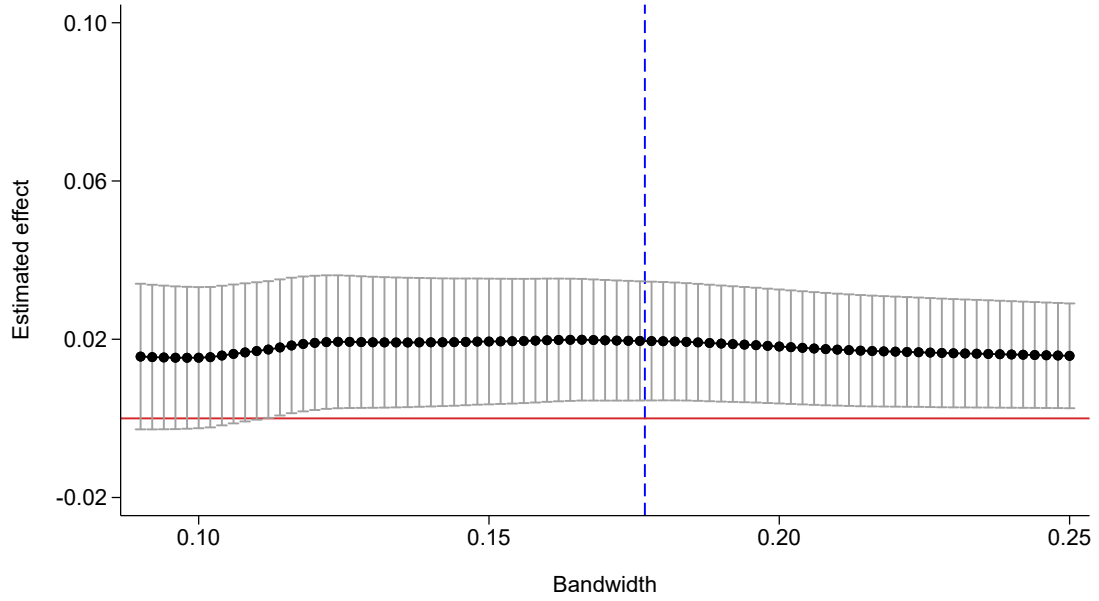


Figure A.5: Electoral Advantage Effect: Robustness to Bandwidth Choice

Notes: This figure plots point estimates obtained from separate local linear regressions using various bandwidths. The specification uses a triangular kernel and includes election-year fixed effects. 90% confidence intervals are computed based on standard errors clustered at the municipality level. The vertical dashed line indicates the CCT optimal bandwidth. For additional details, see footnote to Table 5.

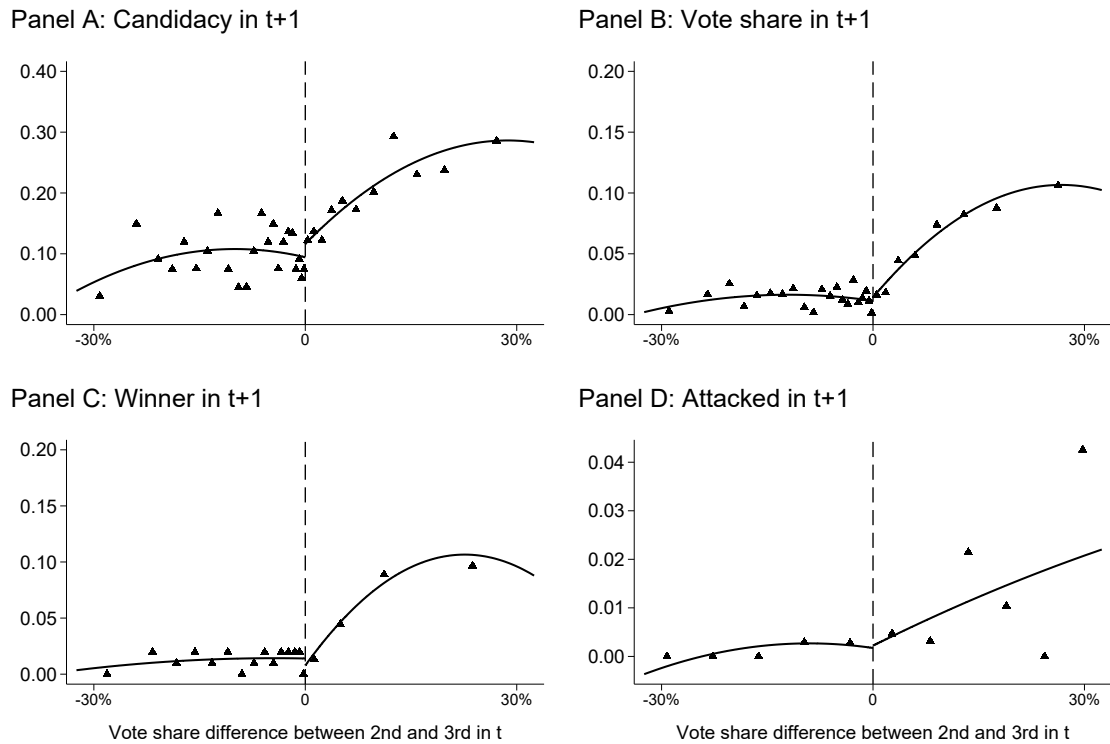


Figure A.6: Electoral Advantage Effect: Placebo Test (3rd vs 4th place)

Notes: This figure plots local averages of candidacy in $t + 1$ (Panel A), vote share in $t + 1$ (Panel B), winner in $t + 1$ (Panel C) and campaign attack in $t + 1$ (Panel D). The sample includes only candidates who placed 3rd and 4th in the election at t , focusing only on races held under single-ballot plurality. Averages are calculated within quantile-spaced bins of the vote share difference (running variable) in Panels A-C and evenly-spaced bins in Panel D, where the number of bins is chosen optimally according to Calonico et al. (2014). For additional details, see footnote to Figure 2.

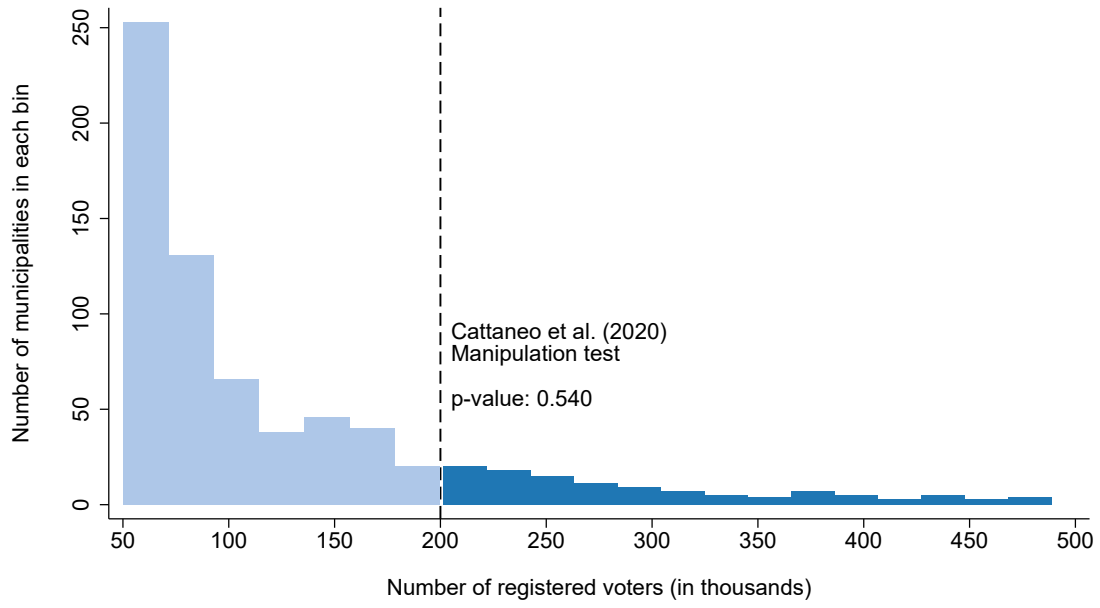


Figure A.7: Distribution of Number of Registered Voters

Notes: This figure plots the distribution of the number of registered voters across municipalities, pooling data from the 2012 and 2016 elections. The sample is restricted to include only observations from races with three or more candidates. The graph also reports the p-value for the manipulation test proposed by Cattaneo et al. (2020).

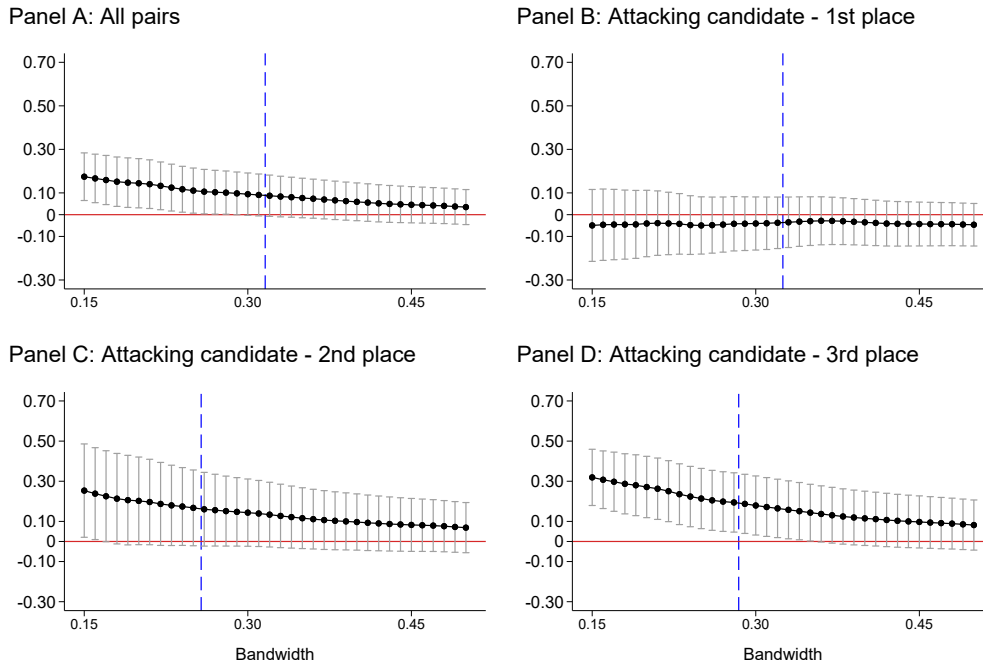


Figure A.8: Single vs Dual Ballot Plurality: Robustness to Bandwidth Choice (All Pairs and by Attacking Candidate)

Notes: This figure plots point estimates obtained from separate local linear regressions using various bandwidth, with the running variable defined as the log difference between the number of registered voters and the 200,000-voter threshold. All specifications use a triangular kernel and include state and election-year fixed effects. 90% confidence intervals are computed based on standard errors clustered at the municipality level. Vertical dashed lines indicate the CCT optimal bandwidths. For additional details, see footnote to Table 6.

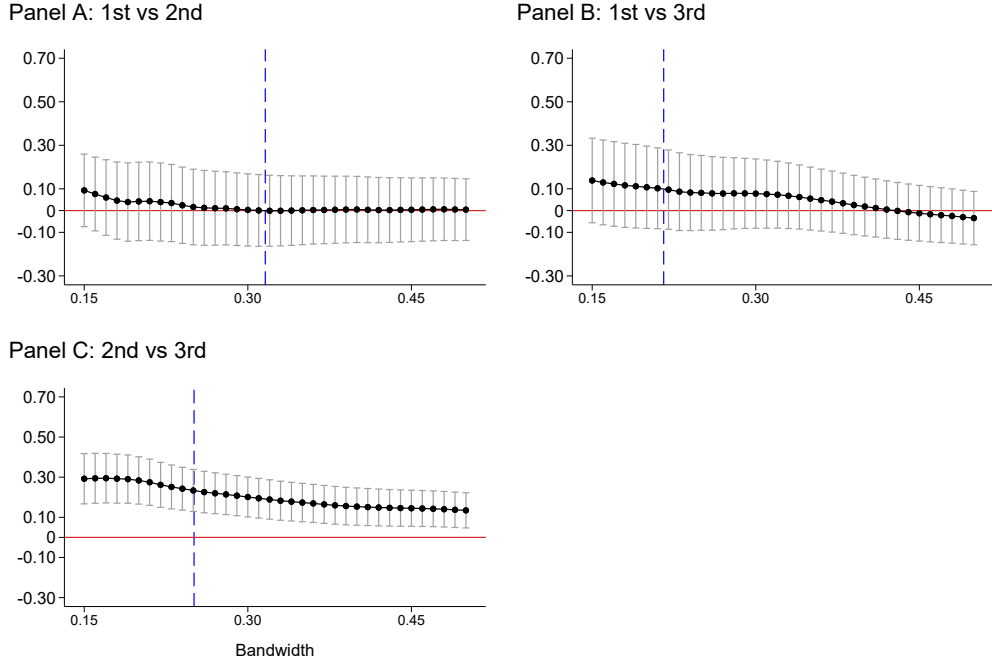


Figure A.9: Single vs Dual Ballot Plurality: Robustness to Bandwidth Choice (By Pairs)

Notes: This figure plots point estimates obtained from separate local linear regressions using various bandwidth, with the running variable defined as the log difference between the number of registered voters and the 200,000-voter threshold. All specifications use a triangular kernel and include state and election-year fixed effects. 90% confidence intervals are computed based on standard errors clustered at the municipality level. Vertical dashed lines indicate the CCT optimal bandwidths. For additional details, see footnote to Table 7.

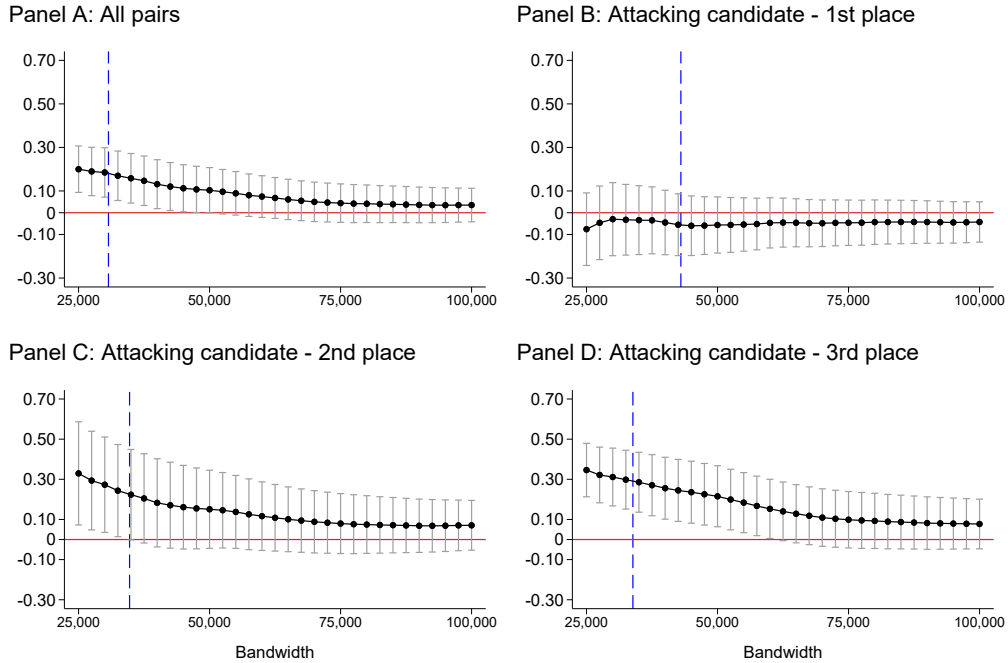


Figure A.10: Single vs Dual Ballot Plurality: Robustness to Bandwidth Choice with Running Variable in Levels (All Pairs and by Attacking Candidate)

Notes: This figure plots point estimates obtained from separate local linear regressions using various bandwidth, with the running variable defined as the difference in levels between the number of registered voters and the 200,000-voter threshold. All specifications use a triangular kernel and include state and election-year fixed effects. 90% confidence intervals are computed based on standard errors clustered at the municipality level. Vertical dashed lines indicate the CCT optimal bandwidths. For additional details, see footnote to Table 6.

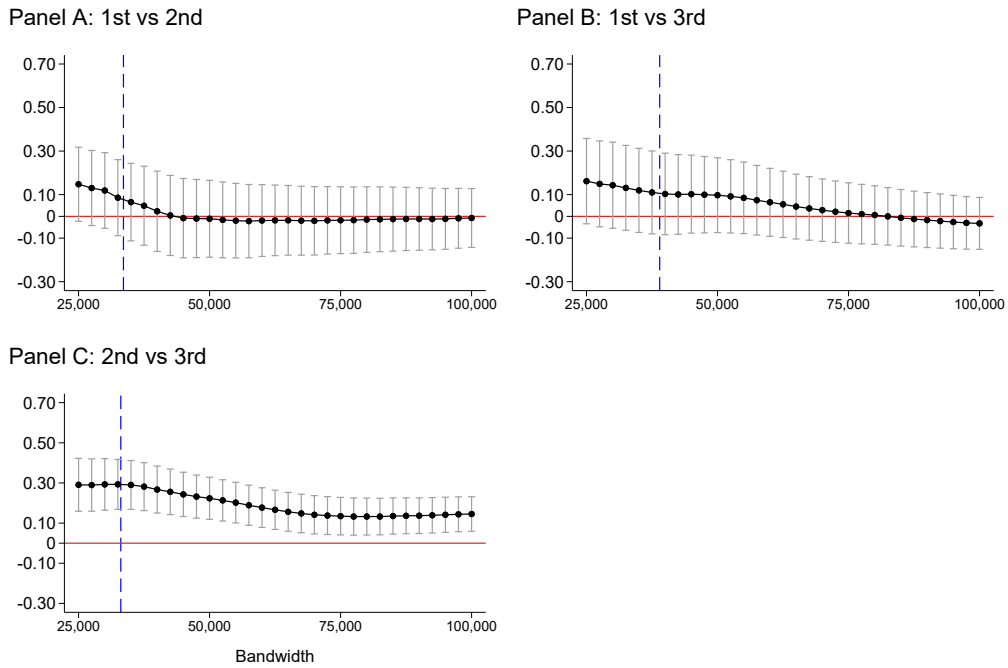


Figure A.11: Single vs Dual Ballot Plurality: Robustness to Bandwidth Choice with Running Variable in Levels (By Pairs)

Notes: This figure plots point estimates obtained from separate local linear regressions using various bandwidth, with the running variable defined as the difference in levels between the number of registered voters and the 200,000-voter threshold. All specifications use a triangular kernel and include state and election-year fixed effects. 90% confidence intervals are computed based on standard errors clustered at the municipality level. Vertical dashed lines indicate the CCT optimal bandwidths. For additional details, see footnote to Table 7.

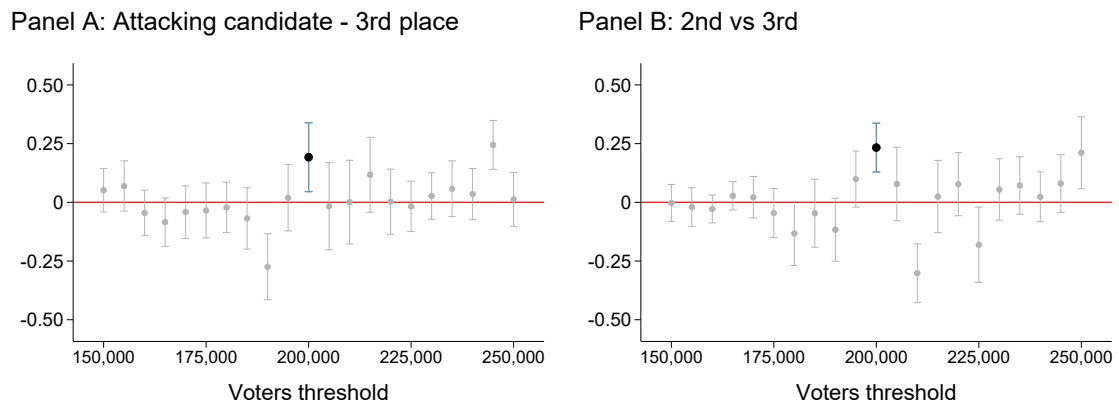


Figure A.12: Single vs Dual Ballot Plurality: Placebo Thresholds

Notes: This figure plots point estimates obtained from separate local linear regressions using false thresholds, above and below the true discontinuity, with the running variable defined as the log difference between the number of registered voters and the 200,000-voter threshold. All specifications use a triangular kernel and include state and election-year fixed effects. 90% confidence intervals are computed based on standard errors clustered at the municipality level. Vertical dashed lines indicate the CCT optimal bandwidths. For additional details, see footnote to Tables 6 and 7.

Table A.1: Distribution of Attack Profiles in Two-Candidate Races

		Population		Income per capita		Gini coefficient	
	Full sample (1)	Below median (2)	Above median (3)	Below median (4)	Above median (5)	Below median (6)	Above median (7)
Panel A: Equilibrium profiles							
Both attack	0.022	0.008	0.041	0.012	0.035	0.020	0.024
Only candidate 2 attacks	0.040	0.021	0.066	0.026	0.056	0.035	0.044
Nobody attacks	0.911	0.955	0.852	0.943	0.872	0.920	0.904
Panel B: Non-equilibrium profiles							
Only candidate 1 attacks	0.027	0.016	0.042	0.019	0.036	0.025	0.028
Observations	6,914	3,983	2,931	3,809	3,105	3,231	3,683

Notes: This table reports the distribution of attack profiles in two-candidate races, using a 5% threshold to define effective candidates and focusing only on races held under single ballot plurality. The profiles in Panel A are consistent with the classes of equilibria predicted to exist in Proposition 2, whereas Panel B reports the sole non-equilibrium profile. The table also show the frequency of attack profiles for municipalities below and above the median in population (10,761), monthly income per capita (R\$ 472.28), and Gini Index (0.49).

Table A.2: Electoral Advantage Effect: Covariate Smoothness

	Linear specification			Alternative specifications	
	SB mean	Optimal BW value	Obs	(1)	(2)
Female	0.127	0.148	2,804	−0.016 [0.025]	−0.009 [0.034]
Age	49.181	0.177	3,242	0.373 [0.754]	1.019 [1.110]
Married	0.719	0.126	2,480	−0.016 [0.036]	−0.015 [0.042]
High School	0.829	0.175	3,218	−0.022 [0.027]	−0.043 [0.041]
College	0.495	0.151	2,844	0.029 [0.038]	0.018 [0.050]
Log of Campaign Expenditure	10.275	0.148	2,612	−0.078 [0.117]	−0.099 [0.136]
Party Filliation: PMDB	0.146	0.181	3,304	−0.030 [0.025]	−0.040 [0.034]
Party Filliation: PT	0.144	0.141	2,704	0.045* [0.026]	0.053 [0.034]
Party Filliation: PSDB	0.103	0.163	3,040	−0.019 [0.022]	−0.028 [0.026]
Bandwidth method				CCT	CCT
Polynomial order				1	2

Notes: This table reports RDD estimates of local polynomial regressions exploiting virtual ties between 2nd and 3rd place candidates. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 5.

Table A.3: Runner-Up Effect and Electoral Advantage Effect on Campaign Attacks: Optimal Bandwidths and Number of Observations

	(1)	(2)	(3)	(4)
Panel A: Candidate Performance				
Candidacy in $t + 1$	0.162 {3,032}	0.179 {3,286}	0.239 {4,198}	0.050 {1,156}
Vote Share in $t + 1$	0.178 {3,264}	0.185 {3,360}	0.244 {4,292}	0.050 {1,156}
Winner in $t + 1$	0.183 {3,330}	0.186 {3,362}	0.180 {3,288}	0.050 {1,156}
Panel B: Campaign Attacks				
Attacked in $t + 1$	0.177 {3,240}	0.169 {3,138}	0.146 {2,774}	0.050 {1,156}
Polynomial Order	1	1	2	0
Candidates' characteristics	No	Yes	No	No

Notes: This table reports the Calonico et al. (2014) optimal bandwidths used in each specification of Table 5, with the number of effective observations shown in braces. For additional details, see footnote to Table 5.

Table A.4: Electoral Advantage Effect on Campaign Attacks: Fuzzy RDD

	Linear specification			Alternative specifications			
	3rd-place mean	Optimal BW value	Obs	(1)	(2)	(3)	(4)
Attacked in $t + 1$	0.012	0.147	2,800	0.497* [0.274]	0.490* [0.266]	0.516* [0.301]	0.400** [0.199]
Bandwidth method				CCT	CCT	CCT	0.05
Polynomial order				1	1	2	0
Candidates' characteristics				No	Yes	No	No

Notes: This table reports fuzzy RDD estimates of local polynomial regressions using the cutoff between 2nd and 3rd place candidates in the election at t as an instrument for vote shares in $t + 1$. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 5.

Table A.5: Electoral Advantage Effect: Placebo Test (3rd vs 4th place)

	Local Linear Regression			Alternative Specifications			
	4th place Mean	Optimal BW	Obs	(1)	(2)	(3)	(4)
Panel A: Candidate Performance							
Candidacy in $t + 1$	0.105	0.062	1,408	0.041 [0.030]	0.040 [0.029]	0.053 [0.035]	0.037* [0.020]
Vote Share in $t + 1$	0.014	0.059	1,368	0.001 [0.006]	0.001 [0.006]	0.002 [0.007]	0.009** [0.005]
Winner in $t + 1$	0.016	0.057	1,348	-0.009 [0.009]	-0.009 [0.009]	-0.009 [0.011]	0.003 [0.007]
Panel B: Campaign Attacks							
Attacked in $t + 1$	0.002	0.043	1,076	0.000 [0.002]	-0.000 [0.002]	-0.000 [0.003]	0.001 [0.002]
Bandwidth Method				CCT	CCT	CCT	0.05
Polynomial Order				1	1	2	0
Candidates' characteristics				No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting virtual ties between 3rd and 4th place candidates. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 5.

Table A.6: Electoral Advantage Effect: Placebo Test (Attacked in t)

	Local Linear Regression			Alternative Specifications			
	3rd place Mean	Optimal BW	Obs	(1)	(2)	(3)	(4)
Attacked in t	0.050	0.156	1,514	0.001 [0.024]	0.001 [0.023]	0.005 [0.029]	0.011 [0.021]
Bandwidth Method				CCT	CCT	CCT	0.05
Polynomial Order				1	1	2	0
Candidates' characteristics				No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting virtual ties between 2nd and 3rd place candidates. The dependent variable is a dummy indicating whether the candidate received a campaign attack in the *same* election at t . All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 5.

Table A.7: Electoral Advantage Effect: Mechanisms (Strong vs Weak Runners-Up)

	Linear specification			Alternative specifications			
	3rd-place mean	Optimal BW value	Obs	(1)	(2)	(3)	(4)
Panel A: Strong Runners-Up ($s_2 + s_3 \geq s_1$)							
Candidacy in $t + 1$	0.299	0.116	1,614	0.137*** [0.049]	0.136*** [0.047]	0.133** [0.058]	0.147*** [0.038]
Winner in $t + 1$	0.123	0.105	1,512	0.064 [0.040]	0.065* [0.038]	0.058 [0.043]	0.085*** [0.030]
Attacked in $t + 1$	0.014	0.127	1,758	0.033** [0.013]	0.032** [0.013]	0.025 [0.016]	0.029*** [0.011]
Panel B: Weak Runners-Up ($s_2 + s_3 < s_1$)							
Candidacy in $t + 1$	0.168	0.153	858	0.043 [0.051]	0.047 [0.052]	0.053 [0.058]	0.047 [0.047]
Winner in $t + 1$	0.056	0.139	792	0.016 [0.033]	0.022 [0.033]	0.018 [0.039]	0.009 [0.029]
Attacked in $t + 1$	0.008	0.170	954	-0.012 [0.011]	-0.010 [0.011]	-0.018 [0.019]	-0.011 [0.011]
Bandwidth method				CCT	CCT	CCT	0.05
Polynomial order				1	1	2	0
Candidates' characteristics				No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting virtual ties between 2nd and 3rd place candidates. Panel A reports estimates obtained from the subsample of candidates where the combined vote shares of 2nd and 3rd place candidates exceeded that of the winner in the election at t , whereas Panel B reports estimates from the subsample where the opposite holds. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 5.

Table A.8: Electoral Advantage Effect: Mechanisms (Centrist vs Non-centrist Runners-Up)

	Linear specification			Alternative specifications			
	3rd-place mean	Optimal BW value	Obs	(1)	(2)	(3)	(4)
Panel A: Centrist Runners-Up							
Candidacy in $t + 1$	0.286	0.151	622	0.138* [0.077]	0.203*** [0.079]	0.187* [0.097]	0.128* [0.070]
Winner in $t + 1$	0.114	0.112	472	0.118* [0.066]	0.111* [0.065]	0.146** [0.075]	0.093* [0.052]
Attacked in $t + 1$	0.021	0.119	484	0.065** [0.026]	0.042* [0.025]	0.079*** [0.028]	0.045** [0.021]
Panel B: Non-centrist Runners-Up							
Candidacy in $t + 1$	0.242	0.140	2,108	0.107*** [0.040]	0.105*** [0.039]	0.102** [0.046]	0.117*** [0.035]
Winner in $t + 1$	0.093	0.128	1,984	0.037 [0.031]	0.042 [0.030]	0.016 [0.041]	0.057** [0.026]
Attacked in $t + 1$	0.011	0.170	2,442	0.012 [0.010]	0.013 [0.010]	0.004 [0.014]	0.011 [0.009]
Bandwidth method				CCT	CCT	CCT	0.05
Polynomial order				1	1	2	0
Candidates' characteristics				No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting virtual ties between 2nd and 3rd place candidates. Panel A reports estimates obtained from the subsample of candidates where the runner-up in t was from a centrist party (based on the ideological scores from Zucco and Power (2023)), whereas Panel B reports estimates from the subsample where the opposite holds. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 5.

Table A.9: Single vs Dual Ballot Plurality: Covariate Smoothness

	Linear specification			Alternative specifications	
	SB mean	Optimal BW value	Obs	(1)	(2)
Latitude	-18.790	0.257	112	-3.118 [4.274]	-3.997 [5.435]
Longitude	-46.598	0.374	170	-2.697 [2.668]	-5.116 [3.922]
Share of Urban Population	0.951	0.342	157	0.010 [0.028]	0.019 [0.034]
Monthly Income per Capita (in Reais)	773.816	0.250	108	165.804* [97.210]	165.645 [112.001]
Illiteracy rate	0.063	0.334	151	-0.005 [0.012]	-0.007 [0.014]
Poverty rate	0.025	0.384	176	-0.010 [0.012]	-0.015 [0.016]
Gini index	0.502	0.261	115	0.010 [0.022]	0.012 [0.027]
Bandwidth method				CCT	CCT
Polynomial order				1	2

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality at the 200,000-voter threshold, using the running variable in logs. The unit of observation is the electoral race. All socioeconomic variables were drawn from the 2010 Population Census, and the sample is restricted to include only races with three or more candidates. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 6.

Table A.10: Single vs Dual Ballot Plurality: Optimal Bandwidths and Number of Observations (All Pairs and by Attacking Candidate)

	(1)	(2)	(3)	(4)	(5)
Panel A: Full Sample					
All pairs	0.285 {744}	0.264 {690}	0.295 {756}	0.292 {756}	0.100 {222}
Panel B: By Attacking Candidate					
1st place Candidate	0.336 {306}	0.258 {228}	0.224 {188}	0.301 {252}	0.100 {74}
2nd place Candidate	0.351 {320}	0.314 {274}	0.194 {156}	0.416 {374}	0.100 {74}
3rd place Candidate	0.240 {210}	0.289 {250}	0.205 {166}	0.434 {388}	0.100 {74}
Polynomial Order	1	1	1	2	0
Municipal characteristics	No	Yes	No	No	No
Candidates' characteristics	No	No	Yes	No	No

Notes: This table reports the Calonico et al. (2014) optimal bandwidths used in each specification of Table 6, with the number of effective observations shown in braces. For additional details, see footnote to Table 6.

Table A.11: Single vs Dual Ballot Plurality: Optimal Bandwidths and Number of Observations (By Pairs)

	(1)	(2)	(3)	(4)	(5)
1st VS 2nd	0.275 {240}	0.271 {236}	0.212 {176}	0.282 {246}	0.100 {74}
1st VS 3rd	0.212 {176}	0.211 {176}	0.213 {176}	0.402 {366}	0.100 {74}
2nd VS 3rd	0.336 {306}	0.175 {142}	0.260 {228}	0.415 {374}	0.100 {74}
Polynomial order	1	1	1	2	0
Municipal characteristics	No	Yes	No	No	No
Candidates' characteristics	No	No	Yes	No	No

Notes: This table reports the Calonico et al. (2014) optimal bandwidths used in each specification of Table 7, with the number of effective observations shown in braces. For additional details, see footnote to Table 7

Table A.12: Single vs Dual Ballot Plurality: Running Variable in Levels (All Pairs and by Attacking Candidate)

	Local Linear Regression			Alternative Specifications				
	SB Mean	Optimal BW	Obs	(1)	(2)	(3)	(4)	(5)
Panel A: Full Sample								
All pairs	0.144	31,957	426	0.143** [0.061]	0.133*** [0.043]	0.176*** [0.064]	0.177** [0.082]	0.127*** [0.045]
Panel B: By Attacking Candidate								
1st place Candidate	0.144	43,018	194	-0.050 [0.090]	-0.115* [0.068]	-0.009 [0.063]	-0.027 [0.121]	-0.020 [0.086]
2nd place Candidate	0.148	43,814	198	0.086 [0.121]	0.140 [0.122]	0.106 [0.107]	0.146 [0.152]	0.197* [0.104]
3rd place Candidate	0.083	36,138	154	0.213** [0.107]	0.250*** [0.083]	0.263*** [0.056]	0.260** [0.127]	0.204** [0.087]
Bandwidth Method				CCT	CCT	CCT	CCT	15,000
Polynomial Order				1	1	1	2	0
Municipal characteristics				No	Yes	No	No	No
Candidates' characteristics				No	No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality at the 200,000-voter threshold, using the running variable in levels. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 6.

Table A.13: Single vs Dual Ballot Plurality: Running Variable in Levels (By Pairs)

	Linear specification			Alternative specifications				
	SB mean	Optimal BW value	Obs	(1)	(2)	(3)	(4)	(5)
1st VS 2nd	0.213	38,117	164	0.026 [0.095]	0.031 [0.098]	0.100 [0.075]	0.176 [0.119]	0.069 [0.063]
1st VS 3rd	0.130	41,282	180	0.061 [0.125]	0.068 [0.075]	0.181* [0.093]	0.095 [0.137]	0.097 [0.114]
2nd VS 3rd	0.049	43,711	198	0.179*** [0.069]	0.237*** [0.056]	0.317*** [0.069]	0.259*** [0.088]	0.216*** [0.055]
Bandwidth method				CCT	CCT	CCT	CCT	15,000
Polynomial order				1	1	1	2	0
Municipal characteristics				No	Yes	No	No	No
Candidates' characteristics				No	No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality at the 200,000-voter threshold, using the running variable in levels. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 7.

Table A.14: Single vs Dual Ballot Plurality: Alternative Mechanisms

	Linear specification			Alternative specifications	
	SB mean	Optimal BW value	Obs	(1)	(2)
Turnout (% voters)	0.831	0.273	119	-0.011 [0.019]	-0.015 [0.024]
Valid votes (% total votes)	0.852	0.234	100	0.012 [0.026]	0.008 [0.029]
Number of candidates	4.929	0.323	143	1.218** [0.481]	-0.537 [0.821]
Log of maximum of cand. campaign expenses	13.988	0.160	67	0.251 [0.243]	0.417 [0.301]
Log of average of cand. campaign expenses	12.986	0.215	89	0.034 [0.235]	0.261 [0.319]
Vote share - 1st	0.507	0.185	72	0.081 [0.070]	0.107 [0.082]
Vote share - 2nd	0.307	0.257	112	0.004 [0.029]	-0.015 [0.036]
Vote share - 3rd	0.125	0.214	88	-0.042 [0.029]	-0.055 [0.035]
Vote share - 3rd and lower placed candidates	0.195	0.214	88	-0.052 [0.059]	-0.082 [0.070]
Vote share - 4th and lower placed candidates	0.068	0.243	106	-0.000 [0.035]	-0.008 [0.041]
Vote share - 5th and lower placed candidates	0.022	0.298	126	0.019 [0.014]	0.016 [0.017]
Bandwidth method				CCT	CCT
Polynomial order				1	2

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality at the 200,000-voter threshold, using the running variable in logs. The unit of observation is the electoral race. The sample is restricted to include only races with three or more candidates. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 6.

Table A.15: Single vs Dual Ballot Plurality: Additional Controls (All Pairs and by Attacking Candidate)

	Local Linear Regression			Alternative Specifications			
	SB Mean	Optimal BW	Obs	(1)	(2)	(3)	(4)
Panel A: Full Sample							
All pairs	0.126	0.285	744	0.052 [0.058]	0.077 [0.058]	0.090* [0.048]	0.103* [0.062]
Panel B: By Attacking Candidate							
1st place Candidate	0.137	0.336	306	-0.052 [0.074]	-0.060 [0.080]	-0.033 [0.070]	-0.074 [0.070]
2nd place Candidate	0.142	0.351	320	0.061 [0.090]	0.127 [0.086]	0.104 [0.093]	0.089 [0.109]
3rd place Candidate	0.110	0.240	210	0.167* [0.101]	0.166* [0.100]	0.183** [0.092]	0.207** [0.097]
Bandwidth method				CCT	CCT	CCT	CCT
Polynomial order				1	1	1	1
Add. controls: No. of candidates				No	Yes	No	No
Add. controls: Campaign expenditures				No	No	Yes	No
Add. controls: Turnout, valid votes and shares				No	No	No	Yes

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality at the 200,000-voter threshold, using the running variable in logs. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. In column 1, we report, for reference, the baseline specification from Table 6, column 1; in column 2, we include a third-order polynomial in the number of candidates; in column 3, we add the logarithm of the maximum and average campaign expenditures; and in column 4, we include turnout, share of valid votes, and the vote shares of the 1st, 2nd, 3rd place candidates, as well as those ranked 4th and lower. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 6.

Table A.16: Single vs Dual Ballot Plurality: Additional Controls (By Pairs)

	Linear specification			Alternative specifications			
	SB mean	Optimal BW value	Obs	(1)	(2)	(3)	(4)
1st VS 2nd	0.201	0.275	240	-0.012 [0.096]	0.057 [0.087]	0.013 [0.095]	-0.053 [0.098]
1st VS 3rd	0.117	0.212	176	0.060 [0.123]	0.046 [0.119]	0.050 [0.106]	0.104 [0.125]
2nd VS 3rd	0.044	0.336	306	0.123* [0.064]	0.131** [0.063]	0.167*** [0.060]	0.248*** [0.071]
Bandwidth method				CCT	CCT	CCT	CCT
Polynomial order				1	1	1	1
Add. controls: No. of candidates				No	Yes	No	No
Add. controls: Campaign expenditures				No	No	Yes	No
Add. controls: Turnout, valid votes and shares				No	No	No	Yes

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality at the 200,000-voter threshold, using the running variable in logs. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. In column 1, we report, for reference, the baseline specification from Table 7, column 1; in column 2, we include a third-order polynomial in the number of candidates; in column 3, we add the logarithm of the maximum and average campaign expenditures; and in column 4, we include turnout, share of valid votes, and the vote shares of the 1st, 2nd, 3rd place candidates, as well as those ranked 4th and lower. Standard errors clustered at the municipality level are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 7.

B Measurement Error and Alternative Measure of Campaign Attacks

In this appendix, we provide a formal discussion of the issues that may arise due to measurement error in our proxy for campaign attacks, including the likely direction of potential biases. We also check the robustness of our findings by employing an alternative measure of attacks based on courts' rulings.

B.1 Measurement Error: Formal Discussion

As discussed in Subsection 3.2, measurement error arises in our analysis from the fact that we observe only the filing of right of reply (RR) lawsuits, rather than the actual occurrence of attacks. Intuitively, measurement error may lead to biased estimates if it is correlated with our variables of interest—namely, the relative candidate ranks. To formally investigate the direction of potential biases, we consider the following setup. Suppose that the true outcome of interest is:

$$Y_{ij}^o = \begin{cases} 1 & \text{if candidate } i \text{ attacked candidate } j \\ 0 & \text{otherwise} \end{cases}$$

while, as researchers, we observe only:

$$Y_{ij}^* = \begin{cases} 1 & \text{if candidate } j \text{ filed a lawsuit against } i \\ 0 & \text{otherwise} \end{cases}$$

where, for simplicity, we omit subscripts for municipality and election year.

We treat lawsuits as an imperfect proxy for actual campaign attacks, and define the misclassification term as:

$$\mathcal{M}_{ij} = Y_{ij}^* - Y_{ij}^o,$$

so that the observed outcome can be written as $Y_{ij}^* = Y_{ij}^o + \mathcal{M}_{ij}$. To simplify the exposition and focus on the more relevant case for our empirical application, we initially assume that misclassification arises only through false positives—that is, cases in which a lawsuit is observed despite no actual attack having occurred.² We define the false positive rate as:

$$\varphi_{ij}^+ = \Pr(Y_{ij}^* = 1 \mid Y_{ij}^o = 0),$$

which is allowed to vary as a function of the ranks of candidates i and j .

Focusing on two-candidate races, and considering, for simplicity, a linear specification without controls, we would like to estimate the following regression:

$$Y_{ij}^o = \alpha + \beta_{21} D21_{ij} + \epsilon_{ij} \tag{17}$$

but in practice we can only estimate:

$$Y_{ij}^* = \alpha + \beta_{21} D21_{ij} + u_{ij} \tag{18}$$

where Y_{ij}^* is the observed, possibly misclassified, measure of campaign attacks. We allow the false positive rate to vary according to the relative positions of candidates:

$$\varphi_{ij}^+ = \begin{cases} \varphi_{21}^+ & \text{if } D21_{ij} = 1 \\ \varphi_{12}^+ & \text{if } D21_{ij} = 0 \end{cases}$$

²The proposed framework can also accommodate false negatives—cases in which a lawsuit is not observed despite an attack having occurred—which we discuss below. For a more general treatment of measurement error in binary outcome variables, see Meyer and Mittag (2017).

where φ_{21}^+ denotes the false positive rate when candidate 2 files a lawsuit against 1, and φ_{12}^+ denotes the corresponding rate when candidate 1 files against 2.

The OLS estimator obtained from regression (18) can be expressed as:

$$\hat{\beta}_{21} = \frac{\text{Cov}(D21_{ij}, Y_{ij}^*)}{\text{Var}(D21_{ij})} = \frac{\text{Cov}(D21_{ij}, Y_{ij}^o + \mathcal{M}_{ij})}{\text{Var}(D21_{ij})} = \beta_{21} + \frac{\text{Cov}(D21_{ij}, \mathcal{M}_{ij})}{\text{Var}(D21_{ij})} \quad (19)$$

Thus, it follows that $\hat{\beta}_{21}$ will overestimate the true parameter β_{21} when $\text{Cov}(D21_{ij}, \mathcal{M}_{ij}) > 0$, that is, when the false positive rate is higher for attacks from candidate 2 against 1 ($D21_{ij} = 1$), than for attacks in the opposite direction, i.e. $\varphi_{21}^+ > \varphi_{12}^+$. Conversely, $\hat{\beta}_{21}$ will underestimate the true parameter when the opposite holds, i.e., $\varphi_{21}^+ < \varphi_{12}^+$.

In the discussion below, we provide suggestive evidence on how false positive rates vary with the relative positions of candidates by leveraging information from courts' rulings. Specifically, we approximate a false positive as a case in which a right of reply lawsuit was filed but ultimately decided against the plaintiff. Consistent with our theoretical framework, we conjecture that the incentive to over-file lawsuits in single-ballot plurality systems is stronger for 2nd-place candidates, for whom campaign resources are particularly valuable. Accordingly, we expect false positive rates to be higher for attacks from candidate 1 against 2—where the 2nd place candidate is the plaintiff—than for attacks in the opposite direction, i.e. $\varphi_{21}^+ < \varphi_{12}^+$, suggesting that, if anything, the OLS estimator is likely to underestimate the true coefficient.

Note that the analysis extends naturally to the case where false negatives are allowed. A false negative corresponds to a situation where $Y_{ij}^* = 0$ and $Y_{ij}^o = 1$, with $\mathcal{M}_{ij} = -1$. Let φ_{21}^- and φ_{12}^- denote the false negative rates when candidate 2 files a lawsuit against 1, and when candidate 1 files a lawsuit against 2, respectively. From equation (19), it still follows that $\hat{\beta}_{21}$ will overestimate the true parameter β_{21} when $\text{Cov}(D21_{ij}, \mathcal{M}_{ij}) > 0$. However, in the case of false negatives, a positive covariance corresponds to a situation in which the false negative rate is lower for attacks from candidate 2 against 1 than for attacks from candidate 1 against 2, i.e., $\varphi_{21}^- < \varphi_{12}^-$.

Unfortunately, an empirical investigation of how false negative rates vary across candidate pairs is not feasible in our setting, as we do not observe actual attacks. However, following a similar reasoning as before, we conjecture that the incentive to under-file lawsuits is stronger for 1st place candidates, as campaign resources are relatively less valuable to them. Accordingly, we expect false negative rates to be higher for attacks from candidate 2 against 1—where the 1st place candidate is the plaintiff—than for attacks in the opposite direction, i.e. $\varphi_{21}^- > \varphi_{12}^-$, suggesting that the OLS estimator is likely to underestimate the true coefficient in this case.

Finally, while the analysis for three-candidate races is more complex, the underlying logic remains the same: observing a pattern in false positive rates that mirrors our main results would suggest that the OLS estimator overestimates the true effect. As noted above, we draw on courts' rulings to assess whether this is the case or not. Moreover, measurement error is unlikely to be a major concern for our RDD analyses, as there is no clear theoretical reason to expect a discontinuous change in misclassification at the thresholds. Still, we check the robustness of our findings by using a stricter definition of attacks based on courts' rulings.

B.2 Alternative Measure of Campaign Attacks Based on Courts' Rulings

Our main dataset was constructed based on 4,306 right of reply lawsuits gathered from the databases of Brazil's Regional Electoral Courts. Of these lawsuits, we were able to successfully retrieve court rulings for 3,575 cases, representing 83% of our initial sample. These decisions were then classified into three categories: (i) favorable to the plaintiff, (ii) unfavorable to the plaintiff, and (iii) dismissed without a judgment on the merits. For most lawsuits, the classification was carried out based on the "case summary" available on the courts' websites, which typically includes details about the judge's decision. Whenever the courts' rulings could not be directly inferred from these summaries, we performed text analysis on the full decisions to identify expressions clearly indicating the nature of the outcome.³

³The analysis was facilitated by the fact that judges usually reiterate their decisions at the end of their rulings, often using expressions such as "I rule in favor" ("*julgo procedente*"), "I rule against" ("*julgo improcedente*"), and "I dismiss the case without judgment on the merits" ("*declaro extinto sem julgamento de mérito*").

Overall, we identified 798 cases that were terminated without a judgment on the merits. These lawsuits typically involved procedural mistakes—such as missed deadlines—or were filed too close to the election date to allow for a timely resolution. Among the remaining cases, 1,141 were classified as favorable to plaintiffs and 1,636 as unfavorable.⁴ Therefore, conditional on a judgment addressing the substance of the case, 41.1% of the decisions were favorable to plaintiffs.

In Panel A of Figure B.1 we plot the proportion of lawsuits decided in favor of and against the plaintiff in the weeks preceding the election. Note that the shares of each type of decision remain stable throughout the campaign period, suggesting no substantial changes in the quality of lawsuits as the Election Day approaches. Moreover, Panel B of Figure B.1 shows a similar pattern for the 2012 and 2016 elections, with the distribution of decisions remaining stable across years.

As discussed in the main text, we expect lawsuits resulting in favorable rulings for plaintiffs to be associated with the harshest attacks.⁵ In this appendix, we test the robustness of our findings by adopting a stricter definition of campaign attacks, whereby an attack is classified as having occurred only when an RR lawsuit results in a favorable decision. In Table B.1 we report summary statistics for this alternative measure, based on our sample of ordered pairs of candidates. Note that attacks are observed in 0.9% of ordered candidate pairs. Moreover, the frequency increases to 1.8% in two-candidate races, but declines to 1.0% and 0.6% in races with three and four or more candidates, respectively. Finally, the likelihood of an attack is 0.8% under single-ballot plurality and 1.5% under dual-ballot plurality.

Next, in Figure B.2 we show that, under our stricter definition, the pattern of attacks closely resembles the baseline results reported in Figure 1, both based on a 5% threshold to define effective candidates. Specifically, in two-candidate races, we find that the frequency of attacks $2 \rightarrow 1$ (2.2 pp) is higher than $1 \rightarrow 2$ (1.4 pp) (diff = 0.8 pp, p-value < 0.01), while in three-candidate races the most likely directions are $2 \rightarrow 1$ (3.0 pp), $1 \rightarrow 2$ (1.9 pp) and $3 \rightarrow 1$ (0.8 pp).⁶ These patterns suggest that systematic differences in lawsuit quality across ordered candidate pairs are unlikely to drive our main results.

To further examine the role of measurement error in our analysis, we use courts’ rulings to approximate false-positive rates across candidate pairs, defining a “false positive” as a case in which an RR lawsuit was filed but ultimately dismissed. Figure B.3 reports false-positive rates (conditional on a lawsuit being filed) across ordered pairs of candidates in both two and three-candidate races, based on a 5% threshold to define effective candidates. Consistent with our previous discussion, 2nd place candidates appear more prone to over-filing, resulting in higher false-positive rates for attacks from candidate 1 against 2—where 2nd place candidates are the plaintiffs—relative to attacks in the opposite direction. Specifically, in two-candidate races, we find that the frequency of false-positives associated with attacks $1 \rightarrow 2$ (70.3 pp) is higher than $2 \rightarrow 1$ (64.9 pp) (diff = 5.4 pp, p-value = 0.11), with a similar pattern also arising in three-candidate races, where false-positive rates are higher for attacks $1 \rightarrow 2$ (70.5 pp) than $2 \rightarrow 1$ (58.0 pp) (diff = 12.5 pp, p-value = 0.02).⁷

Finally, in Figure B.4 we report false-positive rates according to plaintiff characteristics—recalling that in an attack $i \rightarrow j$, the plaintiff is candidate j . We find no significant differences across candidate characteristics, except for campaign expenditures. Interestingly, candidates with lower campaign spending exhibit slightly higher false-positive rates, suggesting that resource-constrained candidates may either face lower success rates due to limited capacity or be prone to over-filing as a way to compensate for their disadvantaged position.⁸ Overall, our results suggest that, if anything, misclassification is more likely to attenuate rather than inflate our estimates regarding the pattern of campaign attacks. In what follows, we present the results from robustness exercises using our stricter measure of campaign attacks, reported separately for each part of the analysis.

Campaign Attacks: General Patterns. We begin by reporting in Tables B.2–B.4 the results of

⁴These decisions also include a small number of cases in which a preliminary ruling (“*decisão liminar*”) was in favor of or against the plaintiff, but the procedure was later terminated without a final judgement due to lack of interest in pursuing the case.

⁵Note, however, that an unfavorable decision does not imply that no attack occurred, but rather that its severity may have been insufficient to warrant a right of reply.

⁶The difference between the frequency of attacks $2 \rightarrow 1$ and $1 \rightarrow 2$ is 1.1 pp (p-value < 0.01), the difference between $2 \rightarrow 1$ and $3 \rightarrow 1$ is 2.2 pp (p-value < 0.01) and the difference between $1 \rightarrow 2$ and $3 \rightarrow 1$ is 1.1 pp (p-value < 0.01).

⁷Note that, excluding attacks $2 \rightarrow 1$, mean comparisons of the frequency of false positives across the remaining directions yield no statistically significant differences.

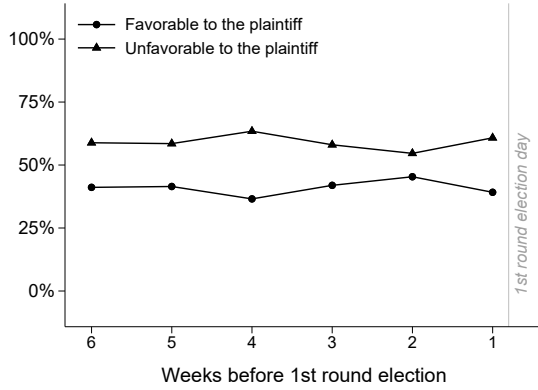
⁸Importantly, all results reported in this Appendix are robust to the inclusion or exclusion of campaign spending from the set of controls, with point estimates varying only marginally across specifications.

our analysis of the general patterns of campaign attacks. Consistent with our main findings, the point estimates obtained based on our alternative measure are in line with the predictions of the model. Specifically, in Table B.2 we show that in two-candidate races the likelihood of an attack $2 \rightarrow 1$ is approximately 0.67 pp higher than that of an attack $1 \rightarrow 2$, which corresponds to an increase of 37.2% relative to the sample mean (see column 1, based on a 5% threshold). Moreover, in Table B.3 we find that in three-candidate races the most likely directions of attacks are, in order, $2 \rightarrow 1$, $1 \rightarrow 2$, and $3 \rightarrow 1$. In particular, an attack $2 \rightarrow 1$ is about 2.59 pp more likely to occur than $2 \rightarrow 3$ (omitted category), which corresponds to an increase of 135.4% relative to the sample mean (see column 1, based on a 5% threshold), while attacks $1 \rightarrow 2$ and $3 \rightarrow 1$ are, respectively, 1.45 pp and 0.56 pp more likely to occur. Finally, our findings in Table B.4 are consistent with the “dilution effect”, with estimates of the interactions with the vote shares of 3rd place candidates being consistently negative and generally statistically significant, particularly for attacks originating from 2nd place candidates. Overall, our main results regarding the general patterns of attacks are robust to using a stricter measure of campaign attacks. Interestingly, we find that while the point estimates are generally smaller in absolute terms, their magnitudes are actually larger relative to the sample means.

Electoral Advantage Effect on Campaign Attacks. Next, we turn to the analysis of the electoral advantage effect using our alternative measure of campaign attacks. In Panels A–C of Figure B.5 and Panel A of Table B.5, we reproduce, for reference, our previously reported results regarding the runner-up effect on subsequent electoral performance, namely candidacy in $t + 1$, vote share in $t + 1$, and likelihood of winning in $t + 1$. These analyses are identical to those presented before. In Panel D of Figure B.5, we then plot the frequency of campaign attacks received in the subsequent election against the vote share difference between 2nd and 3rd place candidates. The graph displays a clear discontinuity at the cutoff, suggesting that barely 2nd place candidates are about 1 pp more likely to receive a “harsh” attack relative to close 3rd place candidates. Consistent with the graphical evidence, the regression analysis reported in Panel B of Table B.5 shows that close runners-up are about 0.9 pp more likely to receive an attack in the next election, which corresponds to an increase of approximately 225% relative to the 3rd place candidates’ mean (see column 1). While the point estimates are less precisely estimated in this case, due to attacks occurring less frequently under our stricter definition, the effects remain highly stable across all specifications reported in columns 2–4, with magnitudes that are actually larger relative to the 3rd place candidates’ mean compared to our main findings.

Single vs. Dual Ballot Plurality. Finally, we turn to the analysis of the impact of a change in the voting rule from single to dual ballot plurality using our alternative measure of campaign attacks. Consistent with previous results, the estimates reported in Table B.6 show that 3rd place candidates become substantially more aggressive under dual ballot plurality. Specifically, according to our preferred specification (column 1), the likelihood of an attack originating from a 3rd place candidate increases by about 6.3 pp, which represents a 165% increase relative to the single ballot mean. Moreover, in line with our previous findings, in Table B.7 we show that the likelihood of attacks between 2nd and 3rd place candidates increases by about 7.5 pp under dual ballot plurality (column 1), with the estimates also suggesting a rise in the frequency of attacks between 1st and 3rd place candidates. Overall, our results are robust to a range of alternative specifications (see columns 2–5, Tables B.6 and B.7). Importantly, the estimated effects remain substantial when compared to the single ballot means and are very similar, in percentage terms, to our main results.

Panel A: Full sample



Panel B: By election year

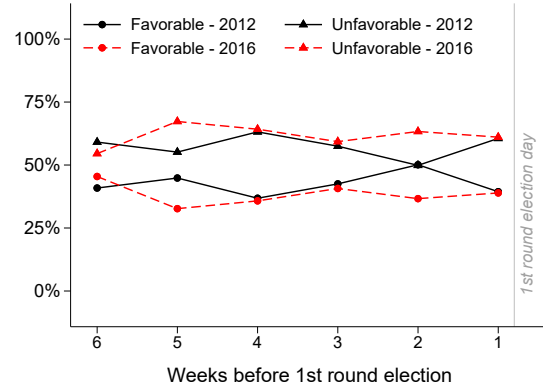


Figure B.1: Right of Reply Lawsuits: Courts' Rulings

Notes: This figure plots the fraction of right of reply lawsuits decided in favor or against the plaintiff in the weeks preceding the Election Day for the full sample (Panel A) and by election year (Panel B).

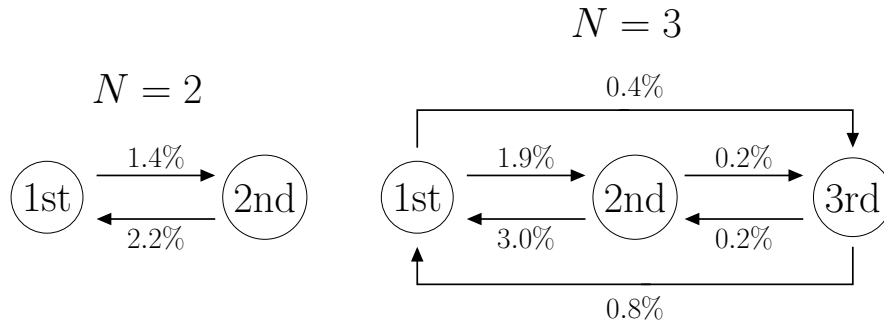


Figure B.2: Frequency of Campaign Attacks (Alternative Measure of Attacks)

Notes: This figure shows the frequency of campaign attacks using our alternative measure based on the courts' rulings in races with two (left panel) and three effective candidates (right panel), using a 5% threshold for the definition of effective candidate and focusing only on races held under single ballot plurality.

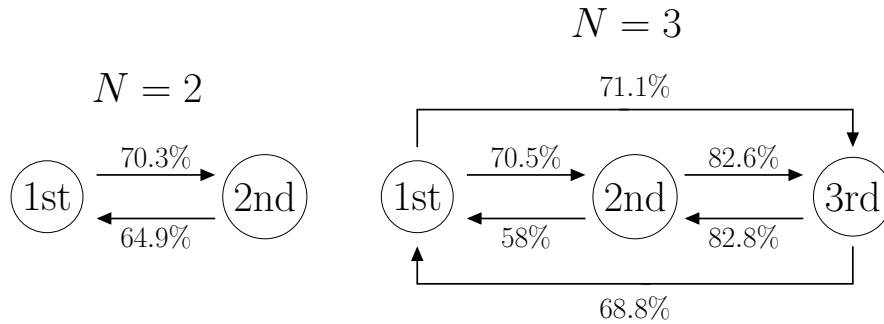


Figure B.3: Frequency of "False Positives"

Notes: This figure shows the frequency of "false positives", defined as a case in which a right of reply lawsuit was filed but ultimately decided against the plaintiff. The analysis is conditional on a lawsuit being filed, using a 5% threshold to define effective candidates and focusing only on races held under single ballot plurality.

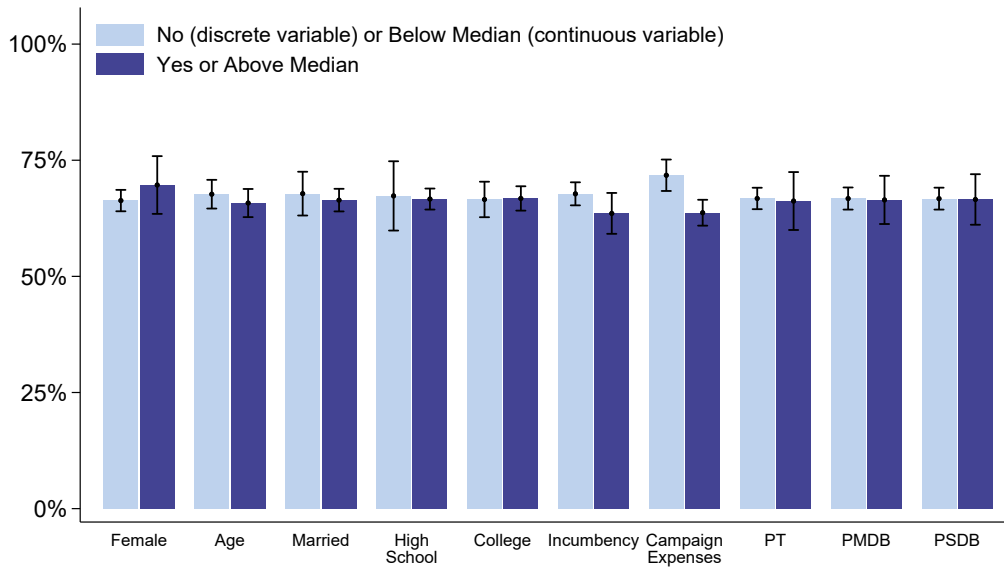


Figure B.4: False-Positive Rates and Plaintiff Characteristics

Notes: This figure reports the frequency of “false positives” according to the following plaintiff characteristics: gender, age, marital status, high school and college education, incumbency status, campaign expenses (above and below median), and party affiliation with PT, PMDB, and PSDB. 95% confidence intervals are reported.

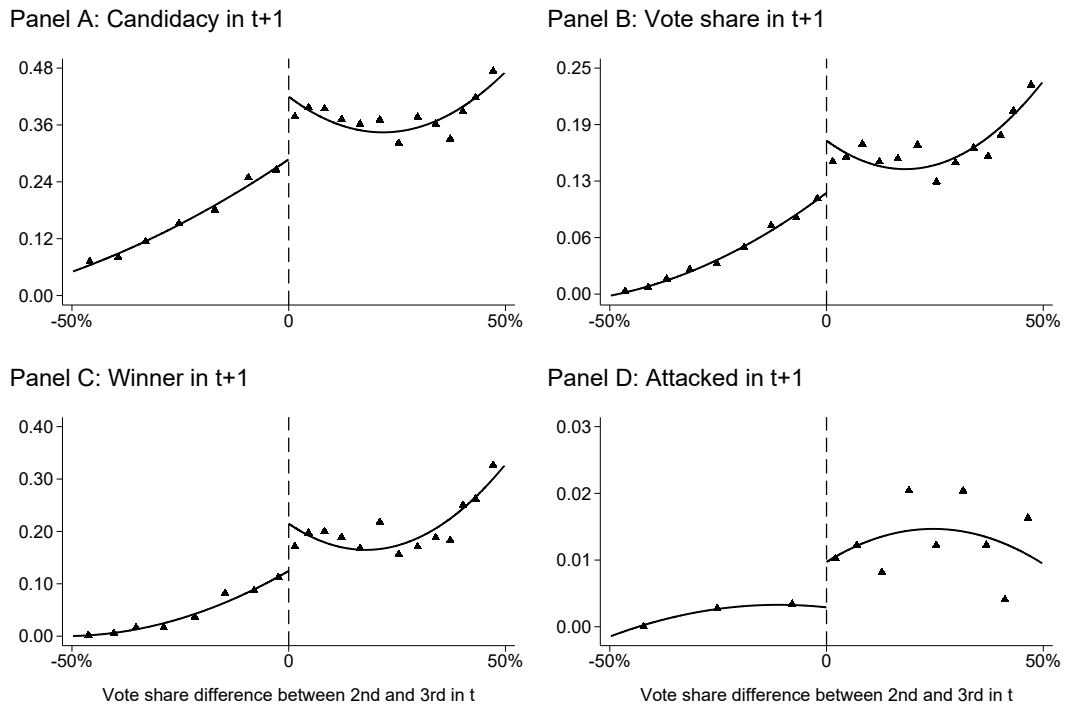


Figure B.5: Runner-Up Effect and Electoral Advantage Effect on Campaign Attacks (Alternative Measure)

Notes: This figure plots local averages of candidacy in $t + 1$ (Panel A), vote-share in $t + 1$ (Panel B), winner in $t + 1$ (Panel C), and campaign attack (alternative measure) in $t + 1$ (Panel D). The sample includes only candidates who placed 2nd and 3rd in the election at t , focusing only on races held under single ballot plurality. For additional details, see footnote to Figure 2.

Table B.1: Summary Statistics: Alternative Measure of Campaign Attacks

	Mean	Std. Dev.	N
Campaign Attack	0.009	0.093	69,252
- Campaign Attack Races with 2 Cands.	0.018	0.132	10,808
- Campaign Attack Races with 3 Cands.	0.010	0.098	18,018
- Campaign Attack Races with 4+ Cands.	0.006	0.077	40,426
- Campaign Attack Single Ballot	0.008	0.089	62,298
- Campaign Attack Dual Ballot	0.015	0.123	6,954

Notes: This table reports descriptive statistics for our sample of ordered pairs of candidates. The variable “Campaign Attack” is a dummy indicating whether an RR lawsuit with a favorable decision to the plaintiff is observed. Descriptive statistics for this dummy are reported for the whole sample and for selected subsamples.

Table B.2: Patterns of Campaign Attacks: Two-Candidate Races (Alternative Measure)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
2nd against 1st (β_{21})	0.0067*** [0.0020]	0.0088*** [0.0020]	0.0096*** [0.0019]	0.0091*** [0.0019]
Effective Candidate Threshold	5%	10%	15%	20%
Dep. Var. Mean	0.018	0.020	0.021	0.020
No. of Electoral Races	6,913	7,668	8,238	8,726
Observations	13,826	15,336	16,476	17,452
Adj. R-squared	0.239	0.234	0.228	0.205

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack, according to our alternative measure, took place between candidates in an ordered pair. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 2.

Table B.3: Patterns of Campaign Attacks: Three-Candidate Races (Alternative Measure)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
1st against 2nd (β_{12})	0.0145*** [0.0027]	0.0112*** [0.0028]	0.0075*** [0.0029]	0.0086** [0.0038]
1st against 3rd (β_{13})	0.0026* [0.0013]	0.0024 [0.0016]	0.0024 [0.0020]	0.0019 [0.0028]
2nd against 1st (β_{21})	0.0259*** [0.0034]	0.0188*** [0.0035]	0.0108*** [0.0032]	0.0076** [0.0037]
3rd against 1st (β_{31})	0.0056*** [0.0017]	0.0052*** [0.0019]	0.0058** [0.0023]	0.0040 [0.0028]
3rd against 2nd (β_{32})	-0.0003 [0.0013]	0.0006 [0.0016]	0.0001 [0.0018]	-0.0009 [0.0022]
$H_0 : \beta_{21} = \beta_{12}$	0.002	0.037	0.329	0.793
$H_0 : \beta_{21} = \beta_{31}$	0.000	0.000	0.172	0.368
$H_0 : \beta_{21} = 0$	0.000	0.000	0.001	0.041
$H_0 : \beta_{12} = \beta_{13}$	0.000	0.003	0.096	0.085
$H_0 : \beta_{31} = \beta_{32}$	0.005	0.052	0.027	0.107
Effective Candidate Threshold	5%	10%	15%	20%
Dep. Var. Mean	0.011	0.009	0.007	0.007
No. of Electoral Races	2,482	2,026	1,591	998
Observations	14,884	12,152	9,542	5,984
Adj. R-squared	0.110	0.058	0.065	0.077

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack, according to our alternative measure, took place between candidates in an ordered pair. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 3.

Table B.4: Patterns of Campaign Attacks: Dillution Effect (Alternative Measure)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
2nd against 1st (β_{21})	0.0216*** [0.0046]	0.0315*** [0.0062]	0.0355*** [0.0087]	0.0356*** [0.0133]
1st against 2nd $\times$ Share 3rd (π_{12})	-0.0646 [0.0403]	-0.0678 [0.0486]	-0.1033 [0.0690]	-0.1320 [0.0935]
2nd against 1st $\times$ Share 3rd (π_{21})	-0.1357*** [0.0437]	-0.1914*** [0.0526]	-0.2453*** [0.0782]	-0.2752*** [0.1017]
Effective Candidate Threshold	0%	2%	5%	10%
Dep. Var. Mean	0.022	0.023	0.022	0.017
No. of Electoral Races	4,609	3,923	3,114	2,311
Observations	9,218	7,846	6,228	4,622
Adj. R-squared	0.148	0.148	0.157	0.134

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack, according to our alternative measure, took place between candidates in an ordered pair. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 4.

Table B.5: Runner-Up Effect and Electoral Advantage Effect on Campaign Attacks (Alternative Measure)

	Local Linear Regression			Alternative Specifications			
	3rd place Mean	Optimal BW	Obs	(1)	(2)	(3)	(4)
Panel A: Candidate Performance							
Candidacy in $t + 1$	0.249	0.162	3,032	0.116*** [0.034]	0.116*** [0.032]	0.120*** [0.040]	0.119*** [0.031]
Vote Share in $t + 1$	0.088	0.178	3,264	0.042*** [0.014]	0.043*** [0.014]	0.041** [0.018]	0.044*** [0.014]
Winner in $t + 1$	0.094	0.183	3,330	0.066*** [0.024]	0.066*** [0.024]	0.039 [0.034]	0.064*** [0.023]
Panel B: Campaign Attacks							
Attacked in $t + 1$	0.004	0.140	2,686	0.009 [0.007]	0.010 [0.007]	0.007 [0.009]	0.007 [0.006]
Bandwidth Method				CCT	CCT	CCT	0.05
Polynomial Order				1	1	2	0
Candidates' characteristics				No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting virtual ties between 2nd and 3rd place candidates. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 5.

Table B.6: Single vs Dual Ballot Plurality (Alternative Measure)

	Local Linear Regression			Alternative Specifications				
	SB Mean	Optimal BW	Obs	(1)	(2)	(3)	(4)	(5)
Panel A: Full Sample								
All pairs	0.042	0.308	792	0.024 [0.030]	0.025 [0.024]	0.077*** [0.029]	0.038 [0.033]	0.039* [0.023]
Panel B: By Attacking Candidate								
1st place Candidate	0.041	0.221	184	-0.003 [0.023]	-0.030 [0.020]	0.031 [0.023]	0.033* [0.017]	-0.013 [0.017]
2nd place Candidate	0.062	0.262	230	0.049 [0.063]	0.040 [0.065]	0.021 [0.062]	0.065 [0.081]	0.068 [0.051]
3rd place Candidate	0.038	0.268	234	0.063 [0.051]	0.058* [0.033]	0.072*** [0.027]	0.065 [0.057]	0.061 [0.043]
Bandwidth Method				CCT	CCT	CCT	CCT	0.10
Polynomial Order				1	1	1	2	0
Municipal characteristics				No	Yes	No	No	No
Candidates' characteristics				No	No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality at the 200,000-voter threshold, using the running variable in logs. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. For additional details, see footnote to Table 6.

Table B.7: Single vs Dual Ballot Plurality: Heterogeneous Effects by Pairs (Alternative Measure)

	Linear specification			Alternative specifications				
	SB mean	Optimal BW value	Obs	(1)	(2)	(3)	(4)	(5)
1st VS 2nd	0.093	0.241	210	-0.046 [0.054]	-0.075 [0.054]	0.038 [0.060]	-0.025 [0.046]	-0.029 [0.043]
1st VS 3rd	0.010	0.223	186	0.065 [0.042]	0.053** [0.026]	0.099*** [0.028]	0.079 [0.048]	0.059* [0.033]
2nd VS 3rd	0.015	0.263	230	0.075 [0.061]	0.082 [0.059]	0.097* [0.055]	0.106 [0.073]	0.086* [0.051]
Bandwidth method				CCT	CCT	CCT	CCT	0.10
Polynomial order				1	1	1	2	0
Municipal characteristics				No	Yes	No	No	No
Candidates' characteristics				No	No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality at the 200,000-voter threshold, using the running variable in logs. All specifications include election-year fixed effects and use a triangular kernel with the Calonico et al. (2014) optimal bandwidth. For additional details, see footnote to Table 7.

C General Patterns of Campaign Attacks: Predicted (“*Ex-Ante*”) Ranks

In our analysis of the general patterns of campaign attacks, we relied on “ex-post” candidate ranks to test the main predictions of our model. As discussed in the main text, this approach may be subject to a reverse causality problem in that campaign attacks could plausibly influence the final position of candidates in a given race. To address this potential concern, in this appendix, we employ a random forest machine-learning algorithm to predict the vote shares of candidates based on a parsimonious model that incorporates a selected set of pre-determined variables. Specifically, we consider as predictors the vote shares obtained by the candidates’ coalitions in the preceding state and federal legislative elections, held two years prior to the municipal elections. Additionally, based on the 2010 Population Census, we include the logarithm of population, logarithm of income per capita, illiteracy rate, and income inequality (Gini), as well as five regional dummies (North, North-East, Center-West, South-East, and South). Finally, we also include candidate-specific characteristics from our baseline specifications in equations (9)–(11), namely: gender, age, age squared, marital status, high school and college degrees, incumbency status, and affiliation with one of the three main parties (PT, PSDB, and PMDB).

The random forest model is an ensemble tree-based learning algorithm that averages predictions over many individual trees, each constructed using bootstrap samples and a random subset of the predictors. This approach enhances variability among trees and helps reduce overfitting in the training process (Breiman, 2001; Athey and Imbens, 2019). Our analysis implements Schonlau and Zou (2020)’s random forest algorithm and is applied to a sample of 29,593 candidates who participated in the 2012 and 2016 municipal elections.^{9,10} The model is used to predict the candidates’ vote shares, from which we derive their expected (“ex-ante”) positions. We then use these predicted ranks to re-estimate the specifications in equations (9)–(11), employing the predicted candidate-pair positions either directly in reduced-form regressions or as instruments in 2SLS specifications. Throughout the analysis, we use the forecasted vote shares to determine the number of effective candidates and apply the corresponding sample restrictions.

Overall, the random forest model performs quite well, yielding a correlation of 0.89 between predicted and actual vote shares. In Figure C.1 we plot the relative importance of the variables included in our model, with scores normalized such that the most predictive feature is set to 1. Interestingly, the variables with the highest predictive power are incumbency status and coalition performance in previous federal and state legislative elections, underscoring the importance of past electoral success in determining local election outcomes in Brazil.

We begin our analysis by presenting results from specifications that use predicted candidate-pair ranks directly in a reduced-form approach (Tables C.1–C.3). Specifically, in Table C.1 we show that 2nd place candidates remain the most aggressive in two-candidate races, with point estimates that are consistently positive and statistically significant—albeit slightly smaller in magnitude than those reported in Table 2. Moreover, consistent with our previous results, in Table C.2 we find that in three-candidate races the most likely directions of attack are, in order, $2 \rightarrow 1$, $1 \rightarrow 2$ and $3 \rightarrow 1$, with point estimates very similar to those reported in Table 3. Finally, consistent with the “dilution effect”, in Table C.3 we show that an increase in the predicted vote shares of (predicted) 3rd place candidates reduces the likelihood of an attack between the two (predicted) front-runners. In particular, in line with the results reported in Table 4, the point estimates for both interactions are always negative, though statistically significant only for attacks originating from 2nd place candidates.

Next, in Tables C.4–C.6 we report results from 2SLS regressions using predicted candidate-pair ranks as instruments for actual (“ex-post”) positions. As before, we find a pattern of campaign attacks that is consistent with our main results. In particular, for two-candidate races, the 2SLS estimates reported in Table C.4 are very close to those presented in Table 2. Similarly, the results for three-candidate races reported in Table C.5 are broadly consistent with those in Table 3, though the corresponding coefficient estimates are generally larger.¹¹ Moreover, using predicted candidate-pair ranks and their interactions

⁹This includes all candidates in our main sample for whom we have complete information on all predictors used in the random forest model.

¹⁰Following a common approach in the literature, we divide the sample into training and testing sets using a 70-30 split and tune the model’s hyperparameters to maximize testing accuracy, following the procedure suggested by Schonlau and Zou (2020). All our results remain robust when using a 50-50 split.

¹¹Note that while the first-stage F-statistics are still high in three-candidate races—particularly in columns (1) and (2) of Table C.5—they are considerably lower than those reported for two-candidate races in Table C.4. This is likely due to the fact that predicting candidate rankings becomes more difficult as the number of candidates increases and races become more competitive. Indeed, note that the first-stage F-statistic for the regression using

with predicted vote shares of (predicted) 3rd place candidates as instruments, Table C.6 reports results consistent with the “dilution effect”, with the point estimates for both interaction terms being always negative, though statistically significant only for attacks originating from 2nd place candidates.

Finally, in a supplemental robustness exercise (available upon request), we show that the results reported in Tables C.1–C.6 remain unchanged when we control for the vote shares received by candidates’ coalitions in the preceding state and federal legislative elections to account for their potential direct influence on the likelihood of attacks. Moreover, we also show that our main results reported in Tables 2 and 3 are robust to controlling for the candidates’ performance during the campaign (relative to their baseline popularity) by including two dummy variables that capture whether the attacking and attacked candidates outperformed their predicted vote shares. The results (available upon request) show that the patterns of attacks remains unchanged, with the point estimates associated with outperformance being close to zero and statistically insignificant.

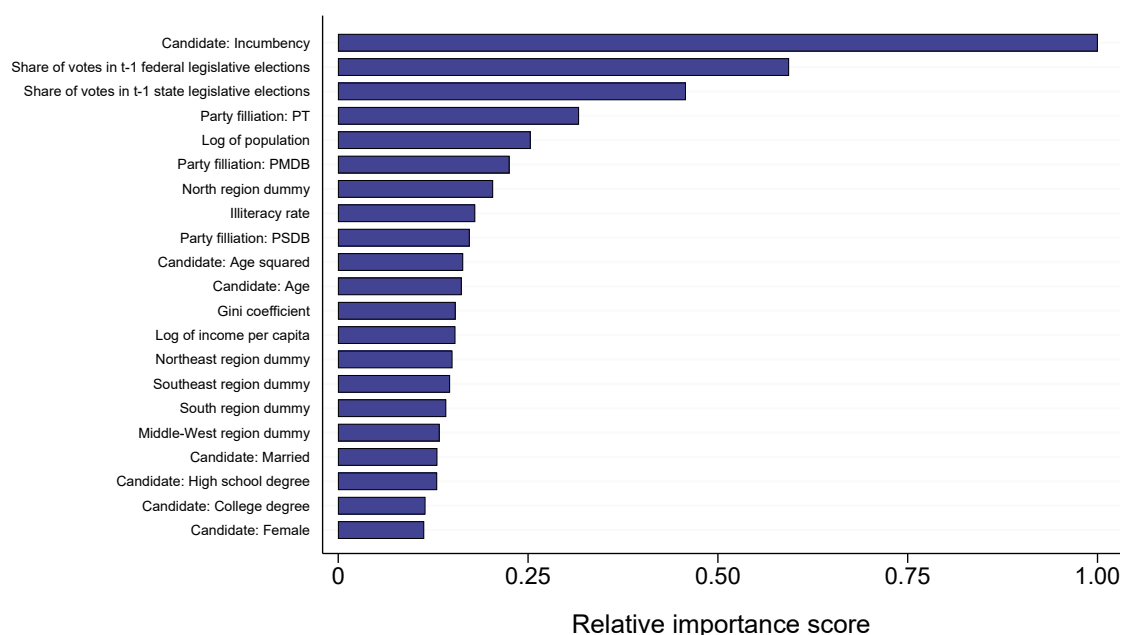


Figure C.1: Importance Scores of Predictors for Candidates’ Final Vote Shares

Notes: This figure plots the relative importance scores of all predictor variables in the Random Forest algorithm. The analysis includes all candidates who participated in the 2012 and 2016 municipal elections. The algorithm is implemented using the *rforest* Stata command (Schonlau and Zou, 2020).

a 5% threshold is 128.7 (Table C.5, column 1) and declines steadily as the threshold increases (Table C.5, columns 2–4).

Table C.1: Patterns of Campaign Attacks: Two-Candidate Races (Predicted Ranks, Reduced-Form)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
2nd against 1st (β_{21})	0.0075* [0.0039]	0.0096** [0.0038]	0.0092** [0.0037]	0.0101*** [0.0035]
Effective Candidate Threshold	5%	10%	15%	20%
Dep. Var. Mean	0.058	0.062	0.062	0.063
No. of Electoral Races	5,017	5,614	6,229	6,699
Observations	10,034	11,227	12,456	13,389
Adj. R-squared	0.326	0.336	0.310	0.294

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair. The sample is restricted to include only observations (ordered pairs of candidates) from races with two effective candidates and considers only the pairs formed by the top two candidates. Forecasted vote shares are used to determine the number of effective candidates and apply the corresponding sample restrictions. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 2.

Table C.2: Patterns of Campaign Attacks: Three-Candidate Races (Predicted Ranks, Reduced-Form)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
1st against 2nd (β_{12})	0.0474*** [0.0047]	0.0437*** [0.0049]	0.0367*** [0.0048]	0.0315*** [0.0055]
1st against 3rd (β_{13})	0.0099*** [0.0023]	0.0096*** [0.0027]	0.0103*** [0.0030]	0.0086*** [0.0038]
2nd against 1st (β_{21})	0.0618*** [0.0054]	0.0515*** [0.0054]	0.0466*** [0.0057]	0.0355*** [0.0064]
3rd against 1st (β_{31})	0.0160*** [0.0032]	0.0207*** [0.0035]	0.0201*** [0.0038]	0.0221*** [0.0049]
3rd against 2nd (β_{32})	0.0029 [0.0022]	0.0041 [0.0027]	0.0064** [0.0029]	0.0069** [0.0035]
$H_0 : \beta_{21} = \beta_{12}$	0.009	0.154	0.079	0.524
$H_0 : \beta_{21} = \beta_{31}$	0.000	0.000	0.000	0.026
$H_0 : \beta_{21} = 0$	0.000	0.000	0.000	0.000
$H_0 : \beta_{12} = \beta_{13}$	0.000	0.000	0.000	0.000
$H_0 : \beta_{31} = \beta_{32}$	0.000	0.000	0.001	0.004
Effective Candidate Threshold	5%	10%	15%	20%
Dep. Var. Mean	0.030	0.030	0.028	0.027
No. of Electoral Races	3,047	2,856	2,422	1,818
Observations	16,553	15,340	12,745	9,336
Adj. R-squared	0.158	0.158	0.167	0.158

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair. The sample is restricted to include only observations (ordered pairs of candidates) from races with three effective candidates and considers only the pairs formed by the top three candidates. Forecasted vote shares are used to determine the number of effective candidates and apply the corresponding sample restrictions. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 3.

Table C.3: Patterns of Campaign Attacks: Dilution Effect (Predicted Ranks, Reduced-Form)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
2nd against 1st (β_{21})	0.0352*** [0.0101]	0.0361*** [0.0101]	0.0318*** [0.0117]	0.0128 [0.0142]
1st against 2nd $\times$ Share 3rd (π_{12})	-0.1151 [0.0724]	-0.1122 [0.0724]	-0.1365* [0.0775]	-0.0498 [0.0844]
2nd against 1st $\times$ Share 3rd (π_{21})	-0.2295*** [0.0763]	-0.2310*** [0.0762]	-0.2359*** [0.0810]	-0.0651 [0.0821]
Effective Candidate Threshold	0%	2%	5%	10%
Dep. Var. Mean	0.064	0.064	0.059	0.052
No. of Electoral Races	4,133	4,132	3,717	3,082
Observations	8,174	8,172	7,346	6,090
Adj. R-squared	0.244	0.246	0.247	0.268

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair. The sample is restricted to include only observations (ordered pairs of candidates) from races with three or more effective candidates and considers only the pairs formed by the top two candidates. Forecasted vote shares are used to determine the number of effective candidates and apply the corresponding sample restrictions. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 4.

Table C.4: Patterns of Campaign Attacks: Two-Candidate Races (Predicted Ranks, 2SLS)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
2nd against 1st (β_{21})	0.0117* [0.0061]	0.0152** [0.0060]	0.0147** [0.0059]	0.0166*** [0.0057]
Effective Candidate Threshold	5%	10%	15%	20%
Dep. Var. Mean	0.058	0.062	0.062	0.063
No. of Electoral Races	5,017	5,614	6,229	6,699
Observations	10,034	11,227	12,456	13,389
First stage F-statistic	3240.1	3632.4	3950.0	4047.9

Notes: This table reports 2SLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair, using predicted candidate-pair ranks as instruments for actual candidate-pair ranks. The sample is restricted to include only observations (ordered pairs of candidates) from races with two effective candidates and considers only the pairs formed by the top two candidates. Forecasted vote shares are used to determine the number of effective candidates and apply the corresponding sample restrictions. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 2.

Table C.5: Patterns of Campaign Attacks: Three-Candidate Races (Predicted Ranks, 2SLS)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
1st against 2nd (β_{12})	0.0561*** [0.0071]	0.0561*** [0.0078]	0.0509*** [0.0085]	0.0526*** [0.0109]
1st against 3rd (β_{13})	0.0101* [0.0059]	0.0076 [0.0076]	0.0116 [0.0098]	0.0078 [0.0136]
2nd against 1st (β_{21})	0.0827*** [0.0086]	0.0683*** [0.0091]	0.0695*** [0.0111]	0.0535*** [0.0144]
3rd against 1st (β_{31})	0.0158*** [0.0058]	0.0252*** [0.0071]	0.0248*** [0.0087]	0.0341*** [0.0129]
3rd against 2nd (β_{32})	0.0028 [0.0044]	0.0014 [0.0060]	0.0076 [0.0077]	0.0065 [0.0107]
$H_0 : \beta_{21} = \beta_{12}$	0.017	0.307	0.192	0.962
$H_0 : \beta_{21} = \beta_{31}$	0.000	0.000	0.002	0.320
$H_0 : \beta_{21} = 0$	0.000	0.000	0.000	0.000
$H_0 : \beta_{12} = \beta_{13}$	0.000	0.000	0.004	0.012
$H_0 : \beta_{31} = \beta_{32}$	0.080	0.013	0.173	0.129
Effective Candidate Threshold	5%	10%	15%	20%
Dep. Var. Mean	0.030	0.030	0.028	0.027
No. of Electoral Races	3,047	2,856	2,422	1,818
Observations	16,553	15,340	12,745	9,336
First stage F-statistic	128.7	94.3	56.5	31.0

Notes: This table reports 2SLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair, using predicted candidate-pair ranks as instruments for actual candidate-pair ranks. The sample is restricted to include only observations (ordered pairs of candidates) from races with three effective candidates and considers only the pairs formed by the top three candidates. Forecasted vote shares are used to determine the number of effective candidates and apply the corresponding sample restrictions. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 3.

Table C.6: Patterns of Campaign Attacks: Dillution Effect (Predicted Ranks, 2SLS)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
2nd against 1st (β_{21})	0.0645*** [0.0193]	0.0662*** [0.0192]	0.0651*** [0.0244]	0.0205 [0.0288]
1st against 2nd $\times$ Share 3rd (π_{12})	-0.2331 [0.2583]	-0.2190 [0.2576]	-0.3022 [0.2922]	-0.1571 [0.3542]
2nd against 1st $\times$ Share 3rd (π_{21})	-0.6023** [0.2578]	-0.6027** [0.2577]	-0.6736** [0.2892]	-0.1503 [0.3450]
Effective Candidate Threshold	0%	2%	5%	10%
Dep. Var. Mean	0.064	0.064	0.059	0.052
No. of Electoral Races	4,133	4,132	3,717	3,082
Observations	8,174	8,172	7,346	6,090
First stage F-statistic	48.0	48.0	28.8	10.7

Notes: This table reports 2SLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair, using predicted candidate-pair ranks and the predicted vote shares of (predicted) 3rd place candidates as instruments for actual candidate-pair ranks and the vote shares of 3rd place candidates. Forecasted vote shares are used to determine the number of effective candidates and apply the corresponding sample restrictions. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 4.

D General Patterns of Campaign Attacks: Additional Robustness Checks and Heterogeneous Effects

In this appendix, we further examine the general patterns of campaign attacks by conducting additional robustness checks and investigating heterogeneous effects across races that differ in electoral and socioeconomic characteristics. We begin by presenting in Table D.1 the results from alternative specifications restricting the sample to races with exactly two or three candidates (columns 1 and 3) or applying a 2% threshold (columns 2 and 4) to define effective candidates. As mentioned in the main text, the point estimates show that our main results are robust to the choice of threshold. Specifically, in two-candidate races (columns 1 and 2), the estimated effects remain positive and statistically significant, albeit somewhat smaller in magnitude. Conversely, in three-candidate races (columns 3 and 4), the point estimates are somewhat larger in magnitude, and the pattern of attacks becomes more pronounced—allowing us to reject all null hypotheses concerning coefficient differences in both specifications.

Next, in Table D.2 we employ an alternative specification to test the “dilution effect”, using the vote share difference between 2nd and 3rd place candidates (*Distance 2nd-3rd*) as the interaction variable. Consistent with our main results, we find that both interaction terms are positive and statistically significant, suggesting that a larger gap between 2nd and 3rd place candidates increases the level of aggressiveness between the two front-runners. Specifically, a 10 pp increase in the vote share difference between 2nd and 3rd place candidates is associated with a 1.6 pp increase in the frequency of attacks $1 \rightarrow 2$ and a 2 pp increase in attacks $2 \rightarrow 1$. Similarly to our main results, we find that the magnitude of both interaction terms increases as the threshold used to define effective candidates rises.¹²

In Table D.4, we investigate the conjecture that the dilution effect is stronger when candidates 1 and 2 lie on opposite sides of the ideological spectrum, in which case a larger share of the votes is expected to leak to the third candidate (see our discussion in footnote 15 and simulation results reported in Table G.3). To empirically test this hypothesis, we run separate regressions by splitting the sample into “polarized races”—where the top two candidates come from parties on opposite sides of the political spectrum (left versus right)—and “non-polarized races”, based on ideological scores from Zucco and Power (2023). Consistent with our conjecture, the results show that the dilution effect is substantially larger in polarized races (columns 2 and 4). Specifically, according to the point estimates reported in column 2 (based on a 0% threshold), a 10 pp increase in the 3rd place candidate’s vote share is associated with a 6.4 pp decrease in the frequency of attacks $2 \rightarrow 1$ and a 4.4 pp decrease in attacks $1 \rightarrow 2$. Conversely, the effect in non-polarized races is much smaller in magnitude and never statistically significant.

We proceed by examining whether the pattern of campaign attacks varies systematically across races held in different socioeconomic and political environments, motivated by the heterogeneous frequency of attacks documented in Figure A.3. To do so, we re-estimate our main specifications focusing on subsamples of electoral races defined by the median values of selected municipal characteristics. Specifically, we split the sample at the median of the following variables: population, income per capita, income inequality (Gini index), and electoral competitiveness, measured by the inverse of the Herfindahl-Hirschman Index.^{13,14}

In Table D.5 we report estimates for two-candidate races using a 5% threshold to define effective candidates. Interestingly, we find that the pattern of campaign attacks is more pronounced in races held in larger (column 2) and wealthier (column 4) municipalities, where media exposure tends to be more valuable and attacks more frequent. Conversely, we do not observe meaningful differences in the pattern of attacks according to income inequality (columns 5 and 6) or electoral competitiveness (columns 7 and 8). Note that while attacks may be more frequent in highly competitive races, the *relative likelihood* of attacks across ordered pairs of candidates may not necessarily be more pronounced. From

¹²In Table D.3, we obtain similar qualitative results when using the vote share difference between 1st and 3rd place candidates (*Distance 1st-3rd*) as the interaction variable. However, the estimated effects are smaller and less precisely estimated, suggesting that the relative distance between 2nd and 3rd place candidates provides a better measure of the competitive threat posed by the 3rd place candidate, especially from the perspective of the runner-up.

¹³The Herfindahl-Hirschman Index (HHI) is calculated based on candidates’ vote shares s_j and is given by $HHI = \sum_j s_j^2$. The index decreases as vote shares become more evenly distributed, indicating greater electoral competitiveness.

¹⁴We obtain similar results when splitting the sample by urban share, illiteracy, and poverty, as these variables are highly correlated with income per capita.

a theoretical perspective, the relationship between electoral competitiveness and the relative pattern of attacks is difficult to pin down. Interestingly, our empirical analysis sheds light on this issue, suggesting that the typical pattern of attacks remains largely unchanged across races, regardless of their level of competitiveness.

In Table D.6 we report estimates from our analysis of three-candidate races, based on a 5% threshold to define effective candidates. As before, we find that the pattern of attacks is more pronounced in larger (column 2) and wealthier (column 4) municipalities. Moreover, although we do not observe strong systematic differences in the overall pattern of attacks by income inequality and electoral competitiveness, attacks $2 \rightarrow 1$ appear to be more likely in municipalities with higher inequality (column 6) and lower competition (column 7).

Next, we investigate whether the pattern of attacks varies with the timing of the campaign. In Table D.7 we separately examine attacks occurring in the “last two weeks” of the campaign and in the preceding three weeks (“weeks 3, 4, and 5”) for races with two (columns 1 and 2) and three (columns 3 and 4) effective candidates (based on a 5% threshold). Our results suggest that the estimated effects become slightly more pronounced as Election Day approaches. Specifically, in two-candidate races, the relative frequency of attacks $2 \rightarrow 1$ rises from 0.69 pp during weeks 3 – 5 (column 2) to 1.00 pp in the final two weeks of the election (column 1), whereas in three-candidate races it increases from 2.58 pp (column 4) to 3.17 pp (column 3).

Finally, we examine whether the patterns of campaign attacks differ across electoral years by conducting a heterogeneity analysis that splits the sample between the 2012 and 2016 municipal elections. Indeed, our study period was marked by substantial socioeconomic and institutional changes that may have influenced campaign dynamics. These include the 2015 electoral reform, which shortened the official campaign period and banned corporate donations; the rapid expansion of social media and messaging platforms—especially WhatsApp—as tools for campaign communication;^{15,16} and the onset of the Lava Jato investigations, alongside a severe economic and political crisis that culminated in President Dilma Rousseff’s impeachment in August 2016, just months before the municipal elections.

To assess whether these changes affected the patterns of campaign attacks, we re-estimate our main specifications separately for the 2012 and 2016 elections (based on 5% and 10% thresholds). The results reported in Tables D.8 and D.9 show that our main findings are robust across election years. Interestingly, although the estimated effects are generally less pronounced in the 2016 elections, the overall frequency of campaign attacks also declined between 2012 and 2016—possibly due to campaign finance restrictions and other regulations introduced by the 2015 reform.¹⁷ Specifically, in Table D.8 we show that in two-candidate races attacks $2 \rightarrow 1$ were about 1.4 pp more likely to occur in 2012 (column 1) and 1.0 pp in 2016 (column 2), corresponding to effects of approximately 18.6% ($= 1.4/7.5$) and 28.5% ($= 1.0/3.5$) relative to their respective sample means. Similarly, in Table D.9 we find that in three-candidate races attacks $2 \rightarrow 1$ were about 7.8 pp more likely to occur in 2012 (column 1) and 3.3 pp in 2016 (column 2), corresponding to effects of 173% ($= 7.8/4.5$) and 157% ($= 3.3/2.1$) relative to their sample means.¹⁸ As a result, when interpreted relative to sample means, the frequency of attacks across different candidate pairs remained broadly stable between the two elections, suggesting that changes in the political and institutional environment did not substantially affect the patterns of campaign attacks.

¹⁵A potential concern related to our measure of campaign attacks is that it may undercount those disseminated through social media platforms, particularly in the more recent elections. The spread of misinformation on social media has received considerable attention from Brazilian electoral authorities, especially the Tribunal Superior Eleitoral, which has shown particular vigilance in combating online attacks and misinformation. Moreover, while enforcing right of reply legislation remains challenging on platforms like WhatsApp, where attacks often originate from sources not directly related to candidates’ campaigns, there has been a growing number of cases in which the legislation was applied to fake content circulated via the platform.

¹⁶Regarding WhatsApp specifically, while user information is not publicly available, Google Trends suggests that the platform was considerably less popular during the 2012 elections. Between 2004 and 2025, search interest in the term “WhatsApp” scored 1 in October 2012, 38 in October 2016, and reached a peak of 100 in October 2021.

¹⁷Importantly, in Panel B of Figure B.1, we show that the proportion of lawsuits decided in favor of or against the plaintiff remained stable across both elections, suggesting that our measure captures attacks of comparable severity over time.

¹⁸We note that, in our main analysis, changes in the overall frequency of attacks across periods are absorbed by election-year fixed effects included in all our specifications.

Table D.1: Patterns of Campaign Attacks: Effective Candidate Threshold (Robustness Check)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
1st against 2nd (β_{12})			0.0442*** [0.0044]	0.0449*** [0.0045]
1st against 3rd (β_{13})			0.0033* [0.0019]	0.0029 [0.0020]
2nd against 1st (β_{21})	0.0059* [0.0035]	0.0077** [0.0033]	0.0624*** [0.0049]	0.0637*** [0.0052]
3rd against 1st (β_{31})			0.0081*** [0.0026]	0.0107*** [0.0029]
3rd against 2nd (β_{32})			-0.0014 [0.0020]	-0.0023 [0.0022]
$H_0 : \beta_{21} = \beta_{12}$			0.000	0.000
$H_0 : \beta_{21} = \beta_{31}$			0.000	0.000
$H_0 : \beta_{21} = 0$			0.000	0.000
$H_0 : \beta_{12} = \beta_{13}$			0.000	0.000
$H_0 : \beta_{31} = \beta_{32}$			0.000	0.000
Effective Candidate Threshold	0%	2%	0%	2%
Dep. Var. Mean	0.052	0.053	0.027	0.030
No. of Electoral Races	5,459	6,131	2,922	2,840
Observations	10,918	12,262	17,524	17,032
Adj. R-squared	0.304	0.298	0.125	0.152

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair. The sample used in columns (1) and (2) includes only observations from races with two effective candidates and considers only the pairs formed by the top two candidates. The sample used in columns (3) and (4) includes only observations from races with three effective candidates and considers only the pairs formed by the top three candidates. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Tables 2 and 3.

Table D.2: Patterns of Campaign Attacks: Dilution Effect (Distance 2nd-3rd)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
2nd against 1st (β_{21})	0.0081 [0.0067]	0.0057 [0.0070]	0.0110 [0.0075]	0.0145* [0.0087]
1st against 2nd $\times$ Distance 2nd-3rd	0.1637*** [0.0421]	0.2037*** [0.0505]	0.2828*** [0.0689]	0.4383*** [0.1054]
2nd against 1st $\times$ Distance 2nd-3rd	0.2095*** [0.0433]	0.2641*** [0.0526]	0.2852*** [0.0741]	0.3834*** [0.1073]
Effective Candidate Threshold	0%	2%	5%	10%
Dep. Var. Mean	0.064	0.064	0.063	0.055
No. of Electoral Races	4,609	3,923	3,114	2,311
Observations	9,218	7,846	6,228	4,622
Adj. R-squared	0.232	0.235	0.264	0.287

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 4.

Table D.3: Patterns of Campaign Attacks: Dilution Effect (Distance 1st-3rd)

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
2nd against 1st (β_{21})	-0.0119 [0.0085]	-0.0145 [0.0089]	-0.0083 [0.0096]	-0.0032 [0.0115]
1st against 2nd $\times$ Distance 1st-3rd	0.0310 [0.0348]	0.0543 [0.0418]	0.0675 [0.0567]	0.0474 [0.0829]
2nd against 1st $\times$ Distance 1st-3rd	0.1118*** [0.0374]	0.1471*** [0.0445]	0.1313** [0.0599]	0.0899 [0.0875]
Effective Candidate Threshold	0%	2%	5%	10%
Dep. Var. Mean	0.064	0.064	0.063	0.055
No. of Electoral Races	4,609	3,923	3,114	2,311
Observations	9,218	7,846	6,228	4,622
Adj. R-squared	0.230	0.232	0.260	0.281

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 4.

Table D.4: Patterns of Campaign Attacks: Dilution Effect (Polarized vs Non-Polarized Races)

	Dependent Variable: Campaign Attack			
	Non-polarized races (1)	Polarized races (2)	Non-polarized races (3)	Polarized races (4)
2nd against 1st (β_{21})	0.0337*** [0.0075]	0.0410*** [0.0146]	0.0392*** [0.0098]	0.0522*** [0.0194]
1st against 2nd $\times$ Share 3rd (π_{12})	0.0198 [0.0818]	-0.4498 [0.3333]	-0.0144 [0.0989]	-0.4276 [0.2629]
2nd against 1st $\times$ Share 3rd (π_{21})	-0.1050 [0.0839]	-0.6431* [0.3403]	-0.1671 [0.1027]	-0.6855** [0.2849]
Effective Candidate Threshold	0%	0%	2%	2%
Dep. Var. Mean	0.062	0.070	0.063	0.070
No. of Electoral Races	3,612	997	3,086	837
Observations	7,224	1,994	6,172	1,674
Adj. R-squared	0.276	0.327	0.270	0.328

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair. The sample is divided between polarized races, where the top two candidates come from parties on opposite sides of the political spectrum (left versus right), and non-polarized races (based on ideological scores from Zucco and Power (2023)). All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 4.

Table D.5: Patterns of Campaign Attacks: Two-Candidate Races (Heterogeneous Effects)

	Population		Income per capita		Gini coefficient		Electoral competitiveness	
	Below median (1)	Above median (2)	Below median (3)	Above median (4)	Below median (5)	Above median (6)	Below median (7)	Above median (8)
2nd against 1st (β_{21})	0.0033 [0.0032]	0.0212*** [0.0055]	0.0064* [0.0036]	0.0178*** [0.0053]	0.0135*** [0.0042]	0.0111** [0.0049]	0.0129*** [0.0046]	0.0108** [0.0044]
Dep. Var. Mean	0.022	0.088	0.032	0.078	0.051	0.060	0.052	0.059
No. of Electoral Races	3,452	3,461	3,455	3,458	3,627	3,286	3,496	3,417
Observations	6,904	6,922	6,910	6,916	7,254	6,572	6,992	6,834
Adj. R-squared	0.167	0.304	0.186	0.327	0.313	0.279	0.275	0.355

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair. Regression subsamples are constructed by dividing the original sample at the median of the following variables: population, income per capita, income inequality (Gini), and degree of electoral competitiveness (HHI). All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 2.

Table D.6: Patterns of Campaign Attacks: Three-Candidate Races (Heterogeneous Effects)

	Population		Income per capita		Gini coefficient		Electoral competitiveness	
	Below median (1)	Above median (2)	Below median (3)	Above median (4)	Below median (5)	Above median (6)	Below median (7)	Above median (8)
1st against 2nd (β_{12})	0.0131*** [0.0037]	0.0810*** [0.0095]	0.0195*** [0.0048]	0.0755*** [0.0091]	0.0433*** [0.0069]	0.0513*** [0.0080]	0.0554*** [0.0075]	0.0388*** [0.0072]
1st against 3rd (β_{13})	-0.0018 [0.0016]	0.0123** [0.0050]	0.0048* [0.0028]	0.0063 [0.0044]	0.0055 [0.0036]	0.0059 [0.0039]	0.0033 [0.0024]	0.0077* [0.0047]
2nd against 1st (β_{21})	0.0123*** [0.0037]	0.0979*** [0.0100]	0.0324*** [0.0057]	0.0779*** [0.0092]	0.0432*** [0.0071]	0.0690*** [0.0083]	0.0737*** [0.0082]	0.0363*** [0.0071]
3rd against 1st (β_{31})	0.0035 [0.0027]	0.0219*** [0.0065]	0.0107*** [0.0036]	0.0154** [0.0060]	0.0095** [0.0045]	0.0180*** [0.0056]	0.0110*** [0.0042]	0.0141** [0.0056]
3rd against 2nd (β_{32})	-0.0011 [0.0023]	-0.0001 [0.0048]	-0.0012 [0.0024]	0.0011 [0.0047]	0.0011 [0.0035]	-0.0023 [0.0041]	-0.0041 [0.0032]	0.0024 [0.0043]
$H_0 : \beta_{21} = \beta_{12}$	0.841	0.093	0.023	0.795	0.984	0.028	0.028	0.729
$H_0 : \beta_{21} = \beta_{31}$	0.039	0.000	0.000	0.000	0.000	0.000	0.000	0.001
$H_0 : \beta_{21} = 0$	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$H_0 : \beta_{12} = \beta_{13}$	0.000	0.000	0.004	0.000	0.000	0.000	0.000	0.000
$H_0 : \beta_{31} = \beta_{32}$	0.101	0.002	0.003	0.030	0.082	0.001	0.001	0.050
Dep. Var. Mean	0.008	0.058	0.017	0.049	0.027	0.039	0.033	0.032
No. of Electoral Races	1,241	1,241	1,241	1,241	1,301	1,181	1,246	1,236
Observations	7,438	7,446	7,442	7,442	7,798	7,086	7,468	7,416
Adj. R-squared	0.063	0.176	0.161	0.159	0.110	0.198	0.166	0.171

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair. Regression subsamples are constructed by dividing the original sample at the median of the following variables: population, income per capita, income inequality (Gini), and degree of electoral competitiveness (HHI). All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 3.

Table D.7: Patterns of Campaign Attacks: Timing (Heterogeneous Effects)

	Dependent Variable: Campaign Attack on restricted days			
	Last two weeks (1)	Weeks 3, 4, and 5 before election (2)	Last two weeks (3)	Weeks 3, 4, and 5 before election (4)
1st against 2nd (β_{12})			0.0287*** [0.0040]	0.0250*** [0.0038]
1st against 3rd (β_{13})			0.0012 [0.0021]	0.0013 [0.0021]
2nd against 1st (β_{21})	0.0100*** [0.0027]	0.0069*** [0.0024]	0.0317*** [0.0042]	0.0258*** [0.0039]
3rd against 1st (β_{31})			0.0074** [0.0029]	0.0043* [0.0025]
3rd against 2nd (β_{32})			-0.0005 [0.0019]	-0.0022 [0.0021]
$H_0 : \beta_{21} = \beta_{12}$			0.509	0.860
$H_0 : \beta_{21} = \beta_{31}$			0.000	0.000
$H_0 : \beta_{21} = 0$			0.000	0.000
$H_0 : \beta_{12} = \beta_{13}$			0.000	0.000
$H_0 : \beta_{31} = \beta_{32}$			0.008	0.012
Dep. Var. Mean	0.032	0.029	0.019	0.017
No. of Electoral Races	6,913	6,913	2,482	2,482
Observations	13,826	13,826	14,884	14,884
Adj. R-squared	0.218	0.246	0.101	0.105

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair during a specific period of the campaign. The sample used in columns (1) and (2) includes only observations from races with two effective candidates and considers only the pairs formed by the top two candidates. The sample used in columns (3) and (4) includes only observations from races with three effective candidates and considers only the pairs formed by the top three candidates. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. For additional details, see footnote to Tables 2 and 3.

Table D.8: Patterns of Campaign Attacks: Two-Candidate Races (2012 and 2016 samples)

	Dependent Variable: Campaign Attack			
	2012 (1)	2016 (2)	2012 (3)	2016 (4)
2nd against 1st (β_{21})	0.0140*** [0.0049]	0.0100*** [0.0039]	0.0161*** [0.0048]	0.0102*** [0.0037]
Effective Candidate Threshold	5%	5%	10%	10%
Dep. Var. Mean	0.075	0.035	0.080	0.037
No. of Electoral Races	3,536	3,377	3,906	3,762
Observations	7,072	6,754	7,812	7,524
Adj. R-squared	0.403	0.265	0.411	0.285

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates in an ordered pair. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 2.

Table D.9: Patterns of Campaign Attacks: Three-Candidate Races (2012 and 2016 samples)

	Dependent Variable: Campaign Attack			
	2012 (1)	2016 (2)	2012 (3)	2016 (4)
1st against 2nd (β_{12})	0.0701*** [0.0083]	0.0257*** [0.0058]	0.0613*** [0.0095]	0.0213*** [0.0061]
1st against 3rd (β_{13})	0.0114*** [0.0040]	−0.0000 [0.0034]	0.0147*** [0.0056]	0.0002 [0.0040]
2nd against 1st (β_{21})	0.0781*** [0.0087]	0.0331*** [0.0063]	0.0622*** [0.0094]	0.0285*** [0.0067]
3rd against 1st (β_{31})	0.0229*** [0.0056]	0.0031 [0.0040]	0.0274*** [0.0072]	0.0035 [0.0044]
3rd against 2nd (β_{32})	0.0018 [0.0044]	−0.0026 [0.0030]	0.0026 [0.0056]	−0.0011 [0.0034]
$H_0 : \beta_{21} = \beta_{12}$	0.372	0.261	0.931	0.293
$H_0 : \beta_{21} = \beta_{31}$	0.000	0.000	0.001	0.000
$H_0 : \beta_{21} = 0$	0.000	0.000	0.000	0.000
$H_0 : \beta_{12} = \beta_{13}$	0.000	0.000	0.000	0.001
$H_0 : \beta_{31} = \beta_{32}$	0.001	0.182	0.001	0.344
Effective Candidate Threshold	5%	5%	10%	10%
Dep. Var. Mean	0.045	0.021	0.045	0.020
No. of Electoral Races	1,210	1,272	961	1,065
Observations	7,256	7,628	5,766	6,386
Adj. R-squared	0.197	0.131	0.183	0.124

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates of an ordered pair. All regressions include candidate-specific characteristics and control for municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. For additional details, see footnote to Table 3.

E Right of Reply Lawsuits in Brazil

In this appendix, we discuss the details of two right of reply lawsuits, focusing on three main aspects: (i) the campaign attack, (ii) the court’s ruling, and (iii) the reply. Our accounts are based on a number of court documents, which we were able to access for these two particular cases and provide interesting insights into how these lawsuits work. Both cases occurred during the 2020 municipal election in the city of Sao Paulo, the largest one in the country, and involved the same two candidates, Celso Russomanno and Guilherme Boulos.¹⁹ Russomanno is a Brazilian congressman who became notorious for a TV show in which he confronted businesses and store owners, and shop workers in defense of consumer rights. Boulos, on the other hand, is the former leader of the Homeless Workers’ Movement (*Movimento dos Trabalhadores sem Teto - MTST*), a social movement that acts against housing shortages in metropolitan areas, often invading unoccupied plots and abandoned buildings. Both cases occurred during the first round of the elections, where Boulos and Russomanno were in direct competition for a place in the runoff against the incumbent mayor, Bruno Covas.

E.1 *Russomanno vs. Boulos*

Campaign Attack. On October 15, Boulos posted on his Facebook and Instagram accounts a message saying, “*Behind the cameras, Russomanno hates the poor*” (see Figure E.1, panel (a)). In the same post, he also shared an excerpt of an interview in which Russomanno stated: “*homeless people and residents of Cracolandia are possibly more resistant to Covid-19 because they don’t bathe*”.²⁰ On the following day, October 16, Russomanno filed a lawsuit against Boulos demanding that the post in question be taken down and that a right of reply be granted in his favor. In his complaint, Russomanno argued that his statement about the homeless being “*more resistant to Covid-19*” had been taken out of context and that the claim that “*Russomanno hates the poor*” was defamatory and untrue. The complaint was short and concise, containing two screenshots of the offending posts and brief references to the relevant legislation and legal precedents.

Court’s Ruling. On October 20, after receiving Boulos’ defense statement, the electoral judge ruled in favor of Russomanno. With respect to the accusation that “*Russomanno hates the poor*”, the magistrate opined that “*while harsh criticism is an integral part of campaigns, in the present case there was excess from the part of the defendant insofar as he made a frivolous assertion, impossible of being verified, without any foundation in reality*”. However, regarding the excerpt from Russomanno’s interview, the judge found no excess from Boulos’ part, noting that “*he just reproduced what the plaintiff actually said*”. The decision granted Russomanno the right to publish a reply on Boulos’ social media, stipulating that it remain visible for a duration twice as long as that of the original attack. The judge also ordered the immediate removal of the offending posts and imposed a daily fine of R\$ 10,000 (US\$ 1,777) in case the defendant did not comply with the decision.

Reply. Figure E.1, panel (b) shows Russomanno’s reply, as posted on Boulos’s Instagram and Facebook accounts. The translation reads as follows:

Right of Reply granted by the Electoral Justice. Guilherme Boulos offended Celso Russomanno and was punished by the Electoral Justice. Celso Russomanno does not hate the poor, so much so that he has always fought for the most disadvantaged class in these 30 years of defending consumer rights, and will continue to fight if elected mayor of Sao Paulo. Real politics are made with proposals and not with offenses!

E.2 *Boulos vs. Russomanno*

Campaign Attack. About two weeks following the events described above, Russomanno posted a 30-second video on his Facebook and Instagram accounts accusing Boulos of (i) leading the invasion of

¹⁹The first-round election in 2020 took place, atypically, on November 15th due to the COVID-19 pandemic.

²⁰“Cracolandia” is a region of the city of Sao Paulo well-known for its high incidence of drug trafficking and public drug use. The story is available at Globo’s G1 Portal, <https://glo.bo/3HMA1X7>.

a building in central Sao Paulo and collecting rent from its occupants²¹, (ii) omitting personal assets from the electoral justice, and (iii) using a supermarket cashier in a campaign ad against Russomanno. Figure E.2 provides a screenshot of the original Instagram post.²² In the complaint filed on October 31, Boulos contended that the accusation of breaking into the building and collecting rent was both false and defamatory. In fact, the invasion of the building in question, along with other unlawful acts, was carried out by another group known as “*Movimento Moradia para Todos e Sem Tetos do Centro*”, whose leaders were all subsequently arrested. He also argued that the omission of personal assets from the electoral justice records was unintentional and very minor—amounting to approximately US\$ 100 — and that, in fact, he had already rectified it. Finally, regarding the supermarket cashier, he argued that she had been humiliated by Russomanno on his TV show years earlier, and that she had voluntarily agreed to participate in the campaign ad against him.

Court’s Ruling. On November 3, after receiving Russomanno’s defense statement, the electoral judge issued a ruling in favor of Boulos. The judge acknowledged that the accusation that Boulos had led the invasion of the building and collected rent was both false and defamatory, constituting a clear offense to his honor. However, he dismissed the allegations of any illicit act related to the other two charges, namely that Boulos omitted assets from the judicial records and that he exploited the supermarket cashier in his campaign. The judge ruled that both claims were factually true and did not offend his integrity. Interestingly, the magistrate stated that “*it is the responsibility of the candidate, through regular campaign channels, to clarify these incidents to his voters*”. The decision granted Boulos the right to publish a reply on Russomanno’s social media, stipulating that it remain visible for a duration twice as long as that of the original attack. As in the previous case, the judge also ordered the immediate removal of the offending video and imposed a R\$ 20,000 (US\$ 3,555) fine in case the defendant did not comply with the decision within 48 hours.

Reply. Boulos’ reply was in the form of a 30-second video in which Cleide Cruz, the supermarket cashier, talks about how she felt humiliated by Russomanno on his TV show.²³ The English transcript of what Cleide says in the video is as follows:

Narrator: *Right of reply granted the Electoral Justice against Russomanno for lies he told against Boulos.*

Cleide Cruz, the supermarket cashier: *Not here, Celso Russomanno. You are lying. You humiliated me that day. You left me feeling terrible, like rubbish. You caused me pain, you caused me problems, Celso Russomanno. You don’t have the dignity to apologize for what you did. Serious people, as you say in the video, do their job without humiliating, offending, or vilifying people.*

Note that in this particular case, the content of the reply does not directly address the part of the attack that was considered defamatory by the court. As we discussed in the main text, the Brazilian legislation does not impose any restriction on the content of the reply, allowing the attacked candidate complete discretion in utilizing the allotted space.

E.3 Litigation in Bad Faith

While the previous examples shed light on how right of reply lawsuits typically unfold in practice, not all such legal actions are pursued in good faith. In this subsection, we examine a few cases in which the judicial process was used for illegitimate or strategic purposes, highlighting that such instances are exceptionally rare in the context of right of reply lawsuits. The Brazilian law expressly defines “*litigância de má-fé*”, or litigation in bad faith, in Article 80 of the Civil Procedure Code (CPC) as conduct that abuses the judicial process for illegitimate purposes.²⁴ This includes, among other actions, altering the

²¹The building in question, “Edifício Wilton Paes de Almeida”, became notorious for having been destroyed by fire and collapsing a few years earlier. For more information, see <https://bit.ly/32G4CFs>.

²²The original video is no longer available; it was removed by a judicial order.

²³The video is available online at <https://bit.ly/3xVpK6q>.

²⁴The Civil Procedure Code is available at <https://www.planalto.gov.br>.

truth of facts, causing unwarranted delays in the proceedings, and interposing clearly unfounded claims or appeals. Under Articles 79 and 81 of the Civil Procedure Code, parties who litigate in bad faith may face sanctions, including fines and compensation for damages suffered by the opposing party. The applicable fine may range from 1% to 10% of the value of the claim at the judge’s discretion.

Given this legal context, we investigate the prevalence of bad-faith litigation in right of reply lawsuits by conducting a comprehensive text analysis of the judicial decisions in our sample. To do so, we focus on cases ruled against the plaintiff for which the full text of the decision is available, totaling 1,532 cases. We then perform a comprehensive search for the term “*litigância de má-fé*”, including common variations in spelling or phrasing, followed by a careful review of each decision to determine the precise context in which the term was used. Out of a sample of 1,532 cases, we identified only 7 cases (< 0.5%) in which the judge found the plaintiff guilty of litigating in bad faith. Interestingly, in the vast majority of cases where the term “*litigância de má-fé*” appeared, it was invoked merely to reject the allegation. Overall, our findings indicate that “spurious” cases account for only a small fraction of right of reply lawsuits, reinforcing the view that litigation in bad-faith is not widespread.

To illustrate the nature of bad-faith litigation in our sample, we describe a few representative cases. In one instance, the court found the plaintiff guilty of filing a right of reply lawsuit despite the absence of “falsehoods or personal offenses”—the judge also criticized the exaggerated indignation displayed in the filing, including the use of boldface and all-caps text. Another judge condemned the plaintiff for attempting to “use the judiciary to intimidate the respondent through procedural maneuvers,” rather than engaging in political debate, emphasising that “the Electoral Court is not a lottery for unfounded or reckless claims.” In a third decision, the court denounced the “entirely unfounded” nature of the filing and warned that “the Electoral Court cannot be mobilized merely to serve the whims of candidates.” Other rulings cited the use of “unfounded claims,” “litigating against undisputed facts,” and “filing frivolous or dilatory appeals” as evidence of bad faith. One judge further reproached a plaintiff for filing multiple unfounded lawsuits against the same opponent, warning that such conduct was “disrupting the electoral process.”



(a) Original Publication

(b) Reply

Figure E.1: *Russomanno vs. Boulos*

Notes: The image in Panel (a) was obtained from the original case file, which can be accessed through the link <https://www.tre-sp.jus.br/servicos-judiciais/processo-judicial-eletronico-pje> (in portuguese) using case number 0600081-61.2020.6.26.0002. Access, however, is restricted to licensed attorneys. The image in Panel (b) can be found at <https://www.instagram.com/p/CGlut7LpQV-/> (accessed on November 9, 2023).



Figure E.2: *Boulos vs. Russomanno: Original Publication*

Notes: This image was obtained from the original case file, which can be accessed through the link <https://www.tre-sp.jus.br/servicos-judiciais/processo-judicial-eletronico-pje> (in portuguese) using case number 0600236-64.2020.6.26.0002. Access, however, is restricted to licensed attorneys.

F Survey Data: Campaign Attacks and Voting Behavior

In this appendix, we provide further details about the data employed in our discussion in Section 6, and complement our analysis by presenting additional regression results. Our dataset consists of a 3,010-subjects nationally representative survey conducted by the Ibope Institute between the two rounds of the 2018 elections (October 21-23, 2018).²⁵ In Panel A of Table F.1 we show that the sample means of several important demographic and socio-economic characteristics of surveyed individuals are very close to the corresponding population means taken from the Tribunal Superior Eleitoral’s (TSE) database of registered voters—although survey respondents do tend to have more high school education. Moreover, in Panel B we present the percentage of respondents who reported voting for Bolsonaro, Haddad, and other candidates in the first round of elections, as well as the distribution of voting intentions for the second round, alongside the actual electoral results. As we discussed in the paper, the 2018 presidential election was sharply polarized between Jair Bolsonaro, the far-right populist candidate, and Fernando Haddad, the leftist candidate and former president Lula’s protégé. Interestingly, as shown in Panel B, the survey results were remarkably close to the actual electoral results.

As mentioned in the main text, the Ibope survey was unique in that, in addition to the usual voting intention questions, it also included a specific query about whether subjects had received a message via WhatsApp containing a campaign attack.²⁶ As shown in Panel C of Table F.1, 25.2% of respondents reported receiving at least one message attacking one of the candidates in the week leading up to the first round of elections.²⁷ Not surprisingly, given the degree of polarization of the 2018 election, among those who received one or more of these messages, 97.3% indicated the attacks were directed at either Bolsonaro or Haddad (not reported in Panel C). Additionally, among those who received an attack message, 23.7% reported that the content of these messages actually influenced their decision on whom to vote for.²⁸ As discussed in the paper, these figures suggest that at least 36.8 million voters were exposed to attack messages in the week preceding the first round of elections and that approximately 8.5 million of these individuals (or about 5.7% of the electorate) were, to some extent, influenced by those attacks.

We further examine the effect of being exposed to a campaign attack on individual electoral behavior by estimating the following regression:

$$V_i = \beta_0 + \beta_1 \text{AttackMessage}_i + X_i\phi + \epsilon_i$$

where V_i is a dummy indicating declared vote for a certain candidate (Bolsonaro, Haddad or others) in the first round of elections or the intention to cast a null vote in the second round, and AttackMessage_i is a dummy representing whether the respondent received an attack message via WhatsApp. Our analysis controls for a detailed set of demographic characteristics, including gender, age, education level, income level, region of residence, race, and religion. Moreover, we also control for the respondent’s current level of life satisfaction, her view on the future of the country (optimistic or pessimistic), the political party (if any) of her preference, the political party (if any) she would never vote for, and interview-day fixed effects.

The results reported in Table F.2 suggest that being exposed to an attack message induces an individual to become less likely to vote for both Bolsonaro and Haddad (columns 1 and 2), although only the effect on Bolsonaro is statistically significant. These results are consistent with the idea that attacks and aggressive campaigning lead voters to demobilize and are in line with experimental evidence provided by Chong et al. (2015). Indeed, as we show in column 3, receiving an attack message—which was most likely directed at either Haddad or Bolsonaro—is associated with a 5.9 pp increase in the likelihood of voting for a third candidate in the first round of elections.²⁹ Furthermore, in column 4, we show that individuals exposed to an attack message are also 3.3 pp more likely to declare an intention to cast a

²⁵The Ibope Institute is a reputable company specialized in public opinion polls. See <https://bit.ly/3IeTmje>.

²⁶WhatsApp is the most widely used messaging app in Brazil. A recent survey found that approximately 98% of smartphone owners in the country reported having WhatsApp installed on their mobile devices (see <https://bit.ly/3IvW845>). Moreover, another survey conducted by the Brazilian Senate in November 2019 revealed that 79% of respondents identified WhatsApp as their primary source of information.

²⁷The exact phrasing of this question was: “*Did you receive a message through WhatsApp containing criticisms or attacks against any candidate in the week before the first round of elections?*”

²⁸The exact phrasing of this question was: “*Did the content you received helped or not to decide your vote?*”

²⁹This finding is consistent with experimental evidence obtained by Galasso et al. (2023), who show that negative campaigning generates a positive spillover effect on third candidates (neither the target nor the attacker).

null vote in the second round.

Next, we investigate whether the effect of receiving an attack message on voting behavior varies with the characteristics of the individuals exposed to it. In particular, we test whether the impact of an attack is more pronounced among the more “disillusioned” voters, i.e., those most likely to feel abandoned by traditional politics and electoral institutions. In Table F.3 we split the sample into two groups of voters according to whether the respondent reported feeling “pessimistic” or “optimistic” about the future of the country. Interestingly, we find that the effects are concentrated in the subsample of “pessimistic” voters. Specifically, in Panel A we show that among “pessimistic” voters an exposure to a campaign attack is associated with a 8.0 pp reduction in the likelihood of voting for Haddad, accompanied by a 10.3 pp increase in the probability of voting for a third candidate in the first round of elections and a 6.5 pp increase in the likelihood of declaring an intention to cast a null vote in the second round. Conversely, in Panel B, we find that among “optimistic” voters, the point estimates are much smaller in magnitude with none of them being statistically significant. Finally, in an additional exercise (available upon request), we show that similar results are obtained when dividing the sample by income level, with all the effects concentrated among poorer voters.

Table F.1: Summary Statistics: Ibope Survey

Panel A: Demographic Characteristics	<u>Ibope Survey</u>	<u>TSE Database</u>
<i>Region of Residence</i>		
- North/Central-West	0.153	0.152
- Northeast	0.260	0.267
- Southeast	0.437	0.435
- South	0.149	0.146
<i>Demographic and Socio-Economic Characteristics</i>		
- Female	0.527	0.525
- 16-17 years old	0.009	0.010
- 18-24 years old	0.166	0.142
- 25-34 years old	0.234	0.212
- 35-44 years old	0.208	0.206
- 45-54 years old	0.176	0.170
- 55-64 years old	0.127	0.133
- 65+ years old	0.081	0.128
- Illiterate	0.041	0.045
- Primary Education or More	0.750	0.700
- High School Education or More	0.553	0.426
- College Education	0.129	0.105
Panel B: 2018 Presidential Elections	<u>Ibope Survey</u>	<u>Election Results</u>
<i>1st Round of Elections</i>		
- Voted for Bolsonaro	0.414	0.420
- Voted for Haddad	0.281	0.267
- Voted for Another Candidate	0.211	0.225
- Null Vote	0.094	0.088
<i>2nd Round of Elections</i>		
- Intention to vote for Bolsonaro	0.512	0.499
- Intention to vote for Haddad	0.380	0.406
- Intention to Cast a Null Vote	0.108	0.096
Panel C: Attacks via WhatsApp	<u>Ibope Survey</u>	
- Received an Attack Message	0.252	
- Attack Influenced Voting Decision	0.237	

Notes: This table reports summary statistics for selected variables available in the Ibope survey, which was conducted between the two rounds of the 2018 elections (October 21-23, 2018). In Panel A, we report sample means for demographic and socio-economic characteristics of survey respondents, alongside the corresponding population means taken from TSE’s database of registered voters. In Panel B, we report the percentage of respondents who claimed to have voted for each candidate in the 1st round of elections, as well as their voting intentions for the 2nd round, alongside the actual election results. In Panel C, we report sample means for answers to questions related to attack messages via WhatsApp. The variable “Received an Attack Message” indicates whether the respondent received a message via WhatsApp attacking one of the candidates in the week preceding the first round of elections. “Attack Influenced Voting Decision” indicates whether the attack influenced the choice of whom to vote for, conditional on receiving an attack.

Table F.2: The Effect of Attack Messages on Voting Behavior

	1st Round of Elections			2nd Round of Elections
	Voted for Bolsonaro (1)	Voted for Haddad (2)	Voted for Another Candidate (3)	Null Vote (4)
Attack Message	-0.037** [0.017]	-0.018 [0.015]	0.059*** [0.018]	0.033** [0.014]
Individual Characteristics	Yes	Yes	Yes	Yes
Observations	3,010	3,010	3,010	3,010
Adj. R-squared	0.336	0.362	0.092	0.089

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether the respondent voted for a certain candidate (Bolsonaro, Haddad, or others) in the first round of elections (columns 1–3) or intends to cast a null vote in the second round (column 4). The variable “Attack Message” indicates whether the respondent received a message via WhatsApp attacking one of the candidates in the week preceding the first round of elections. Individual characteristics include gender, age, education level, income level, region of residence, race, and religion. We also control for the respondent’s current level of life satisfaction, her view on the future of the country (optimistic or pessimistic), the political party (if any) of her preference, the political party (if any) she would never vote for, and interview-day fixed effects. Robust standard errors are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively.

Table F.3: The Effect of Attack Messages on Voting Behavior: Heterogeneous Effects

	1st Round of Elections			2nd Round of Elections
	Voted for Bolsonaro (1)	Voted for Haddad (2)	Voted for Another Candidate (3)	Null Vote (4)
Panel A: Pessimistic Voters				
Attack Message	-0.042 [0.029]	-0.080*** [0.028]	0.103*** [0.031]	0.065** [0.026]
Individual Characteristics	Yes	Yes	Yes	Yes
Observations	1,167	1,167	1,167	1,167
Adj. R-squared	0.256	0.338	0.147	0.084
Panel B: Optimistic Voters				
Attack Message	-0.034 [0.023]	0.023 [0.018]	0.021 [0.022]	0.017 [0.015]
Individual Characteristics	Yes	Yes	Yes	Yes
Observations	1,652	1,652	1,652	1,652
Adj. R-squared	0.343	0.394	0.086	0.091

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether the respondent voted for a certain candidate (Bolsonaro, Haddad, or others) in the first round of elections (columns 1–3) or intends to cast a null vote in the second round of elections (column 4). The variable “Attack Message” indicates whether the respondent received a message via WhatsApp attacking one of the candidates in the week preceding the first round of elections. Panel A reports estimates obtained from a subsample of respondents who declared themselves to be pessimistic about the future of the country. Panel B, in turn, reports estimates obtained from a subsample of respondents who declared themselves to be optimistic about the future of the country. Individual characteristics include gender, age, education level, income level, region of residence, race, and religion. We also control for the respondent’s current level of life satisfaction, the political party (if any) of her preference, the political party (if any) she would never vote for, and interview-day fixed effects. Robust standard errors are reported in brackets. *, ** and *** denote statistical significance at 10%, 5% and 1%, respectively.

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Online Theoretical Appendix for “Electoral Campaign Attacks: Theory and Evidence”

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This Online Theoretical Appendix contains supplemental material for “*Electoral Campaign Attacks: Theory and Evidence*” and is comprised of three sections. Appendix G features additional results that both extend and provide robustness to our main theoretical findings. Appendix H presents the proofs for the main propositions of the paper (Propositions 1-5). Finally, Appendix I collects the proofs for the additional results derived in Appendix G (Propositions G.1-G.10).

G Model: Extensions and Robustness Checks

G.1 Additional Results

This subsection provides a number of additional results that are intended to complement our main analysis. The following proposition describes how the relationship between $\tilde{\Delta}_{12}(n)$ and $\tilde{\Delta}_{31}(n)$ depend on the initial level of support of the candidates.

Proposition G.1. *There exists a threshold $\bar{\phi} > 0$ such that if $\phi < \bar{\phi}$ and s_2^o is sufficiently close to s_1^o then:*

$$\tilde{\Delta}_{31}(n_3) < \tilde{\Delta}_{12}(n_1) \quad \text{for any } n_i \in \mathcal{N}_i$$

Alternatively, if $\phi < \bar{\phi}$ and s_2^o is sufficiently close to s_3^o then:

$$\tilde{\Delta}_{31}(n_3) > \tilde{\Delta}_{12}(n_1) \quad \text{for any } n_i \in \mathcal{N}_i$$

Next, for completeness, we extend the characterization of the benefit function $\tilde{\Delta}_{ij}(n)$ to all remaining pairs of candidates.

Proposition G.2. *There exists a threshold $\bar{\phi} > 0$ such that if $\phi < \bar{\phi}$, then we have:*

- i. The benefit of an attack by candidate 1 on 2 is larger than that of an attack by 3 on 2:*

$$\tilde{\Delta}_{32}(n_3) < \tilde{\Delta}_{12}(n_1) \quad \text{for any } n_i \in \mathcal{N}_i$$

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ii. The benefit of an attack by candidate 3 on 1 is larger than that of an attack by 1 on 3:

$$\tilde{\Delta}_{13}(n_1) < \tilde{\Delta}_{31}(n_3) \quad \text{for any } n_i \in \mathcal{N}_i$$

iii. The benefit of an attack by candidate 1 on 3 is larger than that of an attack by 2 on 3:

$$\tilde{\Delta}_{23}(n_2) < \tilde{\Delta}_{13}(n_1) \quad \text{for any } n_i \in \mathcal{N}_i$$

iv. The benefit of an attack of candidate 3 on 2 is larger than that of an attack of 2 on 3:

$$\tilde{\Delta}_{23}(n_2) < \tilde{\Delta}_{32}(n_3) \quad \text{for any } n_i \in \mathcal{N}_i$$

The following corollary combines the results derived in Propositions 3 and G.2, summarizing our main results in a more concise way.

Corollary G.1. *From Propositions 3 and G.2, it follows that there exists a threshold $\bar{\phi} > 0$ such that if $\phi < \bar{\phi}$, then:*

$$\tilde{\Delta}_{23}(n_2) < \{\tilde{\Delta}_{13}(n_1), \tilde{\Delta}_{32}(n_3)\} < \{\tilde{\Delta}_{12}(n'_1), \tilde{\Delta}_{31}(n'_3)\} < \tilde{\Delta}_{21}(n'_2) \quad \text{for any } n_i, n'_i \in \mathcal{N}_i$$

Interestingly, our analysis shows that while the most likely direction of attack is from candidate 2 against 1, the least likely direction is from 2 against 3. Moreover, note that both the relationship between $\tilde{\Delta}_{12}(n)$ and $\tilde{\Delta}_{31}(n)$, as discussed in the main text (see also Proposition G.1), and that between $\tilde{\Delta}_{13}$ and $\tilde{\Delta}_{32}$ are ambiguous. With respect to the latter, it is possible to show that if s_2^o is sufficiently close to s_1^o , then $\tilde{\Delta}_{13}(n) < \tilde{\Delta}_{32}(n)$; whereas if s_2^o is sufficiently close to s_3^o , then the opposite holds. Intuitively, everything else constant, a larger initial support s_2^o increases the benefit of an attack against candidate 2. The following proposition provides a formal statement for this result.

Proposition G.3. *There exists a threshold $\bar{\phi} > 0$ such that if $\phi < \bar{\phi}$ and s_2^o is sufficiently close to s_1^o , then:*

$$\tilde{\Delta}_{13}(n_1) < \tilde{\Delta}_{32}(n_3) \quad \text{for any } n_i \in \mathcal{N}$$

Conversely, if $\phi < \bar{\phi}$ and s_2^o is sufficiently close to s_3^o , then:

$$\tilde{\Delta}_{13}(n_1) > \tilde{\Delta}_{32}(n_3) \quad \text{for any } n_i \in \mathcal{N}$$

Next, based on the properties of the function $\tilde{\Delta}_{ij}(n)$ derived in Proposition 3, the following result provides a complete characterization of the unique equilibrium of the game with three candidates.^{1,2}

Proposition G.4. *There exists a threshold $\bar{\phi} > 0$ such that if $\phi < \bar{\phi}$ then:*

i. *First, suppose that $\tilde{\Delta}_{31}(n_3) < \tilde{\Delta}_{12}(n_1) < \tilde{\Delta}_{21}(n_2)$ for any $n_i \in \mathcal{N}_i$. There exists a unique Nash equilibrium with the following characteristics:*

- i.a *Candidates 2 and 3 attack 1 and candidate 1 attacks 2 if, and only if, $0 \leq c \leq \tilde{\Delta}_{31}(1, 1, 0)$.*
- i.b *Candidate 2 attacks 1 and candidate 1 attacks 2 if, and only if, $\tilde{\Delta}_{31}(1, 1, 0) < c \leq \tilde{\Delta}_{12}(1, 0, 0)$.*
- i.c *Candidate 2 attacks 1 if, and only if, $\tilde{\Delta}_{12}(1, 0, 0) < c \leq \tilde{\Delta}_{21}(0, 0, 0)$.*
- i.d *No candidate attacks if, and only if, $\tilde{\Delta}_{21}(0, 0, 0) < c$.*

¹Note that from Proposition 3, item iii it follows that there are two cases to consider: (i) $\tilde{\Delta}_{31}(n_3) < \tilde{\Delta}_{12}(n_1) < \tilde{\Delta}_{21}(n_2)$ and (ii) $\tilde{\Delta}_{12}(n_1) < \tilde{\Delta}_{31}(n_3) < \tilde{\Delta}_{21}(n_2)$.

²As a reminder, note that $\tilde{\Delta}_{ij}(n_1, n_2, n_3)$ represents candidate i 's benefit of attacking j when candidate 1 receives n_1 attacks, candidate 2 receives n_2 attacks and candidate 3 receives n_3 attacks. Thus, for example, the condition $c \leq \tilde{\Delta}_{31}(1, 1, 0)$ implies that candidate 3 is willing to attack 1 when $n_1 = 1$, $n_2 = 1$ and $n_3 = 0$.

ii. Second, suppose that $\tilde{\Delta}_{12}(n_1) < \tilde{\Delta}_{31}(n_3) < \tilde{\Delta}_{21}(n_2)$ for any $n_i \in \mathcal{N}_i$. There exists a unique Nash equilibrium with the following characteristics:

- ii.a Candidates 2 and 3 attack 1 and candidate 1 attacks 2 if, and only if, $0 \leq c \leq \tilde{\Delta}_{12}(2, 0, 0)$.
- ii.b Candidate 2 and 3 attack 1 if, and only if, $\tilde{\Delta}_{12}(2, 0, 0) < c \leq \tilde{\Delta}_{31}(1, 0, 0)$.
- ii.c Candidate 2 attacks 1 if, and only if, $\tilde{\Delta}_{31}(1, 0, 0) < c \leq \tilde{\Delta}_{21}(0, 0, 0)$.
- ii.d No candidate attacks if, and only if, $\tilde{\Delta}_{21}(0, 0, 0) < c$.

Figure A.2 depicts the regions of parameters where each class of equilibrium exists for the case where $\tilde{\Delta}_{31}(n_3) < \tilde{\Delta}_{12}(n_1) < \tilde{\Delta}_{21}(n_2)$ (Proposition G.4, item i). Note that, as the cost of attacking increases, we move through four different regions where the following equilibria exist: (a) an equilibrium where 2 and 3 attack 1 and 1 attacks 2, (b) an equilibrium where 2 attacks 1 and 1 attacks 2, (c) an equilibrium where 2 attacks 1, and (d) an equilibrium where nobody attacks.

Next, regarding the comparison between races with two and three candidates, the following proposition shows that the benefit of an attack between candidates 1 and 2 is always smaller in three-candidate races relative to two-candidate races, as a result of the “dillution effect”.

Proposition G.5. *There exists a threshold $\bar{\phi} > 0$ such that if $\phi < \bar{\phi}$ then:*

$$\Delta_{12}(n_1) > \tilde{\Delta}_{12}(n'_1) \quad \text{for any } n_1 \in \{0, 1\} \text{ and } n'_1 \in \mathcal{N}_1$$

and

$$\Delta_{21}(n_2) > \tilde{\Delta}_{21}(n'_2) \quad \text{for any } n_2 \in \{0, 1\} \text{ and } n'_2 \in \mathcal{N}_2$$

We next show that under dual ballot plurality candidates 2 and 3 might prefer to attack each other rather than the front-runner, something which would never happen under single ballot plurality in equilibrium (see Proposition 3, items ii and iii). The following proposition characterizes the conditions under which this might occur.

Proposition G.6. *There exist a threshold $\bar{\phi} > 0$ such that if $\phi < \bar{\phi}$ and s_1^o is sufficiently large, then under dual ballot we have:*

- i. For candidate 2, the benefit of an attack on 3 is larger than that of an attack on 1:

$$\tilde{\Delta}_{21}^{DB}(n) < \tilde{\Delta}_{23}^{DB}(n') \quad \text{for any } n, n' \in \mathcal{N}_2$$

- ii. For candidate 3, the benefit of an attack on 2 is larger than that of an attack on 1:

$$\tilde{\Delta}_{31}^{DB}(n) < \tilde{\Delta}_{32}^{DB}(n') \quad \text{for any } n, n' \in \mathcal{N}_3$$

Intuitively, if the front-runner is sufficiently ahead of other candidates, then it pays for 2 and 3 to attack each other in an attempt to secure a place in the second round. Finally, we show that all three candidates become more aggressive under dual ballot plurality if the race is sufficiently competitive in the sense that the initial support of all candidates are close enough.

Proposition G.7. *There exists a threshold $\bar{\phi} > 0$ such that if $\phi < \bar{\phi}$ and the distance $s_1^o - s_3^o > 0$ is sufficiently small, then for any i and j with $i \neq j$ we have:*

$$\tilde{\Delta}_{ij}(n_i) < \tilde{\Delta}_{ij}^{DB}(n'_i) \quad \text{for any } n_i, n'_i \in \mathcal{N}_i$$

G.2 Voter Demobilization

In this subsection, we extend our basic model to consider an alternative setting where campaign attacks lead to voter demobilization. Our main goal is to incorporate in a reduced form fashion the idea that campaign attacks and mudslinging may generate voter apathy and mistrust in politics (Ansolabehere and Iyengar (1995)). To do so, we allow voters to cast a null vote or simply choose not to vote, denoting this alternative (outside option) by $\emptyset$ with $s_\emptyset^o \geq 0$. Our proposed framework captures two sources of voter demobilization while maintaining the assumption that the overall effect of an attack is given by $\phi \in (0, 1)$. First, we suppose that a fraction $\alpha\phi$ of voters of the attacked candidate become demobilized and switch to not supporting anyone, while a fraction $(1 - \alpha)\phi$ move to other candidates, with $\alpha \in [0, 1]$. Second, we assume that a fraction $\gamma\phi$ of voters of the attacking candidate also become disengaged and switch to not voting (i.e. “backlash effect”), with $\gamma \in [0, 1]$.

Let $a_{ij} \in \{0, 1\}$ denote candidate i ’s binary decision to attack j . In two-candidate races the final support of candidate i is now given by:

$$x_i(a_{ij}, a_{ji}) = (1 - \phi a_{ji})s_i^o + a_{ij}((1 - \alpha)\phi s_j^o - \gamma\phi s_i^o) + \epsilon_i$$

Thus, on the one hand, if candidate i receives an attack from j , she loses a fraction ϕ of her initial support. On the other hand, if candidate i attacks j , she is able to steal a fraction $(1 - \alpha)\phi$ of j ’s initial support, but at the same time loses a fraction $\gamma\phi$ of her own support due to the backlash effect. The overall level of voter demobilization is thus given by:

$$x_\emptyset(a_{12}, a_{21}) = s_\emptyset^o + a_{12}\phi(\alpha s_2^o + \gamma s_1^o) + a_{21}\phi(\alpha s_1^o + \gamma s_2^o) + \epsilon_\emptyset$$

Finally, the probability that a candidate i wins the election is still given by expression (2) while the share of abstentions and null votes is:

$$p_\emptyset(a_{12}, a_{21}) = \frac{\exp(s_\emptyset(a_{12}, a_{21}))}{\exp(s_1(a_{12}, a_{21})) + \exp(s_2(a_{12}, a_{21})) + \exp(s_\emptyset(a_{12}, a_{21}))}$$

Note that this general version of the model reduces to our basic setup when $\alpha = 0$ and $\gamma = 0$.

Similarly, in three-candidate races the final level of support of candidate i is given by:

$$x_i(a) = (1 - \phi n_i)s_i^o + \sum_{j \neq i} \frac{n_j(1 - \alpha)\phi s_i^o}{2} - (\sum_{j \neq i} a_{ij})\gamma\phi s_i^o + \epsilon_i$$

where n_i denotes the number of attacks received by candidate i , while $\sum_{j \neq i} a_{ij}$ captures whether i attacked one of her opponents. In this case, the overall level of voter demobilization can be written as:

$$x_\emptyset(a) = s_\emptyset^o + \sum_i \sum_{j \neq i} a_{ij}\phi(\alpha s_j^o + \gamma s_i^o) + \epsilon_\emptyset$$

Moreover, the probability that a candidate i wins the election under single ballot plurality is still given by expression (6). As before, our basic setup is obtained when $\alpha = 0$ and $\gamma = 0$.

We conduct our analysis in two parts. First, we consider a version of the model without backlash effects, $\gamma = 0$. Interestingly, in this case, we are able to show that all results go through for any $\alpha \in [0, 1]$.³ Thus, it follows that our main conclusions remain unchanged even if an attack serves only to demobilize voters of rival candidates. In particular, none of our results depend on voters switching candidates, i.e. it is enough that they become disengaged for a campaign attack to have its desired effect.⁴ Next, we consider the general version of the model with backlash effects, $\gamma > 0$. While the model in this case becomes significantly less tractable, it is still possible to show that our main findings remain unchanged for any $\alpha \in [0, 1]$ provided that γ is not too large. Specifically, for two-candidate races we show that the restriction on γ is given by $\gamma < \frac{(1 - \alpha)s_2^o}{s_1^o}$, which simply guarantees that the benefit of an attack for

³All proofs can be adapted in a direct manner. Additional details are available upon request.

⁴In this sense, this version of the setup resembles models of negative vote buying, where voters are paid to abstain (Morgan and Várdy (2012)).

candidate 1 is always strictly positive. Intuitively, the benefit of an attack for candidate 1 is increasing in $(1 - \alpha)s_2^o$, i.e. the amount of support that she is able to steal from 2, and decreasing in γs_1^o , i.e. the size of the backlash effect for 1. For three-candidate races, the exact condition is much harder to pin down, but we can still show that, conditional on attacking someone, each candidate prefers to target her highest-ranked opponent for any $\gamma > 0$. Intuitively, while the backlash effect may reduce the overall willingness to attack, it does not affect the choice of whom to attack. The other results can be shown to hold when $\gamma > 0$ is sufficiently close to zero.⁵

G.3 Multiple Attacks per Candidate

In this subsection, we extend our basic model to allow candidates to target multiple opponents. Observe that the set of all possible profiles of attacks is now given by $\mathcal{N} = \{0, 1, 2\}^3$. Let $\mathcal{N}_{i,\ell} \subset \mathcal{N}$ denote the set of possible 3-tuples which the vector n may assume when we impose the restriction that player i does not attack ℓ , where ℓ denotes an individual opponent or a pair of rivals.⁶ Observe that, as in our main analysis, for any $n \in \mathcal{N}_{i,j}$, the benefit obtained by candidate i when she attacks a single opponent j is:

$$\tilde{\Delta}_{i,j}(n) = p_i(n_i, n_j + 1, n_k) - p_i(n_i, n_j, n_k),$$

whereas, for any $n \in \mathcal{N}_{i,jk}$, the benefit *per attack* obtained by candidate i when she targets both j and k is:

$$\tilde{\Delta}_{i,jk}(n) = \frac{p_i(n_i, n_j + 1, n_k + 1) - p_i(n_i, n_j, n_k)}{2},$$

where we normalize the total benefit by the number of attacks in order to make the measures of cost and benefit comparable.

As before, we focus our analysis on the case where the parameter ϕ is small, so that all results derived in Proposition 3 still hold in the present setting. The following proposition complements the characterization of the benefit function $\tilde{\Delta}_{i,\ell}(n)$ by extending it to consider multiple attacks.

Proposition G.8. *There exists a threshold $\bar{\phi} > 0$ such that if $\phi < \bar{\phi}$, then we have:*

i. *The benefit of attacking for candidate 1 is such that:*

$$\tilde{\Delta}_{1,3}(n_{1,3}) < \tilde{\Delta}_{1,23}(n_{1,23}) < \tilde{\Delta}_{1,2}(n_{1,2}) \quad \text{for any } n_{1,\ell} \in \mathcal{N}_{1,\ell}$$

ii. *The benefit of attacking for candidate 2 is such that:*

$$\tilde{\Delta}_{2,3}(n_{2,3}) < \tilde{\Delta}_{2,13}(n_{2,13}) < \tilde{\Delta}_{2,1}(n_{2,1}) \quad \text{for any } n_{2,\ell} \in \mathcal{N}_{2,\ell}$$

iii. *The benefit of attacking for candidate 3 is such that:*

$$\tilde{\Delta}_{3,2}(n_{3,2}) < \tilde{\Delta}_{3,12}(n_{3,12}) < \tilde{\Delta}_{3,1}(n_{3,1}) \quad \text{for any } n_{3,\ell} \in \mathcal{N}_{3,\ell}$$

Thus, similarly to Proposition 3, each candidate prefers to target her highest ranked opponent alone rather than any other rival or pair of rivals. Moreover, every candidate prefers to target both of her rivals together rather than the lowest ranked opponent alone, i.e. $\tilde{\Delta}_{1,3}(n_{1,3}) < \tilde{\Delta}_{1,23}(n_{1,23})$, $\tilde{\Delta}_{2,3}(n_{2,3}) < \tilde{\Delta}_{2,13}(n_{2,13})$ and $\tilde{\Delta}_{3,2}(n_{3,2}) < \tilde{\Delta}_{3,12}(n_{3,12})$. In other words, an attack against the lowest ranked opponent is more valuable when coupled with an attack against the highest ranked opponent. We also note that the relationship among $\tilde{\Delta}_{1,23}(n)$, $\tilde{\Delta}_{2,13}(n)$ and $\tilde{\Delta}_{3,12}(n)$ is ambiguous in general, so that it is not possible to completely pin down the structure of the equilibrium. However, from Propositions 3

⁵ Additional details are available upon request.

⁶ For the sake of clarity, we alter the notation slightly by adding a comma in the subindex to separate the attacking candidate i from the attacked candidates ℓ .

and G.8, it follows that in any equilibrium two properties must always hold: (i) the most likely direction of attack is from candidate 2 against 1, and (ii) no candidate attacks her lowest ranked opponent alone.

For specific parameter values, we are able to provide a complete characterization of the unique equilibrium of the game in pure strategies. Suppose, for instance, that $s_1^o = 0.4$, $s_2^o = 0.3$, $s_3^o = 0.2$ and $\phi = 0.1$, in which case we have the following ordering:⁷

$$\tilde{\Delta}_{1,23}(n_{1,\ell}) < \tilde{\Delta}_{2,13}(n_{2,\ell}) < \tilde{\Delta}_{1,2}(n'_{1,\ell}) < \tilde{\Delta}_{3,12}(n_{3,\ell}) < \tilde{\Delta}_{3,1}(n'_{3,\ell}) < \tilde{\Delta}_{2,1}(n_{2,\ell}),$$

for any $n_{i,\ell}, n'_{i,\ell} \in \mathcal{N}_{i,\ell}$. In this case, the equilibrium is such that:

- i. Every candidate attacks every other candidate if, and only if, $c \leq \tilde{\Delta}_{1,23}(2, 1, 1)$.
- ii. Candidate 2 attacks 1 and 3, candidate 3 attacks 1 and 2 and candidate 1 attacks 2 if, and only if, $\tilde{\Delta}_{1,23}(2, 1, 1) < c \leq \tilde{\Delta}_{2,13}(1, 2, 0)$.
- iii. Candidate 2 attacks 1, candidate 3 attacks 1 and 2 and candidate 1 attacks 2 if, and only if, $\tilde{\Delta}_{2,13}(1, 2, 0) < c \leq \tilde{\Delta}_{1,2}(2, 1, 0)$.
- iv. Candidate 2 attacks 1 and candidate 3 attacks 1 and 2 if, and only if, $\tilde{\Delta}_{1,2}(2, 1, 0) < c \leq \tilde{\Delta}_{3,12}(1, 0, 0)$.
- v. Candidate 2 attacks 1 and candidate 3 attacks 1 if, and only if, $\tilde{\Delta}_{3,12}(1, 0, 0) < c \leq \tilde{\Delta}_{3,1}(1, 0, 0)$.
- vi. Candidate 2 attacks 1 if, and only if, $\tilde{\Delta}_{3,1}(1, 0, 0) < c \leq \tilde{\Delta}_{2,1}(0, 0, 0)$.
- vii. No candidate attacks if, and only if, $c > \tilde{\Delta}_{2,1}(0, 0, 0)$.

Figure G.1 depicts the region of parameters where each class of equilibrium exists in this case. Note that, as the cost of attacking increases, we move through seven distinct regions corresponding to the different classes of equilibria describe above. Note that the equilibrium structure is similar to the one characterized in Proposition G.4, although the fact that candidates are now allowed to target multiple opponents adds significant complexity to the equilibrium.

G.4 Tullock Contest Success Function

In this subsection, we consider an alternative specification for the probability of winning based on the widely used contest success function proposed by Tullock (1980). Specifically, suppose that in a two-candidate race the final level of support of a candidate i is given by:

$$x_i(n_i, n_j) = \log \left(((1 - \phi n_i) s_i^o + n_j \phi s_j^o)^\mu \right) + \epsilon_i,$$

where $\mu > 0$ and ϵ_i is an iid shock with Type I Extreme Value distribution. As before, let $s_i(n_i, n_j) := (1 - \phi n_i) s_i^o + n_j \phi s_j^o$. In this case, the probability that candidate i wins is given by:

$$p_i^T(n_i, n_j) = \frac{s_i(n_i, n_j)^\mu}{s_i(n_i, n_j)^\mu + s_j(n_i, n_j)^\mu} \quad (20)$$

This particular functional form is known as Tullock CSF. Note that the parameter μ captures the sensitivity of a candidate's winning chances with respect to changes in her electoral support.

Similarly, in a three-candidate race, we assume that the final level of support of a candidate i is:

$$x_i(n) = \log \left(\left((1 - \phi n_i) s_i^o + \sum_{j \neq i} \frac{n_j \phi s_j^o}{2} \right)^\mu \right) + \epsilon_i,$$

⁷For $s_1^o = 0.4$, $s_2^o = 0.3$, $s_3^o = 0.2$ and $\phi = 0.1$, we have: $\tilde{\Delta}_{1,23}(2, 1, 1) = 0.0043$, $\tilde{\Delta}_{2,13}(1, 2, 0) = 0.0050$, $\tilde{\Delta}_{1,2}(2, 1, 0) = 0.0051$, $\tilde{\Delta}_{3,12}(1, 0, 0) = 0.0056$, $\tilde{\Delta}_{3,1}(1, 0, 0) = 0.0064$ and $\tilde{\Delta}_{2,1}(0, 0, 0) = 0.0072$. Additional results using alternative parameter values are available upon request.

where $\mu > 0$ and ϵ_i is an iid shock with Type I EV distribution. Let $s_i(n) := (1 - \phi n_i) s_i^o + \sum_{j \neq i} \frac{n_j \phi s_j^o}{2}$. In this case, the probability that candidate i wins is given by:

$$\hat{p}_i^T(n) = \frac{s_i(n)^\mu}{\sum_{k=1}^3 s_k(n)^\mu} \quad (21)$$

The benefit functions for two and three-candidate races are defined as in the main text and denoted by $\Delta_{ij}^T(n)$ and $\tilde{\Delta}_{ij}^T(n)$, respectively. In what follows, we investigate the robustness of our findings under this alternative setting. Overall, the Tullock CSF is considerably less tractable, although we are still able to show that our main qualitative results hold in this case.

Two-candidate Races. We begin our analysis by studying races with two candidates. We show that a version of Proposition 1 holds in this case, provided that ϕ is sufficiently small.

Proposition G.9. *There exists a threshold $\bar{\phi} > 0$ such that if $\phi < \bar{\phi}$, then candidate 2 is more aggressive than candidate 1 in the sense that:*

$$\Delta_{12}^T(n_1) < \Delta_{21}^T(n_2) \quad \text{for any } n_1, n_2 \in \{0, 1\}$$

Therefore, it follows that our previous result that candidate 2 is always the more aggressive candidate holds also for a Tullock CSF. Thus, the equilibrium structure remains very similar to the one characterized in Proposition 2. In particular, it is possible to show that, as the cost of attacking increases, we move through three different parameter regions where the following equilibria exist: (i) an equilibrium where both candidates attack, (ii) an equilibrium where only candidate 2 attacks, and (iii) an equilibrium where nobody attacks.

Three-candidate Races. Next, for races with three candidates, we show that a version of Proposition 3 holds for the Tullock CSF, provided that $\mu > 1$.

Proposition G.10. *There exists a threshold $\bar{\phi} > 0$ such that if $\phi < \bar{\phi}$ and $\mu > 1$, then we have:*

i. *For candidate 1, the benefit of an attack on 2 is larger than that of an attack on 3:*

$$\tilde{\Delta}_{13}^T(n) < \tilde{\Delta}_{12}^T(n') \quad \text{for any } n, n' \in \mathcal{N}_1$$

ii. *For candidate 2, the benefit of an attack on 1 is larger than that of an attack on 3:*

$$\tilde{\Delta}_{23}^T(n) < \tilde{\Delta}_{21}^T(n') \quad \text{for any } n, n' \in \mathcal{N}_2$$

iii. *For candidate 3, the benefit of an attack on 1 is larger than that of an attack on 2:*

$$\tilde{\Delta}_{32}^T(n) < \tilde{\Delta}_{31}^T(n') \quad \text{for any } n, n' \in \mathcal{N}_3$$

iv. *Candidate 2 is the most aggressive candidate in the sense that:*

$$\max\{\tilde{\Delta}_{12}^T(n_1), \tilde{\Delta}_{31}^T(n_3)\} < \tilde{\Delta}_{21}^T(n_2) \quad \text{for any } n_i \in \mathcal{N}_i$$

Therefore, as in our main analysis, all candidates prefer to target their highest ranked opponent and candidate 2 is always the most aggressive, provided that $\mu > 1$. Finally, the model becomes too intractable to allow us to examine the robustness of the comparative static results concernig the dilution effect and the comparison between single and dual ballot plurality systems, derived in Propositions 4 and 5 respectively. In the next subsection, we present simulation results suggesting that those findings also hold under a Tullock CSF.

G.5 Simulations

Most of our theoretical results—especially those for three-candidate races—are derived under the assumption that the impact of a campaign attack, as captured by the parameter ϕ , is sufficiently small. In this subsection, we present simulation results showing that our main findings hold for “reasonable”, i.e. not vanishingly small, values of ϕ . We begin our analysis by examining two-candidate races with $s_1^o = 0.6$ and $s_2^o = 0.4$.⁸ We compute the benefits of attacking for candidates 1 and 2 under both Logit and Tullock CSFs, for $\phi \in \{0.05, 0.1, 0.15\}$ and $\mu \in \{1.5, 2\}$.⁹ The simulation results are reported in Table G.1. Consistent with Propositions 1 and G.9, for both Logit and Tullock CSFs, we always have $\Delta_{12}(n_1) < \Delta_{21}(n_2)$ for any $n_1, n_2 \in \{0, 1\}$, implying that candidate 2 is always the most aggressive. Moreover, in all cases, $\Delta_{12}(0) < \Delta_{12}(1) < \Delta_{21}(1) < \Delta_{21}(0)$.

Next, we study three-candidate races with $s_1^o = 0.4$, $s_2^o = 0.3$ and $s_3^o = 0.2$, considering $\phi \in \{0.05, 0.1, 0.15\}$ and $\mu \in \{1.5, 2\}$. For this and the following exercises, we evaluate the benefit function $\tilde{\Delta}_{ij}(n)$ at $n = (0, 0, 0)$, that is, assuming no candidate initially receives an attack.¹⁰ The results presented in Table G.2 show that, in all cases, $\tilde{\Delta}_{13} < \tilde{\Delta}_{12}$, $\tilde{\Delta}_{23} < \tilde{\Delta}_{21}$ and $\tilde{\Delta}_{32} < \tilde{\Delta}_{31}$, with $\max\{\tilde{\Delta}_{12}, \tilde{\Delta}_{31}\} < \tilde{\Delta}_{21}$, in line with Propositions 3 and G.10. That is, each candidate prefers to target her highest-ranked opponent, with candidate 2 being always the most aggressive. Furthermore, consistent with Proposition G.2 and Corollary G.1, the ordering $\tilde{\Delta}_{23} < \{\tilde{\Delta}_{13}, \tilde{\Delta}_{32}\} < \{\tilde{\Delta}_{12}, \tilde{\Delta}_{31}\} < \tilde{\Delta}_{21}$ holds for both Logit and Tullock CSFs.

We next examine how the incentives for the top two candidates to attack each other vary as the strength of the 3rd place candidate increases. Specifically, Figure G.2 plots the benefits of an attack between candidates 1 and 2 as a function of candidate 3’s initial support, assuming $s_1^o = 0.4$, $s_2^o = 0.3$ and $\phi = 0.1$, for both the Logit CSF (Panel A) and the Tullock CSF with $\mu = 1.5$ (Panel B). Consistent with the “dillution effect” derived in Proposition 4, we find that $\tilde{\Delta}_{12}$ and $\tilde{\Delta}_{21}$ are strictly decreasing in s_3^o , implying that both candidates become less aggressive towards each other as the 3rd place candidate becomes stronger. Interestingly, although Proposition 4 applies only to the Logit CSF, our simulation suggest that the dillution effect extends to the Tullock CSF as well.

Moreover, we examine the situation discussed in footnote 15, where candidates 1 and 2 lie on opposite sides of the ideological spectrum, in which case a larger share of the votes can be expected to leak to the third candidate. Formally, let α_{ij} denote the share of electoral support that candidate i receives when j is attacked. Our baseline model assumes that $\alpha_{ij} = 0.5$ for any pair i and j (see equation 5). To capture polarization between candidates 1 and 2, we assume $\alpha_{12} = \alpha_{21} = 0.4$, thereby limiting the direct transfer of support between the two front-runners, with $\alpha_{32} = \alpha_{31} = 0.6$ and $\alpha_{13} = \alpha_{23} = 0.5$. In Table G.3, we compare the percentage change in the benefits of an attack between candidates 1 and 2 under the benchmark case (symmetric redistribution) and the scenario where 1 and 2 are polarized (asymmetric redistribution), as the support for candidate 3 increases from $s_3^o = 0.1$ to $s_3^o = 0.25$, assuming a Logit CSF with $\phi = 0.1$, $s_1^o = 0.4$ and $s_2^o = 0.3$. Consistent with our conjecture, the results indicate that the dilution effect is amplified when candidates 1 and 2 are ideologically distant.

Finally, we compare the benefits of attacks under single and dual ballot plurality systems for both the Logit CSF and the Tullock CSF with $\mu = 1.5$, assuming $s_1^o = 0.4$, $s_2^o = 0.3$, $s_3^o = 0.2$ and $\phi = 0.1$. Consistent with Proposition 5, the simulation results reported in Table G.4 show that the ordering $\max\{\tilde{\Delta}_{12}, \tilde{\Delta}_{13}\} < \max\{\tilde{\Delta}_{21}, \tilde{\Delta}_{23}\} < \max\{\tilde{\Delta}_{31}, \tilde{\Delta}_{32}\}$ holds under dual ballot plurality for both CSFs, implying that candidate 3 is always the most aggressive candidate, followed by candidates 2 and 1. Moreover, switching from single to dual ballot plurality produces a particularly large increase in $\tilde{\Delta}_{23}$ and $\tilde{\Delta}_{32}$. This result is in line with Proposition G.6, which shows that candidates 2 and 3 may become more aggressive toward each other under dual ballot plurality. In particular, note that for the Tullock CSF we have $\tilde{\Delta}_{23} > \tilde{\Delta}_{21}$ and $\tilde{\Delta}_{32} > \tilde{\Delta}_{31}$ under dual ballot.¹¹

⁸Our simulation results are robust to alternative initial levels of support, provided that $s_1^o > s_2^o$. Additional results are available upon request.

⁹We remind the reader that μ is a parameter of the Tullock CSF. Additional results with alternative values for ϕ and μ are available upon request.

¹⁰Similar results (available upon request) are obtained for alternative values of $n = (n_1, n_2, n_3)$.

¹¹Interestingly, the results for the Tullock CSF also show that the benefits of campaign attacks between candidates 1 and 2 decrease when we move from single to dual ballot plurality.

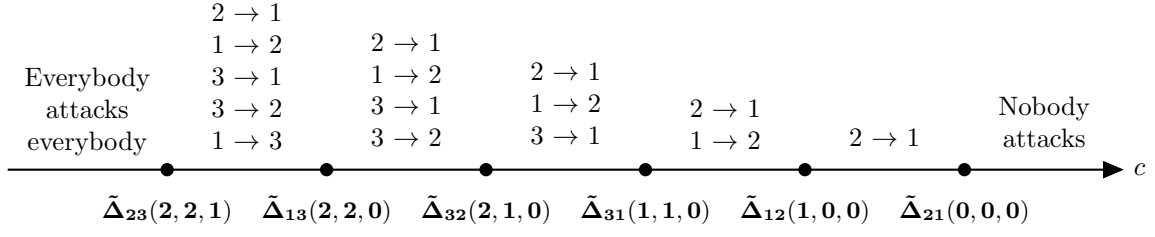
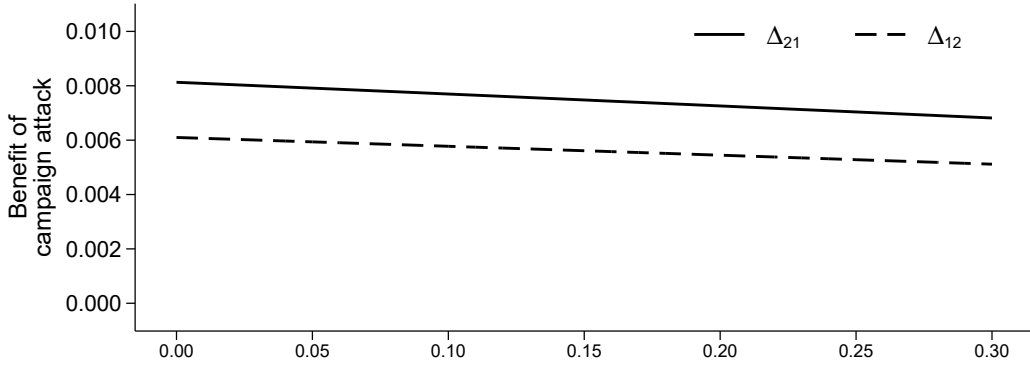


Figure G.1: Three-Candidate Races with Multiple Targets: Equilibria

Notes: This figure depicts the unique equilibrium that exists in each region of the parameters for the case where candidates are allowed to target multiple opponents, and supposing that $s_1^o = 0.4$, $s_2^o = 0.3$, $s_3^o = 0.2$ and $\phi = 0.1$. $i \rightarrow j$ denotes an attack from candidate i against j .

Panel A: Logit CSF



Panel B: Tullock CSF

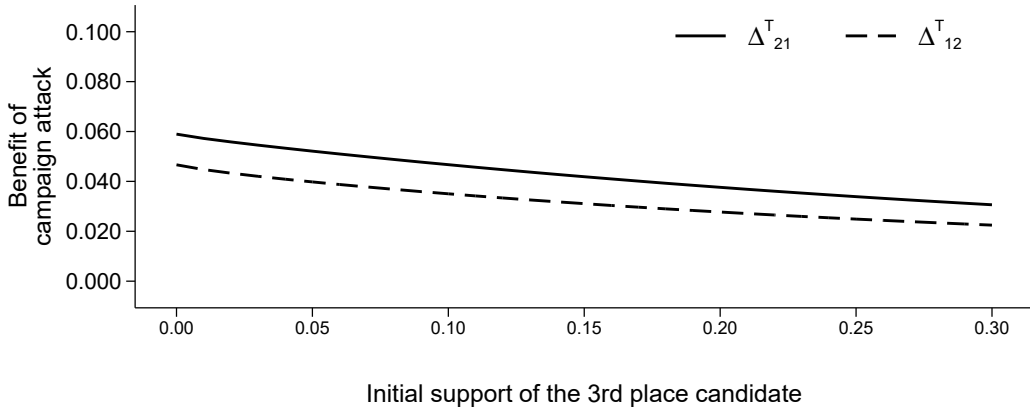


Figure G.2: Simulations: Dillution Effect

Notes: This figure plots simulated values for the benefits of campaign attacks between candidates 1 and 2 as a function of the initial support of candidate 3 for both the Logit CSF (Panel A) and the Tullock CSF with $\mu = 1.5$ (Panel B). Our simulations assume that $s_1^o = 0.4$, $s_2^o = 0.3$ and $\phi = 0.1$, with s_3^o varying in the interval $[0, 0.3]$. The benefit functions are evaluated at $n = (0, 0, 0)$.

Table G.1: Simulations: Two-Candidate Races

	Logit CSF			Tullock CSF					
				$\mu = 1.5$			$\mu = 2$		
	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$
$\Delta_{12}(0)$	0.0098	0.0197	0.0294	0.0282	0.0557	0.0825	0.0346	0.0673	0.0979
$\Delta_{12}(1)$	0.0099	0.0199	0.0299	0.0290	0.0588	0.0891	0.0370	0.0765	0.1173
$\Delta_{21}(0)$	0.0148	0.0298	0.0448	0.0433	0.0876	0.1325	0.0549	0.1128	0.1723
$\Delta_{21}(1)$	0.0148	0.0296	0.0443	0.0425	0.0845	0.1259	0.0525	0.1036	0.1529

Notes: This table reports simulated values for the benefits of campaign attacks in two-candidate races under both Logit and Tullock CSFs. Our simulations assume that $s_1^o = 0.6$ and $s_2^o = 0.4$, and consider $\phi \in \{0.05, 0.1, 0.15\}$ and $\mu \in \{1.5, 2\}$.

Table G.2: Simulations: Three-Candidate Races

	Logit CSF			Tullock CSF					
				$\mu = 1.5$			$\mu = 2$		
	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$
$\tilde{\Delta}_{12}(0, 0, 0)$	0.0027	0.0054	0.0081	0.0140	0.0277	0.0409	0.0202	0.0394	0.0574
$\tilde{\Delta}_{13}(0, 0, 0)$	0.0016	0.0032	0.0049	0.0071	0.0140	0.0206	0.0078	0.0150	0.0216
$\tilde{\Delta}_{21}(0, 0, 0)$	0.0036	0.0072	0.0108	0.0188	0.0376	0.0563	0.0273	0.0548	0.0822
$\tilde{\Delta}_{23}(0, 0, 0)$	0.0014	0.0029	0.0044	0.0066	0.0131	0.0195	0.0069	0.0136	0.0200
$\tilde{\Delta}_{31}(0, 0, 0)$	0.0032	0.0065	0.0098	0.0147	0.0297	0.0449	0.0170	0.0346	0.0527
$\tilde{\Delta}_{32}(0, 0, 0)$	0.0022	0.0044	0.0066	0.0099	0.0199	0.0299	0.0103	0.0207	0.0310

Notes: This table reports simulated values for the benefits of campaign attacks in three-candidate races under both Logit and Tullock CSFs. Our simulations assume that $s_1^o = 0.4$, $s_2^o = 0.3$ and $s_3^o = 0.2$, and consider $\phi \in \{0.05, 0.1, 0.15\}$ and $\mu \in \{1.5, 2\}$. The benefit functions are evaluated at $n = (0, 0, 0)$.

Table G.3: Simulations: Dilution Effect (Symmetric vs Asymmetric Redistribution)

	Symmetric Redistribution: Benchmark			Asymmetric Redistribution: Polarized 1 & 2		
	$s_3^o = 0.1$	$s_3' = 0.25$	Dilution Effect (%)	$s_3^o = 0.1$	$s_3' = 0.25$	Dilution Effect (%)
$\tilde{\Delta}_{12}(0, 0, 0)$	0.0057	0.0052	-0.0854	0.0047	0.0042	-0.1057
$\tilde{\Delta}_{21}(0, 0, 0)$	0.0076	0.0070	-0.0857	0.0063	0.0057	-0.1040

Notes: This table reports simulated values for the benefits of campaign attacks between candidates 1 and 2 under the benchmark case (symmetric redistribution, with $\alpha_{ij} = 0.5$ for any pair i and j) and an alternative scenario where candidates 1 and 2 are polarized (asymmetric redistribution, with $\alpha_{12} = \alpha_{21} = 0.4$, $\alpha_{32} = \alpha_{31} = 0.6$, and $\alpha_{13} = \alpha_{23} = 0.5$). Our simulations are based on a Logit CSF and assume that $\phi = 0.1$, $s_1^o = 0.4$, $s_2^o = 0.3$, with the initial share of candidate 3 varying from $s_3^o = 0.1$ to $s_3' = 0.1$. The dilution effect reports the percentage variation in the benefits of an attack, calculated relative to the values shown in the previous two columns. The benefit functions are evaluated at $n = (0, 0, 0)$.

Table G.4: Simulations: Single vs Dual Ballot Plurality

	Logit CSF			Tullock CSF ($\mu = 1.5$)		
	Single Ballot	Dual Ballot	Variation (%)	Single Ballot	Dual Ballot	Variation (%)
$\tilde{\Delta}_{12}(0, 0, 0)$	0.0054	0.0057	+0.05	0.0277	0.0114	-0.58
$\tilde{\Delta}_{13}(0, 0, 0)$	0.0032	0.0041	+0.25	0.0140	0.0189	+0.35
$\tilde{\Delta}_{21}(0, 0, 0)$	0.0072	0.0077	+0.06	0.0376	0.0139	-0.62
$\tilde{\Delta}_{23}(0, 0, 0)$	0.0029	0.0044	+0.51	0.0131	0.0354	+1.68
$\tilde{\Delta}_{31}(0, 0, 0)$	0.0065	0.0083	+0.26	0.0297	0.0428	+0.44
$\tilde{\Delta}_{32}(0, 0, 0)$	0.0044	0.0067	+0.51	0.0199	0.0533	+1.66

Notes: This table reports simulated values for the benefits of campaign attacks in three-candidate races under single and dual ballot plurality for both the Logit CSF and the Tullock CSF with $\mu = 1.5$. Our simulations assume that $s_1^o = 0.4$, $s_2^o = 0.3$, $s_3^o = 0.2$ and $\phi = 0.1$. The benefit functions are evaluated at $n = (0, 0, 0)$.

H Model: Proofs

H.1 Proposition 1

Item i. We want to show that $\Delta_{12}(0) < \Delta_{12}(1)$, which can be expressed as:

$$\frac{\exp(s_1^o + \phi s_2^o)}{\exp(s_1^o + \phi s_2^o) + \exp((1-\phi)s_2^o)} - \frac{\exp(s_1^o)}{\exp(s_1^o) + \exp(s_2^o)} < \frac{\exp((1-\phi)s_1^o + \phi s_2^o)}{\exp((1-\phi)s_1^o + \phi s_2^o) + \exp((1-\phi)s_2^o + \phi s_1^o)} - \frac{\exp((1-\phi)s_1^o)}{\exp((1-\phi)s_1^o) + \exp(s_2^o + \phi s_1^o)}$$

After some algebra, we can rewrite the above expression as:

$$\frac{\exp(s_1^o + s_2^o)(\exp(2\phi s_2^o) - 1)}{(\exp(s_1^o) + \exp(s_2^o))(\exp(s_2^o) + \exp(s_1^o + 2\phi s_2^o))} < \frac{\exp((1+2\phi)s_1^o + s_2^o)(\exp(2\phi s_2^o) - 1)}{(\exp(s_1^o) + \exp(2\phi s_1^o + s_2^o))(\exp(2\phi s_1^o + s_2^o) + \exp(s_1^o + 2\phi s_2^o))}$$

Rearranging and simplifying, we get:

$$\exp(s_1^o + s_2^o)(\exp(2\phi s_1^o) - 1)(\exp(2\phi s_2^o) - 1)(\exp(2(s_1^o + \phi s_2^o)) - \exp(2(\phi s_1^o + s_2^o))) > 0$$

which always holds, since $\exp(2\phi s_1^o) > 1$, $\exp(2\phi s_2^o) > 1$ and $\exp(2(s_1^o + \phi s_2^o)) > \exp(2(\phi s_1^o + s_2^o))$ for any $0 < \phi < 1$ and $s_1^o > s_2^o$.

Item ii. We want to show that $\Delta_{21}(1) < \Delta_{21}(0)$, which can be expressed as:

$$\frac{\exp(\phi s_1^o + (1-\phi)s_2^o)}{\exp((1-\phi)s_1^o + \phi s_2^o) + \exp(\phi s_1^o + (1-\phi)s_2^o)} - \frac{\exp((1-\phi)s_2^o)}{\exp(s_1^o + \phi s_2^o) + \exp((1-\phi)s_2^o)} < \frac{\exp(\phi s_1^o + s_2^o)}{\exp((1-\phi)s_1^o) + \exp(\phi s_1^o + s_2^o)} - \frac{\exp(s_2^o)}{\exp(s_1^o) + \exp(s_2^o)}$$

After some algebra, we can rewrite the above expression as:

$$\frac{\exp(s_1^o + (1+2\phi)s_2^o)(\exp(2\phi s_1^o) - 1)}{(\exp(s_2^o) + \exp(s_1^o + 2\phi s_2^o))(\exp(2\phi s_1^o + s_2^o) + \exp(s_1^o + 2\phi s_2^o))} < \frac{\exp(s_1^o + s_2^o)(\exp(2\phi s_1^o) - 1)}{(\exp(s_1^o) + \exp(s_2^o))(\exp(s_1^o) + \exp(2\phi s_1^o + s_2^o))}$$

Rearranging and simplifying, we get:

$$\exp(s_1^o + s_2^o)(\exp(2\phi s_1^o) - 1)(\exp(2\phi s_2^o) - 1)(\exp(2(s_1^o + \phi s_2^o)) - \exp(2(\phi s_1^o + s_2^o))) > 0$$

which always holds, since $\exp(2\phi s_1^o) > 1$, $\exp(2\phi s_2^o) > 1$ and $\exp(2(s_1^o + \phi s_2^o)) > \exp(2(\phi s_1^o + s_2^o))$ for any $0 < \phi < 1$ and $s_1^o > s_2^o$.

Item iii. We want to show that $\Delta_{12}(1) < \Delta_{21}(1)$, which can be expressed as:

$$\frac{\exp((1-\phi)s_1^o + \phi s_2^o)}{\exp((1-\phi)s_1^o + \phi s_2^o) + \exp((1-\phi)s_2^o + \phi s_1^o)} - \frac{\exp((1-\phi)s_1^o)}{\exp((1-\phi)s_1^o) + \exp(s_2^o + \phi s_1^o)} < \frac{\exp(\phi s_1^o + (1-\phi)s_2^o)}{\exp((1-\phi)s_1^o + \phi s_2^o) + \exp(\phi s_1^o + (1-\phi)s_2^o)} - \frac{\exp((1-\phi)s_2^o)}{\exp(s_1^o + \phi s_2^o) + \exp((1-\phi)s_2^o)}$$

After some algebra, we can rewrite the above expression as:

$$\frac{\exp((1+2\phi)s_1^o + s_2^o)(\exp(2\phi s_2^o) - 1)}{(\exp(s_1^o) + \exp(2\phi s_1^o + s_2^o))(\exp(2\phi s_1^o + s_2^o) + \exp(s_1^o + 2\phi s_2^o))} < \frac{\exp(s_1^o + (1+2\phi)s_2^o)(\exp(2\phi s_1^o) - 1)}{(\exp(s_2^o) + \exp(s_1^o + 2\phi s_2^o))(\exp(2\phi s_1^o + s_2^o) + \exp(s_1^o + 2\phi s_2^o))}$$

Rearranging and simplifying, we get:

$$\begin{aligned} & \exp(2\phi s_1^o + s_2^o) + 2\exp(s_1^o + 2\phi s_1^o + 2\phi s_2^o) + \exp(4\phi s_1^o + s_2^o + 2\phi s_2^o) \\ & - \exp(s_1^o + 2\phi s_2^o) - \exp(s_1^o + 2\phi s_1^o + 4\phi s_2^o) - 2\exp(2\phi s_1^o + s_2^o + 2\phi s_2^o) > 0 \end{aligned}$$

which in turn can be re-expressed as:

$$\exp((1+2\phi)(s_1^o + s_2^o))(\exp(2\phi s_1^o) - \exp(2\phi s_2^o)) > 0$$

which always holds since $\exp(2\phi s_1^o) - \exp(2\phi s_2^o) > 0$ for any $0 < \phi < 1$ and $s_1^o > s_2^o$. ■

H.2 Proposition 2

Note that from Proposition 1, we have $\Delta_{12}(0) < \Delta_{12}(1) < \Delta_{21}(1) < \Delta_{21}(0)$. Thus, we need to consider five cases for the cost parameter $c \in \mathbb{R}_+$:

- i.* Case 1: Suppose that $c \leq \Delta_{12}(0)$. In this case, both candidates are always willing to attack their opponent. Thus, in equilibrium, both candidates attack.
- ii.* Case 2: Suppose that $\Delta_{12}(0) < c \leq \Delta_{12}(1)$. In this case, candidate 2 is always willing to attack, whereas candidate 1 is willing to attack only if attacked. Thus, in equilibrium, both candidates attack.
- iii.* Case 3: Suppose that $\Delta_{12}(1) < c \leq \Delta_{21}(1)$. In this case, candidate 2 is always willing to attack, whereas candidate 1 is never willing to attack. Thus, in equilibrium, only candidate 2 attacks.
- iv.* Case 4: Suppose that $\Delta_{21}(1) < c \leq \Delta_{21}(0)$. In this case, candidate 2 is willing to attack only if not attacked, whereas candidate 1 is never willing to attack. Thus, in equilibrium, only candidate 2 attacks.
- v.* Case 5: Suppose that $\Delta_{21}(0) < c$. In this case, both candidates are never willing to attack. Thus, in equilibrium, no candidate attacks.

Therefore, in equilibrium, we have: (i) both candidates attack if and only if $c \leq \Delta_{12}(1)$; (ii) only candidate 2 attacks if and only if $\Delta_{12}(1) < c \leq \Delta_{21}(0)$; and (iii) no candidate attacks if and only if $\Delta_{21}(0) < c$. ■

H.3 Proposition 3

Item i. We want to show that if ϕ is small enough then $\tilde{\Delta}_{12}(n) > \tilde{\Delta}_{13}(n')$ for any $n, n' \in \mathcal{N}_1$. To do so, let $f_1(\phi, n, n') := \tilde{\Delta}_{12}(n) - \tilde{\Delta}_{13}(n')$. We will show that $f_1(0, n, n') = 0$ and $\partial f_1(0, n, n')/\partial \phi > 0$ for any $n, n' \in \mathcal{N}_1$, so that by taking a first-order Taylor expansion of the function $f_1(\phi, n, n')$ around $\phi = 0$ it follows that $f_1(\phi, n, n') > 0$ for ϕ sufficiently small. It is immediate to see that $f_1(0, n, n') = 0$, given that $\tilde{\Delta}_{12}(n) = \tilde{\Delta}_{13}(n') = 0$ when $\phi = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_1(\phi, n, n')}{\partial \phi} = \frac{3\exp(s_1^o)(s_2^o \exp(s_2^o) - s_3^o \exp(s_3^o))}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o \exp(s_2^o) > s_3^o \exp(s_3^o)$.

Item ii. We want to show that if ϕ is small enough then $\tilde{\Delta}_{21}(n) > \tilde{\Delta}_{23}(n')$ for any $n, n' \in \mathcal{N}_2$. To do so, let $f_2(\phi, n, n') := \tilde{\Delta}_{21}(n) - \tilde{\Delta}_{23}(n')$. We will show that $f_2(0, n, n') = 0$ and $\partial f_2(0, n, n')/\partial \phi > 0$ for any $n, n' \in \mathcal{N}_2$. It is immediate to see that $f_2(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_2(0, n, n')}{\partial \phi} = \frac{3 \exp(s_2^o) (s_1^o \exp(s_1^o) - s_3^o \exp(s_3^o))}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o \exp(s_1^o) > s_3^o \exp(s_3^o)$.

Item iii. We want to show that if ϕ is small enough then $\tilde{\Delta}_{31}(n) > \tilde{\Delta}_{32}(n')$ for any $n, n' \in \mathcal{N}_3$. To do so, let $f_3(\phi, n, n') := \tilde{\Delta}_{31}(n) - \tilde{\Delta}_{32}(n')$. We will show that $f_3(0, n, n') = 0$ and $\partial f_3(0, n, n')/\partial \phi > 0$ for any $n, n' \in \mathcal{N}_3$. It is immediate to see that $f_3(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_3(0, n, n')}{\partial \phi} = \frac{3 \exp(s_3^o) (s_1^o \exp(s_1^o) - s_2^o \exp(s_2^o))}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o \exp(s_1^o) > s_2^o \exp(s_2^o)$.

Item iv. First, we want to show that if ϕ is small enough then $\tilde{\Delta}_{21}(n') > \tilde{\Delta}_{12}(n)$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_2$. To do so, let $h_{12}(\phi, n, n') := \tilde{\Delta}_{21}(n') - \tilde{\Delta}_{12}(n)$. We will show that $h_{12}(0, n, n') = 0$ and $\partial h_{12}(0, n, n')/\partial \phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_2$. It is immediate to see that $h_{12}(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial h_{12}(0, n, n')}{\partial \phi} = \frac{3 \exp(s_1^o + s_2^o) (s_1^o - s_2)}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o > s_2^o$. Next, we want to show that if ϕ is small enough then $\tilde{\Delta}_{21}(n') > \tilde{\Delta}_{31}(n'')$ for any $n' \in \mathcal{N}_2$ and $n'' \in \mathcal{N}_3$. To do so, let $h_{23}(\phi, n', n'') := \tilde{\Delta}_{21}(n') - \tilde{\Delta}_{31}(n'')$. We will show that $h_{23}(0, n', n'') = 0$ and $\partial h_{23}(0, n', n'')/\partial \phi > 0$ for any $n' \in \mathcal{N}_2$ and $n'' \in \mathcal{N}_3$. It is immediate to see that $h_{23}(0, n', n'') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial h_{23}(0, n', n'')}{\partial \phi} = \frac{3 \exp(s_1^o) (\exp(s_2^o) - \exp(s_3^o)) s_1^o}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $\exp(s_2^o) > \exp(s_3^o)$. Therefore, we conclude that $\tilde{\Delta}_{21}(n') > \max\{\tilde{\Delta}_{12}(n), \tilde{\Delta}_{31}(n'')\}$ for any $n \in \mathcal{N}_1$, $n' \in \mathcal{N}_2$ and $n'' \in \mathcal{N}_3$. ■

H.4 Proposition 4

First, we want to show that if ϕ is small enough then $\partial \tilde{\Delta}_{12}(n)/\partial s_3^o < 0$ for any $n = (n_1, n_2, n_3) \in \mathcal{N}_1$. Taking the derivative of $\tilde{\Delta}_{12}(n)$ w.r.t. s_3^o , re-arranging and simplifying, we get:

$$\frac{\partial \tilde{\Delta}_{12}(n)}{\partial s_3^o} = \frac{\Gamma_1 \Gamma_2}{2(\Gamma_3)^2 (\Gamma_4)^2} (3n_3 \phi - 2)$$

where

$$\Gamma_1 = \exp((1 + n_1 \phi) s_1^o + 3n_2 \phi s_2^o + (1 + n_3 \phi) s_3^o) \left(\exp\left(\frac{3}{2} \phi s_2^o\right) - 1 \right) > 0$$

$$\begin{aligned}\Gamma_2 = & \exp\left(\frac{3}{2}\phi(n_1s_1^o + n_3s_3^o)\right) + \exp\left(\frac{3}{2}\phi(n_1s_1^o + s_2^o + n_3s_3^o)\right) \\ & + 2\exp\left(\frac{1}{2}(3n_1\phi s_1^o - (2-3(1+n_2)\phi)s_2^o + 2s_3^o)\right) \\ & + 2\exp\left(\frac{1}{2}(2s_1^o - (2-3(1+n_2)\phi)s_2^o + 3n_3\phi s_3^o)\right) > 0\end{aligned}$$

$$\begin{aligned}\Gamma_3 = & \exp\left(\frac{1}{2}\phi(n_1s_1^o + n_3s_3^o)\right) + \exp\left(\frac{1}{2}(2(1-n_1\phi)s_1^o - (2-3(1+n_2)\phi)s_2^o + n_3\phi s_3^o)\right) \\ & + \exp\left(\frac{1}{2}(n_1\phi s_1^o - (2-3(1+n_2)\phi)s_2^o + 2(1-n_3\phi)s_3^o)\right) > 0\end{aligned}$$

$$\begin{aligned}\Gamma_4 = & \exp\left(\frac{1}{2}(3n_1\phi s_1^o + 3n_2\phi s_2^o + 2s_3^o)\right) + \exp\left(\frac{1}{2}(3n_1\phi s_1^o + 2s_2^o + 3n_3\phi s_3^o)\right) \\ & + \exp\left(\frac{1}{2}(2s_1^o + 3n_2\phi s_2^o + 3n_3\phi s_3^o)\right) > 0\end{aligned}$$

Note that the sign of the derivative is completely determined by the term $3n_3\phi - 2$, which is guaranteed to be negative as long as ϕ is sufficiently small. More specifically, since $n_3 \leq 2$, the sufficient condition is $\phi < \frac{1}{3}$.

Next, we want to show that if ϕ is small enough then $\partial\tilde{\Delta}_{21}(n)/\partial s_3^o < 0$ for any $n = (n_1, n_2, n_3) \in \mathcal{N}_2$. Taking the derivative of $\tilde{\Delta}_{21}(n)$ w.r.t. s_3^o , re-arranging and simplifying, we get:

$$\frac{\partial\tilde{\Delta}_{21}(n)}{\partial s_3^o} = \frac{\Psi_1\Psi_2}{2(\Psi_3)^2(\Psi_4)^2}(3n_3\phi - 2)$$

where

$$\Psi_1 = \exp(3n_1\phi s_1^o + (1+3n_2\phi)s_2^o + (1+n_3\phi)s_3^o) \left(\exp\left(\frac{3}{2}\phi s_1^o\right) - 1 \right) > 0$$

$$\begin{aligned}\Psi_2 = & \exp\left(\frac{3}{2}\phi(n_2s_2^o + n_3s_3^o)\right) + \exp\left(\frac{3}{2}\phi(s_1^o + n_2s_2^o + n_3s_3^o)\right) \\ & + 2\exp\left(\frac{1}{2}(-(2-3(1+n_1)\phi)s_1^o - 3n_2\phi s_2^o + 2s_3^o)\right) \\ & + 2\exp\left(\frac{1}{2}(-(2-3(1+n_1)\phi)s_1^o + s_2^o + 3n_3\phi s_3^o)\right) > 0\end{aligned}$$

$$\begin{aligned}\Psi_3 = & \exp\left(-\frac{1}{2}(-(2-3(1+n_1)\phi)s_1^o + 3n_2\phi s_2^o + 2s_3^o)\right) + \exp\left(\frac{3}{2}(n_2\phi s_2^o + n_3\phi s_3^o)\right) \\ & + \exp\left(-\frac{1}{2}(-(2-3(1+n_1)\phi)s_1^o + 2s_2^o + 3n_3\phi s_3^o)\right) > 0\end{aligned}$$

$$\begin{aligned}\Psi_4 = & \exp\left(\frac{1}{2}(3n_1\phi s_1^o + 3n_2\phi s_2^o + 2s_3^o)\right) + \exp\left(\frac{1}{2}(3n_1\phi s_1^o + 2s_2^o + 3n_3\phi s_3^o)\right) \\ & + \exp\left(\frac{1}{2}(2s_1^o + 3n_2\phi s_2^o + 3n_3\phi s_3^o)\right) > 0\end{aligned}$$

Note that, as before, the sign of the derivative is completely determined by the term $3n_3\phi - 2$, which is guaranteed to be negative as long as ϕ is sufficiently small. More specifically, since $n_3 \leq 2$, the sufficient condition is $\phi < \frac{1}{3}$. ■

H.5 Proposition 5

First, we want to show that if ϕ is small enough then $\max\{\tilde{\Delta}_{31}^{DB}(n_3), \tilde{\Delta}_{32}^{DB}(n'_3)\} > \max\{\tilde{\Delta}_{21}^{DB}(n_2), \tilde{\Delta}_{23}^{DB}(n'_2)\}$ for any $n_i, n'_i \in \mathcal{N}_i$. To do so, we will show that: (i) $\tilde{\Delta}_{31}^{DB}(n_3) > \tilde{\Delta}_{21}^{DB}(n_2)$ for any $n_i \in \mathcal{N}_i$ and (ii) $\tilde{\Delta}_{32}^{DB}(n'_3) > \tilde{\Delta}_{23}^{DB}(n'_2)$ for any $n_i \in \mathcal{N}_i$. First, let $f_{23}(\phi, n, n') := \tilde{\Delta}_{31}^{DB}(n') - \tilde{\Delta}_{21}^{DB}(n)$. We will show that $f_{23}(0, n, n') = 0$ and $\partial f_{23}(0, n, n')/\partial\phi > 0$ for any $n \in \mathcal{N}_2$ and $n' \in \mathcal{N}_3$, so that by taking a first-order Taylor expansion of the function $f_{23}(\phi, n, n')$ around $\phi = 0$ it follows that $f_{23}(\phi, n, n') > 0$ for ϕ sufficiently small. It is immediate to see that $f_{23}(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{23}(0, n, n')}{\partial\phi} = \frac{3s_1^o \exp(s_1^o + s_2^o + s_3^o) \Gamma(\exp(s_2^o) - \exp(s_3^o))}{2},$$

where

$$\Gamma = \exp(2s_2^o) + \exp(2s_3^o) + 3\exp(2s_1^o) + 3\exp(s_2^o + s_3^o) + 4\exp(s_1^o + s_2^o) + 4\exp(s_1^o + s_3^o) > 0$$

Note that the derivative above is always strictly positive, since $\exp(s_2^o) > \exp(s_3^o)$. Next, let $g_{23}(\phi, n, n') := \tilde{\Delta}_{32}^{DB}(n') - \tilde{\Delta}_{23}^{DB}(n)$. We will show that $g_{23}(0, n, n') = 0$ and $\partial g_{23}(0, n, n')/\partial\phi > 0$ for any $n \in \mathcal{N}_2$ and $n' \in \mathcal{N}_3$. It is immediate to see that $g_{23}(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g_{23}(0, n, n')}{\partial\phi} = \frac{3\exp(s_1^o + s_2^o + s_3^o)(\exp(s_1^o) + 2\exp(s_2^o) + 2\exp(s_3^o))(s_2^o - s_3^o)}{2(\exp(s_2^o) + \exp(s_3^o))^2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o > s_3^o$. Therefore, from the arguments above, it follows that $\max\{\tilde{\Delta}_{31}^{DB}(n_3), \tilde{\Delta}_{32}^{DB}(n'_3)\} > \max\{\tilde{\Delta}_{21}^{DB}(n_2), \tilde{\Delta}_{23}^{DB}(n'_2)\}$.

Second, we want to show that if ϕ is small enough then $\max\{\tilde{\Delta}_{21}^{DB}(n_2), \tilde{\Delta}_{23}^{DB}(n'_2)\} > \max\{\tilde{\Delta}_{12}^{DB}(n_1), \tilde{\Delta}_{13}^{DB}(n'_1)\}$ for any $n_i, n'_i \in \mathcal{N}_i$. To do so, we will show that: (i) $\tilde{\Delta}_{21}^{DB}(n_2) > \tilde{\Delta}_{12}^{DB}(n_1)$ for any $n_i \in \mathcal{N}_i$ and (ii) $\tilde{\Delta}_{23}^{DB}(n'_2) > \tilde{\Delta}_{13}^{DB}(n'_1)$ for any $n_i \in \mathcal{N}_i$. First, let $g_{12}(\phi, n, n') := \tilde{\Delta}_{21}^{DB}(n') - \tilde{\Delta}_{12}^{DB}(n)$. We will show that $g_{12}(0, n, n') = 0$ and $\partial g_{12}(0, n, n')/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_2$. It is immediate to see that $g_{12}(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g_{12}(0, n, n')}{\partial\phi} = \frac{3\exp(s_1^o + s_2^o + s_3^o)(2\exp(s_1^o) + 2\exp(s_2^o) + \exp(s_3^o))(s_2^o - s_3^o)}{2(\exp(s_2^o) + \exp(s_3^o))^2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o > s_3^o$. Similarly, let $f_{12}(\phi, n, n') := \tilde{\Delta}_{23}^{DB}(n') - \tilde{\Delta}_{13}^{DB}(n)$. We will show that $f_{12}(0, n, n') = 0$ and $\partial f_{12}(0, n, n')/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_2$. It is immediate to see that $f_{12}(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{12}(0, n, n')}{\partial\phi} = \frac{3s_3^o \exp(s_1^o + s_2^o + s_3^o) \Psi(\exp(s_1^o) - \exp(s_2^o))}{2},$$

where

$$\Psi = \exp(2s_1^o) + \exp(2s_2^o) + 3\exp(s_1^o + s_2^o) + 3\exp(2s_3^o) + 4\exp(s_1^o + s_3^o) + 4\exp(s_2^o + s_3^o) > 0$$

Note that the derivative above is always strictly positive, since $\exp(s_1^o) > \exp(s_2^o)$. Therefore, from the arguments above, it follows that $\max\{\tilde{\Delta}_{21}^{DB}(n_2), \tilde{\Delta}_{23}^{DB}(n'_2)\} > \max\{\tilde{\Delta}_{12}^{DB}(n_1), \tilde{\Delta}_{13}^{DB}(n'_1)\}$. ■

I Model: Additional Proofs

I.1 Proposition G.1

First, we want to show that if ϕ is small enough and s_2^o is sufficiently close to s_1^o , then $\tilde{\Delta}_{12}(n) > \tilde{\Delta}_{31}(n')$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. To do so, let $f(\phi, n, n') := \tilde{\Delta}_{12}(n) - \tilde{\Delta}_{31}(n')$. We will show that $f(0, n, n') = 0$ and $\lim_{s_2^o \rightarrow s_1^o} \partial f(0, n, n') / \partial \phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$, so that by taking a first-order Taylor expansion of the function $f(\phi, n, n')$ around $\phi = 0$ it follows that $f(\phi, n, n') > 0$ for ϕ sufficiently small and s_2^o close enough to s_1^o . It is immediate to see that $f(0, n_1, n_2) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f(0, n, n')}{\partial \phi} = \frac{3 \exp(s_1^o) (s_2^o \exp(s_2^o) - s_1^o \exp(s_3^o))}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is positive provided that s_2^o is close enough to s_1^o . Specifically, observe that $\lim_{s_2^o \rightarrow s_1^o} (s_2^o \exp(s_2^o) - s_1^o \exp(s_3^o)) > 0$. Therefore, $\lim_{s_2^o \rightarrow s_1^o} \partial f(0, n, n') / \partial \phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. Next, following a similar argument, we can show that if ϕ is small enough and s_2^o is sufficiently close to s_3^o , then $f(\phi, n, n') < 0$. In fact, note that $\lim_{s_2^o \rightarrow s_3^o} (s_2^o \exp(s_2^o) - s_1^o \exp(s_3^o)) < 0$, so that $\lim_{s_2^o \rightarrow s_3^o} \partial f(0, n, n') / \partial \phi < 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. ■

I.2 Proposition G.2

Item i. We want to show that if ϕ is small enough then $\tilde{\Delta}_{12}(n) > \tilde{\Delta}_{32}(n')$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. To do so, let $f(\phi, n, n') := \tilde{\Delta}_{12}(n) - \tilde{\Delta}_{32}(n')$. We will show that $f(0, n, n') = 0$ and $\partial f(0, n, n') / \partial \phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$, so that by taking a first-order Taylor expansion of the function $f(\phi, n, n')$ around $\phi = 0$ it follows that $f(\phi, n, n') > 0$ for ϕ sufficiently small. It is immediate to see that $f(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f(0, n, n')}{\partial \phi} = \frac{3 \exp(s_2^o) s_2^o (\exp(s_1^o) - \exp(s_3^o))}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $\exp(s_1^o) > \exp(s_3^o)$.

Item ii. We want to show that if ϕ is small enough then $\tilde{\Delta}_{13}(n) < \tilde{\Delta}_{31}(n')$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. To do so, let $g(\phi, n, n') := \tilde{\Delta}_{31}(n') - \tilde{\Delta}_{13}(n)$. We will show that $g(0, n, n') = 0$ and $\partial g(0, n, n') / \partial \phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. It is immediate to see that $g(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g(0, n, n')}{\partial \phi} = \frac{3 \exp(s_1^o + s_3^o) (s_1^o - s_3^o)}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o > s_3^o$.

Item iii. We want to show that if ϕ is small enough then $\tilde{\Delta}_{13}(n) > \tilde{\Delta}_{23}(n')$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_2$. To do so, let $h(\phi, n, n') := \tilde{\Delta}_{13}(n) - \tilde{\Delta}_{23}(n')$. We will show that $h(0, n, n') = 0$ and $\partial h(0, n, n') / \partial \phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_2$. It is immediate to see that $h(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial h(0, n, n')}{\partial \phi} = \frac{3 \exp(s_3^o) s_3^o (\exp(s_1^o) - \exp(s_2^o))}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $\exp(s_1^o) > \exp(s_2^o)$.

Item iv. We want to show that if ϕ is small enough then $\tilde{\Delta}_{23}(n) < \tilde{\Delta}_{32}(n')$ for any $n \in \mathcal{N}_2$ and $n' \in \mathcal{N}_3$. To do so, let $t(\phi, n, n') := \tilde{\Delta}_{32}(n') - \tilde{\Delta}_{23}(n)$. We will show that $t(0, n, n') = 0$ and $\partial t(0, n, n') / \partial \phi > 0$ for any $n \in \mathcal{N}_2$ and $n' \in \mathcal{N}_3$. It is immediate to see that $t(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial t(0, n, n')}{\partial \phi} = \frac{3 \exp(s_2^o + s_3^o) (s_2^o - s_3^o)}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o > s_3^o$. ■

I.3 Proposition G.3

First, we want to show that if ϕ is small enough and s_2^o is sufficiently close to s_1^o then $\tilde{\Delta}_{13}(n) < \tilde{\Delta}_{32}(n')$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. To do so, let $f(\phi, n, n') := \tilde{\Delta}_{32}(n') - \tilde{\Delta}_{13}(n)$. We will show that $f(0, n, n') = 0$ and $\lim_{s_2^o \rightarrow s_1^o} \partial f(0, n, n') / \partial \phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$, so that by taking a first-order Taylor expansion of the function $f(\phi, n, n')$ around $\phi = 0$ it follows that $f(\phi, n, n') > 0$ for ϕ and $s_2^o - s_1^o$ sufficiently small. It is straightforward to show that $f(0, n, n') = 0$. Furthermore, after some algebra, we obtain:

$$\frac{\partial f(0, n, n')}{\partial \phi} = \frac{3 \exp(s_3^o)(s_2^o \exp(s_2^o) - s_3^o \exp(s_1^o))}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2}$$

which is positive provided that s_2^o is close enough to s_1^o . Specifically, observe that $\lim_{s_2^o \rightarrow s_1^o} (s_2^o \exp(s_2^o) - s_3^o \exp(s_1^o)) = (s_1^o - s_3^o) \exp(s_2^o) > 0$, so that $\lim_{s_2^o \rightarrow s_1^o} \partial f(0, n, n') / \partial \phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. On the other hand, we can show that if s_2^o is sufficiently close to s_3^o , then $\tilde{\Delta}_{13}(n) < \tilde{\Delta}_{32}(n')$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. In fact, note that $\lim_{s_2^o \rightarrow s_3^o} (s_2^o \exp(s_2^o) - s_3^o \exp(s_1^o)) = -(\exp(s_1^o) - \exp(s_3^o))s_3^o < 0$, so that $\lim_{s_2^o \rightarrow s_3^o} \partial f(0, n, n') / \partial \phi < 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. ■

I.4 Proposition G.4

Let us denote an attack from candidate i against j by $i \rightarrow j$. From Proposition 3, we know that $\tilde{\Delta}_{21}(n_2) > \max\{\tilde{\Delta}_{12}(n_1), \tilde{\Delta}_{31}(n_3)\}$ for any $n_i \in \mathcal{N}_i$. Thus, there are two cases to consider, namely: (i) $\tilde{\Delta}_{31}(n_3) < \tilde{\Delta}_{12}(n_1) < \tilde{\Delta}_{21}(n_2)$ and (ii) $\tilde{\Delta}_{12}(n_1) < \tilde{\Delta}_{31}(n_3) < \tilde{\Delta}_{21}(n_2)$.

Item i. Suppose, first, that $\tilde{\Delta}_{31}(n_3) < \tilde{\Delta}_{12}(n_1) < \tilde{\Delta}_{21}(n_2)$ for any $n_i \in \mathcal{N}_i$. Here, we need to consider four cases for the cost parameter $c \in \mathbb{R}_+$:

- a. Case 1: Suppose that $c \leq \tilde{\Delta}_{31}(1, 1, 0)$. In this case, candidates 2 and 3 are willing to attack 1 and candidate 1 is willing to attack 2. Thus, in equilibrium, the following attacks occur: $2 \rightarrow 1$, $1 \rightarrow 2$ and $3 \rightarrow 1$.
- b. Case 2: Suppose that $\tilde{\Delta}_{31}(1, 1, 0) < c \leq \tilde{\Delta}_{12}(1, 0, 0)$. In this case, candidate 2 is willing to attack 1, candidate 1 is willing to attack 2 and candidate 3 is not willing to attack 1. Thus, in equilibrium, the following attacks occur: $2 \rightarrow 1$ and $1 \rightarrow 2$.
- c. Case 3: Suppose that $\tilde{\Delta}_{12}(1, 0, 0) < c \leq \tilde{\Delta}_{21}(0, 0, 0)$. In this case, candidate 2 is willing to attack 1, candidate 1 is not willing to attack 2 and candidate 3 is not willing to attack 1. Thus, in equilibrium, the following attack occur: $2 \rightarrow 1$.
- d. Case 4: Suppose that $\tilde{\Delta}_{21}(0, 0, 0) < c$. In this case, no candidate is willing to attack an opponent. Thus, in equilibrium, no candidate attacks.

Item ii. Next, suppose that $\tilde{\Delta}_{12}(n_1) < \tilde{\Delta}_{31}(n_3) < \tilde{\Delta}_{21}(n_2)$ for any $n_i \in \mathcal{N}_i$. As before, we need to consider the following four cases:

- a. Case 1: Suppose that $c \leq \tilde{\Delta}_{12}(2, 0, 0)$. In this case, candidates 2 and 3 are willing to attack 1 and candidate 1 is willing to attack 2. Thus, in equilibrium, the following attacks occur: $2 \rightarrow 1$, $3 \rightarrow 1$ and $1 \rightarrow 2$.
- b. Case 2: Suppose that $\tilde{\Delta}_{12}(2, 0, 0) < c \leq \tilde{\Delta}_{31}(1, 0, 0)$. In this case, candidate 2 is willing to attack 1, candidate 3 is willing to attack 1 and candidate 1 is not willing to attack 2. Thus, in equilibrium, the following attacks occur: $2 \rightarrow 1$ and $3 \rightarrow 1$.

- c. Case 3: Suppose that $\tilde{\Delta}_{31}(1, 0, 0) < c \leq \tilde{\Delta}_{21}(0, 0, 0)$. In this case, candidate 2 is willing to attack 1, candidate 1 is not willing to attack 2 and candidate 3 is not willing to attack 1. Thus, in equilibrium, the following attack occurs: $2 \rightarrow 1$.
- d. Case 4: Suppose that $\tilde{\Delta}_{21}(0, 0, 0) < c$. In this case, no candidate is willing to attack an opponent. Thus, in equilibrium, no candidate attacks. ■

I.5 Proposition G.5

First, we want to show that if ϕ is small enough, then $\Delta_{12}(n) > \tilde{\Delta}_{12}(n')$ for any $n \in \{0, 1\}$ and $n' \in \mathcal{N}_1$. To do so, let $f_1(\phi, n, n') := \Delta_{12}(n) - \tilde{\Delta}_{12}(n')$. We will show that $f_1(0, n, n') = 0$ and $\partial f_1(0, n, n')/\partial \phi > 0$ for any $n \in \{0, 1\}$ and $n' \in \mathcal{N}_1$. It is immediate to see that $f_1(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_1(0, n, n')}{\partial \phi} = \frac{1}{2} \exp(s_1^o + s_2^o) \left(\frac{4}{(\exp(s_1^o) + \exp(s_2^o))^2} - \frac{3}{(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} \right) s_2^o,$$

which is always positive, given that $(\exp(s_1^o) + \exp(s_2^o))^2 < (\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2$, so that the term inside parenthesis is positive.

Next, we want to show that if ϕ is small enough, then $\Delta_{21}(n) > \tilde{\Delta}_{21}(n')$ for any $n \in \{0, 1\}$ and $n' \in \mathcal{N}_2$. To do so, let $f_2(\phi, n, n') := \Delta_{21}(n) - \tilde{\Delta}_{21}(n')$. We will show that $f_2(0, n, n') = 0$ and $\partial f_2(0, n, n')/\partial \phi > 0$ for any $n \in \{0, 1\}$ and $n' \in \mathcal{N}_2$. It is immediate to see that $f_2(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_2(0, n, n')}{\partial \phi} = \frac{1}{2} \exp(s_1^o + s_2^o) \left(\frac{4}{(\exp(s_1^o) + \exp(s_2^o))^2} - \frac{3}{(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} \right) s_1^o,$$

which, using the same argument as above, is always positive. ■

I.6 Proposition G.6

Item i. We want to show that if ϕ is small enough and s_1^o is sufficiently large, then $\tilde{\Delta}_{23}^{DB}(n) > \tilde{\Delta}_{21}^{DB}(n')$ for any $n, n' \in \mathcal{N}_2$. To do so, let $f_2(\phi, n, n') := \tilde{\Delta}_{23}^{DB}(n) - \tilde{\Delta}_{21}^{DB}(n')$. We will show that $f_2(0, n, n') = 0$ and $\lim_{s_1^o \rightarrow \infty} \partial f_2(0, n, n')/\partial \phi > 0$ for any $n, n' \in \mathcal{N}_2$, so that by taking a first-order Taylor expansion of the function $f_2(\phi, n, n')$ around $\phi = 0$ it follows that $f_2(\phi, n, n') > 0$ for ϕ sufficiently small and s_1^o large enough. It is immediate to see that $f_2(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\begin{aligned} \frac{\partial f_2(0, n, n')}{\partial \phi} = & \Gamma \{ \exp(3s_1^o)s_3^o + 4\exp(2s_1^o + s_2^o)s_3^o + 2\exp(2s_1^o + s_3^o)s_3^o \\ & - \exp(3s_3^o)s_1^o - 2\exp(s_1^o + 2s_3^o)s_1^o - 4\exp(s_2^o + 2s_3^o)s_1^o - \exp(s_1^o + 2s_2^o)(2s_1^o - 5s_3^o) \\ & - \exp(2s_2^o + s_3^o)(5s_1^o - 2s_3^o) - 2\exp(3s_2^o)(s_1^o - s_3^o) - 4\exp(s_1^o + s_2^o + s_3^o)(s_1^o - s_3^o) \} \end{aligned}$$

where

$$\Gamma = \frac{3 \exp(s_1^o + s_2^o + s_3^o)}{2 (\exp(s_1^o) + \exp(s_2^o))^2 (\exp(s_2^o) + \exp(s_3^o))^2 (\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} > 0$$

Note that since we are interested in the sign of the derivative when s_1^o is arbitrarily large, it is enough to consider only the terms inside the curly brackets where s_1^o appears as an exponent, given that these terms will dominate the expression when $s_1^o \rightarrow \infty$. Therefore, we are left with:

$$\begin{aligned} & \exp(3s_1^o)s_3^o + 4\exp(2s_1^o + s_2^o)s_3^o + 2\exp(2s_1^o + s_3^o)s_3^o - 2\exp(s_1^o + 2s_3^o)s_1^o \\ & - \exp(s_1^o + 2s_2^o)(2s_1^o - 5s_3^o) - 4\exp(s_1^o + s_2^o + s_3^o)(s_1^o - s_3^o) \end{aligned}$$

Next, observe that the exponential terms become dominant when $s_1^o \rightarrow \infty$, so that we can also ignore the terms multiplying them. Thus, we get:

$$\exp(3s_1^o) + \exp(2s_1^o + s_2^o) + \exp(2s_1^o + s_3^o) - \exp(s_1^o + 2s_2^o) - \exp(s_1^o + 2s_3^o) - \exp(s_1^o + s_2^o + s_3^o),$$

which is always positive when s_1^o is sufficiently large, since $\lim_{s_1^o \rightarrow \infty} \exp(3s_1^o)$, $\lim_{s_1^o \rightarrow \infty} \exp(2s_1^o + s_2^o)$ and $\lim_{s_1^o \rightarrow \infty} \exp(2s_1^o + s_3^o)$ are all greater than $\lim_{s_1^o \rightarrow \infty} \exp(s_1^o + 2s_2^o)$, $\lim_{s_1^o \rightarrow \infty} \exp(s_1^o + 2s_3^o)$ and $\lim_{s_1^o \rightarrow \infty} \exp(s_1^o + s_2^o + s_3^o)$. Therefore, $\lim_{s_1^o \rightarrow \infty} \partial f_2(0, n, n') / \partial \phi > 0$ for any $n \in \mathcal{N}_2$.

Item ii. We want to show that if ϕ is small enough and s_1^o is sufficiently large, then $\tilde{\Delta}_{32}^{DB}(n) > \tilde{\Delta}_{31}^{DB}(n')$ for any $n, n' \in \mathcal{N}_3$. To do so, let $f_3(\phi, n, n') := \tilde{\Delta}_{32}^{DB}(n) - \tilde{\Delta}_{31}^{DB}(n')$. We will show that $f_3(0, n, n') = 0$ and $\lim_{s_1^o \rightarrow \infty} \partial f_3(0, n, n') / \partial \phi > 0$ for any $n, n' \in \mathcal{N}_3$. It is immediate to see that $f_3(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\begin{aligned} \frac{\partial f_3(0, n, n')}{\partial \phi} = & \Psi \{ \exp(3s_1^o)s_2^o + 4\exp(2s_1^o + s_3^o)s_2^o + 2\exp(2s_1^o + s_2^o)s_2^o \\ & - \exp(3s_2^o)s_1^o - 2\exp(s_1^o + 2s_2^o)s_1^o - 4\exp(2s_2^o + s_3^o)s_1^o - \exp(s_1^o + 2s_3^o)(2s_1^o - 5s_2^o) \\ & - \exp(s_2^o + 2s_3^o)(5s_1^o - 2s_2^o) - 2\exp(3s_3^o)(s_1^o - s_2^o) - 4\exp(s_1^o + s_2^o + s_3^o)(s_1^o - s_2^o) \} \end{aligned}$$

where

$$\Psi = \frac{3 \exp(s_1^o + s_2^o + s_3^o)}{2 (\exp(s_1^o) + \exp(s_3^o))^2 (\exp(s_2^o) + \exp(s_3^o))^2 (\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} > 0$$

Proceeding as above, the sign of the derivative when $s_1^o \rightarrow \infty$ is determined by the following expression:

$$\exp(3s_1^o) + \exp(2s_1^o + s_3^o) + \exp(2s_1^o + s_2^o) - \exp(s_1^o + 2s_2^o) - \exp(s_1^o + 2s_3^o) - \exp(s_1^o + s_2^o + s_3^o),$$

which is always positive when s_1^o is sufficiently large. Therefore, $\lim_{s_1^o \rightarrow \infty} \partial f_3(0, n, n') / \partial \phi > 0$ for any $n \in \mathcal{N}_3$. ■

I.7 Proposition G.7

We want to show that if ϕ is small enough and the distance $s_1^o - s_3^o$ is sufficiently small, then $\tilde{\Delta}_{ij}^{DB}(n) > \tilde{\Delta}_{ij}^{DB}(n')$ for any i and j with $i \neq j$ and $n, n' \in \mathcal{N}_i$. To do so, let $f_{ij}(\phi, n, n') := \tilde{\Delta}_{ij}^{DB}(n) - \tilde{\Delta}_{ij}^{DB}(n')$. We will show that $f_{ij}(0, n, n') = 0$ and $\lim_{(s_1^o, s_2^o, s_3^o) \rightarrow (s^o, s^o, s^o)} \partial f_{ij}(0, n, n') / \partial \phi > 0$ for any $n, n' \in \mathcal{N}_i$, so that by taking a first-order Taylor expansion of the function $f_{ij}(\phi, n, n')$ around $\phi = 0$ it follows that $f_{ij}(\phi, n, n') > 0$ for ϕ and $s_1^o - s_3^o$ sufficiently small. In what follows, we consider all possible combinations for i and j .

- i. Case 1: Let $i = 1$ and $j = 2$. It is immediate to see that $f_{12}(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{12}(0, n, n')}{\partial \phi} = \frac{3}{2} \exp(s_1^o + s_2^o) \left(\frac{1}{(\exp(s_1^o) + \exp(s_2^o))^2} - \frac{2}{(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} \right) s_2^o$$

Then, note that $\lim_{(s_1^o, s_2^o, s_3^o) \rightarrow (s^o, s^o, s^o)} \frac{\partial f_{12}(0, n, n')}{\partial \phi} = \frac{3}{2} \exp(2s^o) \left(\frac{1}{4 \exp(s^o)} - \frac{2}{9 \exp(s^o)} \right) s^o > 0$.

- ii. Case 2: Let $i = 1$ and $j = 3$. It is immediate to see that $f_{13}(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{13}(0, n, n')}{\partial \phi} = \frac{3}{2} \exp(s_1^o + s_3^o) \left(\frac{1}{(\exp(s_1^o) + \exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} \right) s_3^o$$

Then, note that $\lim_{(s_1^o, s_2^o, s_3^o) \rightarrow (s^o, s^o, s^o)} \frac{\partial f_{13}(0, n, n')}{\partial \phi} = \frac{3}{2} \exp(2s^o) \left(\frac{1}{4 \exp(s^o)} - \frac{2}{9 \exp(s^o)} \right) s^o > 0$.

iii. Case 3: Let $i = 2$ and $j = 1$. It is immediate to see that $f_{21}(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{21}(0, n, n')}{\partial \phi} = \frac{3}{2} \exp(s_1^o + s_2^o) \left(\frac{1}{(\exp(s_1^o) + \exp(s_2^o))^2} - \frac{2}{(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} \right) s_1^o$$

Then, note that $\lim_{(s_1^o, s_2^o, s_3^o) \rightarrow (s^o, s^o, s^o)} \frac{\partial f_{21}(0, n, n')}{\partial \phi} = \frac{3}{2} \exp(2s^o) \left(\frac{1}{4 \exp(s^o)} - \frac{2}{9 \exp(s^o)} \right) s^o > 0$.

iv. Case 4: Let $i = 2$ and $j = 3$. It is immediate to see that $f_{23}(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{23}(0, n, n')}{\partial \phi} = \frac{3}{2} \exp(s_2^o + s_3^o) \left(\frac{1}{(\exp(s_2^o) + \exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} \right) s_3^o$$

Note that this expression is always positive since $\frac{1}{(\exp(s_2^o) + \exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} > 0$ for any $s_1^o > s_2^o > s_3^o$. Thus, the result that $\tilde{\Delta}_{23}^{DB}(n) > \tilde{\Delta}_{23}(n')$ for any $n, n' \in \mathcal{N}_2$ does not require that s_1^o be close to s_3^o .

v. Case 5: Let $i = 3$ and $j = 1$. It is immediate to see that $f_{31}(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{31}(0, n, n')}{\partial \phi} = \frac{3}{2} \exp(s_1^o + s_3^o) \left(\frac{1}{(\exp(s_1^o) + \exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} \right) s_1^o$$

Then, note that $\lim_{(s_1^o, s_2^o, s_3^o) \rightarrow (s^o, s^o, s^o)} \frac{\partial f_{31}(0, n, n')}{\partial \phi} = \frac{3}{2} \exp(2s^o) \left(\frac{1}{4 \exp(s^o)} - \frac{2}{9 \exp(s^o)} \right) s^o > 0$.

vi. Case 6: Let $i = 3$ and $j = 2$. It is immediate to see that $f_{32}(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{32}(0, n, n')}{\partial \phi} = \frac{3}{2} \exp(s_2^o + s_3^o) \left(\frac{1}{(\exp(s_2^o) + \exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} \right) s_2^o$$

Note that this expression is always positive since $\frac{1}{(\exp(s_2^o) + \exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2} > 0$ for any $s_1^o > s_2^o > s_3^o$. Thus, the result that $\tilde{\Delta}_{32}^{DB}(n) > \tilde{\Delta}_{32}(n')$ for any $n, n' \in \mathcal{N}_3$ for any does not require that s_1^o be close to s_3^o . ■

I.8 Proposition G.8

Item i. First, we want to show that if ϕ is small enough then $\tilde{\Delta}_{1,2}(n) > \tilde{\Delta}_{1,23}(n')$ for any $n \in \mathcal{N}_{1,2}$ and $n' \in \mathcal{N}_{1,23}$. To do so, let $f_1(\phi, n, n') := \tilde{\Delta}_{1,2}(n) - \tilde{\Delta}_{1,23}(n')$. We will show that $f_1(0, n, n') = 0$ and $\partial f_1(0, n, n') / \partial \phi > 0$ for any $n \in \mathcal{N}_{1,2}$ and $n' \in \mathcal{N}_{1,23}$, so that by taking a first-order Taylor expansion of the function $f_1(\phi, n, n')$ around $\phi = 0$ it follows that $f_1(\phi, n, n') > 0$ for ϕ sufficiently small. It is immediate to see that $f_1(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_1(0, n, n')}{\partial \phi} = \frac{3 \exp(s_1^o) (s_2^o \exp(s_2^o) - s_3^o \exp(s_3^o))}{4 (\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o \exp(s_2^o) > s_3^o \exp(s_3^o)$.

Next, we want to show that if ϕ is small enough then $\tilde{\Delta}_{1,23}(n) > \tilde{\Delta}_{1,3}(n')$ for any $n \in \mathcal{N}_{1,23}$ and $n' \in \mathcal{N}_{1,3}$. To do so, let $g_1(\phi, n, n') := \tilde{\Delta}_{1,23}(n) - \tilde{\Delta}_{1,3}(n')$. We will show that $g_1(0, n, n') = 0$ and

$\partial g_1(0, n, n')/\partial \phi > 0$ for any $n \in \mathcal{N}_{1,23}$ and $n' \in \mathcal{N}_{1,3}$, so that by taking a first-order Taylor expansion of the function $g_1(\phi, n, n')$ around $\phi = 0$ it follows that $g_1(\phi, n, n') > 0$ for ϕ sufficiently small. It is immediate to see that $g_1(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g_1(0, n, n')}{\partial \phi} = \frac{3 \exp(s_1^o) (s_2^o \exp(s_2^o) - s_3^o \exp(s_3^o))}{4(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o \exp(s_2^o) > s_3^o \exp(s_3^o)$.

Item ii. Second, we want to show that if ϕ is small enough then $\tilde{\Delta}_{2,1}(n) > \tilde{\Delta}_{2,13}(n')$ for any $n \in \mathcal{N}_{2,1}$ and $n' \in \mathcal{N}_{2,13}$. To do so, let $f_2(\phi, n, n') := \tilde{\Delta}_{2,1}(n) - \tilde{\Delta}_{2,13}(n')$. We will show that $f_2(0, n, n') = 0$ and $\partial f_2(0, n, n')/\partial \phi > 0$ for any $n \in \mathcal{N}_{2,1}$ and $n' \in \mathcal{N}_{2,13}$, so that by taking a first-order Taylor expansion of the function $f_2(\phi, n, n')$ around $\phi = 0$ it follows that $f_2(\phi, n, n') > 0$ for ϕ sufficiently small. It is immediate to see that $f_2(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_2(0, n, n')}{\partial \phi} = \frac{3 \exp(s_2^o) (s_1^o \exp(s_1^o) - s_3^o \exp(s_3^o))}{4(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o \exp(s_1^o) > s_3^o \exp(s_3^o)$.

Next, we want to show that if ϕ is small enough then $\tilde{\Delta}_{2,13}(n) > \tilde{\Delta}_{2,3}(n')$ for any $n \in \mathcal{N}_{2,13}$ and $n' \in \mathcal{N}_{2,3}$. To do so, let $g_2(\phi, n, n') := \tilde{\Delta}_{2,13}(n) - \tilde{\Delta}_{2,3}(n')$. We will show that $g_2(0, n, n') = 0$ and $\partial g_2(0, n, n')/\partial \phi > 0$ for any $n \in \mathcal{N}_{2,13}$ and $n' \in \mathcal{N}_{2,3}$, so that by taking a first-order Taylor expansion of the function $g_2(\phi, n, n')$ around $\phi = 0$ it follows that $g_2(\phi, n, n') > 0$ for ϕ sufficiently small. It is immediate to see that $g_2(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g_2(0, n, n')}{\partial \phi} = \frac{3 \exp(s_2^o) (s_1^o \exp(s_1^o) - s_3^o \exp(s_3^o))}{4(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o \exp(s_1^o) > s_3^o \exp(s_3^o)$.

Item iii. Third, we want to show that if ϕ is small enough then $\tilde{\Delta}_{3,1}(n) > \tilde{\Delta}_{3,12}(n')$ for any $n \in \mathcal{N}_{3,1}$ and $n' \in \mathcal{N}_{3,12}$. To do so, let $f_3(\phi, n, n') := \tilde{\Delta}_{3,1}(n) - \tilde{\Delta}_{3,12}(n')$. We will show that $f_3(0, n, n') = 0$ and $\partial f_3(0, n, n')/\partial \phi > 0$ for any $n \in \mathcal{N}_{3,1}$ and $n' \in \mathcal{N}_{3,12}$, so that by taking a first-order Taylor expansion of the function $f_3(\phi, n, n')$ around $\phi = 0$ it follows that $f_3(\phi, n, n') > 0$ for ϕ sufficiently small. It is immediate to see that $f_3(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_3(0, n, n')}{\partial \phi} = \frac{3 \exp(s_3^o) (s_1^o \exp(s_1^o) - s_2^o \exp(s_2^o))}{4(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o \exp(s_1^o) > s_2^o \exp(s_2^o)$.

Next, we want to show that if ϕ is small enough then $\tilde{\Delta}_{3,12}(n) > \tilde{\Delta}_{3,2}(n')$ for any $n \in \mathcal{N}_{3,12}$ and $n' \in \mathcal{N}_{3,2}$. To do so, let $g_3(\phi, n, n') := \tilde{\Delta}_{3,12}(n) - \tilde{\Delta}_{3,2}(n')$. We will show that $g_3(0, n, n') = 0$ and $\partial g_3(0, n, n')/\partial \phi > 0$ for any $n \in \mathcal{N}_{3,12}$ and $n' \in \mathcal{N}_{3,2}$, so that by taking a first-order Taylor expansion of the function $g_3(\phi, n, n')$ around $\phi = 0$ it follows that $g_3(\phi, n, n') > 0$ for ϕ sufficiently small. It is immediate to see that $g_3(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g_3(0, n, n')}{\partial \phi} = \frac{3 \exp(s_3^o) (s_1^o \exp(s_1^o) - s_2^o \exp(s_2^o))}{4(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o \exp(s_1^o) > s_2^o \exp(s_2^o)$. ■

I.9 Proposition G.9

We want to show that if ϕ is small enough then $\Delta_{12}^T(n) < \Delta_{21}^T(n')$ for any $\mu > 0$ and $n, n' \in \{0, 1\}$. To do so, let $h(\phi, n, n') := \Delta_{21}^T(n') - \Delta_{12}^T(n)$. We will show that $h(0, n, n') = 0$ and $\partial h(0, n, n')/\partial \phi > 0$ for any $n, n' \in \{0, 1\}$, so that by taking a first-order Taylor expansion of the function $h(\phi, n, n')$ around $\phi = 0$ it follows that $h(\phi, n, n') > 0$ for ϕ sufficiently small. It is immediate to see that $h(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial h(0, n, n')}{\partial \phi} = \frac{\mu(s_1^o)^{\mu-1}(s_2^o)^{\mu-1}(s_1^o + s_2^o)(s_1^o - s_2^o)}{((s_1^o)^\mu + (s_2^o)^\mu)^2},$$

which is always positive, since $s_1^o > s_2^o$. ■

I.10 Proposition G.10

Item i. We want to show that if ϕ is small enough then $\tilde{\Delta}_{12}^T(n) > \tilde{\Delta}_{13}^T(n')$ for any $\mu > 1$ and $n, n' \in \mathcal{N}_1$. To do so, let $f_1(\phi, n, n') := \tilde{\Delta}_{12}^T(n) - \tilde{\Delta}_{13}^T(n')$. We will show that $f_1(0, n, n') = 0$ and $\partial f_1(0, n, n')/\partial \phi > 0$ for any $\mu > 1$ and $n, n' \in \mathcal{N}_1$, so that by taking a first-order Taylor expansion of function $f_1(\phi, n, n')$ around $\phi = 0$ it follows that $f_1(\phi, n, n') > 0$ for ϕ sufficiently small and $\mu > 1$. It is immediate to see that $f_1(0, n, n') = 0$. Furthermore, after some algebra, we obtain:

$$\frac{\partial f_1(0, n, n')}{\partial \phi} = \frac{\mu(s_1^o)^{\mu-1}}{2s_2^o s_3^o ((s_1^o)^\mu + (s_2^o)^\mu + (s_3^o)^\mu)^2} [(2s_1^o (s_2^o)^{\mu+1} s_3^o + s_1^o (s_2^o)^\mu (s_3^o)^2 + (s_2^o)^{\mu+2} s_3^o + (s_2^o)^2 (s_3^o)^{\mu+1}) - (2s_1^o s_2^o (s_3^o)^{\mu+1} + s_1^o (s_2^o)^2 (s_3^o)^\mu + s_2^o (s_3^o)^{\mu+2} + (s_2^o)^{\mu+1} (s_3^o)^2)]$$

Note that the term inside brackets determines the sign of the derivative and can be expressed as:

$$s_1^o (s_2^o)^\mu s_3^o (2s_2^o + s_3^o) - s_1^o s_2^o (s_3^o)^\mu (s_2^o + 2s_3^o) + ((s_2^o)^2 s_3^o - s_2^o (s_3^o)^2)((s_2^o)^\mu + (s_3^o)^\mu),$$

which is guaranteed to be positive provided that $\mu > 1$.

Item ii. We want to show that if ϕ is small enough then $\tilde{\Delta}_{21}^T(n) > \tilde{\Delta}_{23}^T(n')$ for any $\mu > 1$ and $n, n' \in \mathcal{N}_2$. To do so, let $f_2(\phi, n, n') := \tilde{\Delta}_{21}^T(n) - \tilde{\Delta}_{23}^T(n')$. We will show that $f_2(0, n, n') = 0$ and $\partial f_2(0, n, n')/\partial \phi > 0$ for any $\mu > 1$ and $n, n' \in \mathcal{N}_2$. It is immediate to see that $f_2(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_2(0, n, n')}{\partial \phi} = \frac{\mu(s_2^o)^{\mu-1}}{2s_1^o s_3^o ((s_1^o)^\mu + (s_2^o)^\mu + (s_3^o)^\mu)^2} [(2(s_1^o)^{\mu+1} s_2^o s_3^o + (s_1^o)^\mu s_2^o (s_3^o)^2 + (s_1^o)^{\mu+2} s_3^o + (s_1^o)^2 (s_3^o)^{\mu+1}) - (2s_1^o s_2^o (s_3^o)^{\mu+1} + (s_1^o)^2 s_2^o (s_3^o)^\mu + s_1^o (s_3^o)^{\mu+2} + (s_1^o)^{\mu+1} (s_3^o)^2)]$$

Note that the term inside brackets can be rewritten as:

$$(s_1^o)^\mu s_2^o s_3^o (2s_1^o + s_3^o) - s_1^o s_2^o (s_3^o)^\mu (s_1^o + 2s_3^o) + ((s_1^o)^2 s_3^o - s_1^o (s_3^o)^2)((s_1^o)^\mu + (s_3^o)^\mu),$$

which is guaranteed to be positive provided that $\mu > 1$.

Item iii. We want to show that if ϕ is small enough then $\tilde{\Delta}_{31}^T(n) > \tilde{\Delta}_{32}^T(n')$ for any $\mu > 1$ and $n, n' \in \mathcal{N}_3$. To do so, let $f_3(\phi, n, n') := \tilde{\Delta}_{31}^T(n) - \tilde{\Delta}_{32}^T(n')$. We will show that $f_3(0, n, n') = 0$ and $\partial f_3(0, n, n')/\partial \phi > 0$ for any $\mu > 1$ and $n, n' \in \mathcal{N}_3$. It is immediate to see that $f_3(0, n, n') = 0$. Furthermore, after some

algebra, we get:

$$\frac{\partial f_3(0, n, n')}{\partial \phi} = \frac{\mu(s_3^o)^{\mu-1}}{2s_1^o s_2^o ((s_1^o)^\mu + (s_2^o)^\mu + (s_3^o)^\mu)^2} [(2(s_1^o)^{\mu+1} s_2^o s_3^o + (s_1^o)^\mu (s_2^o)^2 s_3^o + (s_1^o)^{\mu+2} s_2^o + (s_1^o)^2 (s_2^o)^{\mu+1}) - (2s_1^o (s_2^o)^{\mu+1} s_3^o + (s_1^o)^2 (s_2^o)^\mu s_3^o + s_1^o (s_2^o)^{\mu+2} + (s_1^o)^{\mu+1} (s_2^o)^2)]$$

Note that the term inside brackets can be rewritten as:

$$(s_1^o)^\mu s_2^o s_3^o (2s_1^o + s_2^o) - s_1^o (s_2^o)^\mu s_3^o (s_1^o + 2s_2^o) + ((s_1^o)^2 s_2^o - s_1^o (s_2^o)^2) ((s_1^o)^\mu + (s_2^o)^\mu),$$

which is guaranteed to be positive provided that $\mu > 1$.

Item iv. Finally, we want to show that if ϕ is small enough then $\tilde{\Delta}_{21}^T(n') > \max\{\tilde{\Delta}_{12}^T(n), \tilde{\Delta}_{31}^T(n'')\}$ for any $\mu > 1$ and $n \in \mathcal{N}_1$, $n' \in \mathcal{N}_2$ and $n'' \in \mathcal{N}_3$. First, let $h_{12}(\phi, n, n') := \tilde{\Delta}_{21}^T(n') - \tilde{\Delta}_{12}^T(n)$. We will show that $h_{12}(0, n, n') = 0$ and $\partial h_{12}(0, n, n')/\partial \phi > 0$ for any $\mu > 1$, $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_2$. It is immediate to see that $f_3(0, n, n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial h_{12}(0, n, n')}{\partial \phi} = \frac{\mu}{2s_1^o s_2^o ((s_1^o)^\mu + (s_2^o)^\mu + (s_3^o)^\mu)^2} [((s_1^o)^{\mu+2} (s_2^o)^\mu s_3^o + (s_1^o)^{\mu+1} (s_2^o)^2 (s_3^o)^\mu + (s_1^o)^2 (s_2^o)^\mu (s_3^o)^{\mu+1}) - ((s_1^o)^\mu (s_2^o)^{\mu+2} s_3^o + (s_1^o)^\mu (s_2^o)^2 (s_3^o)^{\mu+1} + (s_1^o)^2 (s_2^o)^{\mu+1} (s_3^o)^\mu)]$$

Observe that the term inside brackets can be rewritten as:

$$(s_1^o)^\mu (s_2^o)^\mu s_3^o ((s_1^o)^2 - (s_2^o)^2) + (s_1^o)^\mu s_2^o (s_3^o)^\mu (s_1^o - s_2^o) s_2^o - s_1^o (s_2^o)^\mu (s_3^o)^\mu (s_2^o - s_3^o) s_1^o,$$

But note that since $(s_1^o)^\mu (s_2^o)^\mu s_3^o > (s_1^o)^\mu s_2^o (s_3^o)^\mu > s_1^o (s_2^o)^\mu (s_3^o)^\mu$ for any $\mu > 1$, then a sufficient condition for this expression to be positive is:

$$((s_1^o)^2 - (s_2^o)^2) + (s_1^o - s_2^o) s_2^o - (s_2^o - s_3^o) s_1^o > 0,$$

which always holds.

Next, let $h_{23}(\phi, n', n'') := \tilde{\Delta}_{21}^T(n') - \tilde{\Delta}_{31}^T(n'')$. We will show that $h_{23}(0, n', n'') = 0$ and $\partial h_{23}(0, n', n'')/\partial \phi > 0$ for any $\mu > 1$, $n' \in \mathcal{N}_2$ and $n'' \in \mathcal{N}_3$. It is immediate to see that $h_{23}(0, n', n'') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial h_{23}(0, n', n'')}{\partial \phi} = \frac{\mu}{2s_1^o s_2^o ((s_1^o)^\mu + (s_2^o)^\mu + (s_3^o)^\mu)^2} [((s_1^o)^{\mu+2} (s_2^o)^\mu s_3^o + 2(s_1^o)^{\mu+1} (s_2^o)^{\mu+1} s_3^o + 2(s_1^o)^2 (s_2^o)^\mu (s_3^o)^{\mu+1}) - ((s_1^o)^{\mu+2} s_2^o (s_3^o)^\mu - 2(s_1^o)^{\mu+1} s_2^o (s_3^o)^{\mu+1} - 2(s_1^o)^2 (s_2^o)^{\mu+1} (s_3^o)^\mu)]$$

Observe that the term inside brackets can be written as:

$$s_1^o s_2^o s_3^o \{s_1^o ((s_1^o)^\mu (s_2^o)^{\mu-1} - (s_1^o)^\mu (s_3^o)^{\mu-1}) + 2(s_1^o)^\mu ((s_2^o)^\mu - (s_3^o)^\mu) - 2s_1^o ((s_2^o)^\mu (s_3^o)^{\mu-1} - (s_2^o)^{\mu-1} (s_3^o)^\mu)\},$$

which is always positive since $2(s_1^o)^\mu ((s_2^o)^\mu - (s_3^o)^\mu) > 2s_1^o ((s_2^o)^\mu (s_3^o)^{\mu-1} - (s_2^o)^{\mu-1} (s_3^o)^\mu)$ for any $\mu > 1$. ■

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