

Online Appendix for “Degree-Weighted Social Learning”

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O1 Supplementary Proofs of Theorem 1

Proof of Proposition B.1. To bound $\|\mathbf{A} - \mathbf{R}\|$, we rewrite it as:

$$\mathbf{A} - \mathbf{R} = \mathbf{Y} + \sum_{1 \leq i < j \leq n} \mathbf{X}^{i,j},$$

where

$$\mathbf{X}^{i,j} = (A_{ij} - R_{ij})(\mathbf{e}_i \mathbf{e}_j^\top + \mathbf{e}_j \mathbf{e}_i^\top),$$

$$\mathbf{Y} = \text{diag}(A_{11} - R_{11}, \dots, A_{nn} - R_{nn}),$$

and $\mathbf{e}_i$ is the standard basis vector in $\mathbb{R}^n$.

Observe that $\mathbf{X}^{i,j}$ are independent, have mean zero, and are also independent of $\mathbf{Y}$, which itself has mean zero. Then, applying the matrix Bernstein inequality (Theorem 5.4.1 in Vershynin, 2018), we obtain:

$$\mathbb{P} \left\{ \left\| \mathbf{Y} + \sum_{1 \leq i < j \leq n} \mathbf{X}^{i,j} \right\| \geq t \right\} \leq 2n \exp \left(-\frac{t^2/2}{\sigma^2 + Kt/3} \right), \quad (\text{O.1})$$

where $\|\mathbf{Y}\|, \|\mathbf{X}^{i,j}\| \leq K = 1$, and $\sigma^2 = \left\| \mathbb{E} \mathbf{Y}^2 + \sum_{1 \leq i < j \leq n} \mathbb{E} (\mathbf{X}^{i,j})^2 \right\|$.

Since

$$(\mathbf{X}^{i,j})^2 = (A_{ij} - R_{ij})^2 (\mathbf{e}_i \mathbf{e}_i^\top + \mathbf{e}_j \mathbf{e}_j^\top),$$

we obtain

$$\mathbb{E} (\mathbf{X}^{i,j})^2 = (R_{ij} - R_{ij}^2) (\mathbf{e}_i \mathbf{e}_i^\top + \mathbf{e}_j \mathbf{e}_j^\top).$$

Similarly,

$$\mathbb{E}\mathbf{Y}^2 = \sum_{i=1}^n (R_{ii} - R_{ii}^2) \mathbf{e}_i \mathbf{e}_i^\top.$$

Since each A_{ij} follows a Bernoulli distribution, we have $R_{ij} \in [0, 1]$, and thus, $R_{ij} - R_{ij}^2 \leq 1/4$. It follows that

$$\sigma^2 \leq n/4.$$

Setting $t = 3\sqrt{n \log n}$ in (O.1), we obtain:

$$\begin{aligned} \mathbb{P} \left\{ \left\| \mathbf{Y} + \sum_{1 \leq i < j \leq n} \mathbf{X}^{i,j} \right\| \geq 3\sqrt{n \log n} \right\} &\leq 2n \exp \left(-\frac{\frac{9n \log n}{2}}{\frac{n}{4} + \frac{3\sqrt{n \log n}}{3}} \right) \\ &\leq 2n \exp(-3 \log n) \\ &= \frac{2}{n^2}. \end{aligned}$$

□

Proof of Proposition B.2. The proof of Proposition B.2 follows from Lemmas B.4 to O1.1.

Lemmas B.4 and B.5 have already been stated; we now proceed to prove them. □

Proof of Lemma B.4. We first prove (B17). Recall that $K > 1$. The inequality

$$\phi(\alpha, \min_i d_i(\mathbf{R})) \leq \phi(\alpha, K \min_i d_i(\mathbf{R}))$$

follows from the monotonicity of ϕ for $\alpha \geq 0$. For the reverse inequality, we consider

$$\begin{aligned} \log \frac{\phi(\alpha, K \min_i d_i(\mathbf{R}))}{\phi(\alpha, \min_i d_i(\mathbf{R}))} &= \log \phi(\alpha, K \min_i d_i(\mathbf{R})) - \log \phi(\alpha, \min_i d_i(\mathbf{R})) \\ &= (K - 1) \min_i d_i(\mathbf{R}) \frac{\partial}{\partial d} \log \phi(\alpha, d_0), \end{aligned}$$

for some $d_0 \in [\min_i d_i(\mathbf{R}), K \min_i d_i(\mathbf{R})]$.

By Property 3' in (B3),

$$\begin{aligned} (K-1) \min_i d_i(\mathbf{R}) \frac{\partial}{\partial d} \log \phi(\alpha, d_0) &= (K-1) \min_i d_i(\mathbf{R}) \frac{\partial \phi / \partial d(\alpha, d_0)}{\phi(\alpha, d_0)} \\ &\leq (K-1) K d_0 \frac{\partial \phi / \partial d(\alpha, d_0)}{\phi(\alpha, d_0)} \leq C. \end{aligned}$$

Thus, we obtain

$$\frac{\phi(\alpha, K \min_i d_i(\mathbf{R}))}{\phi(\alpha, \min_i d_i(\mathbf{R}))} = \exp \left(\log \frac{\phi(\alpha, K \min_i d_i(\mathbf{R}))}{\phi(\alpha, \min_i d_i(\mathbf{R}))} \right) \leq e^C.$$

The proof of (B18) follows similarly using Property 3' in (B4). Finally, (B19) and (B20) follow from (B17) and (B18), respectively, by reversing all inequalities, since $\partial \phi / \partial d \leq 0$ for $\alpha < 0$. $\square$

Proof of Lemma B.5. Recall that $d_j(\mathbf{A}) = \sum_{i=1}^n A_{ij}$ and $d_j(\mathbf{R}) = \sum_{i=1}^n R_{ij}$. Applying Hoeffding's inequality, we obtain

$$\begin{aligned} \mathbb{P}\{|d_j(\mathbf{A}) - d_j(\mathbf{R})| \geq t\} &= \mathbb{P}\left\{\left|\sum_{i=1}^n (A_{ij} - R_{ij})\right| \geq t\right\} \\ &\leq \mathbb{P}\left\{\sum_{i=1}^n (A_{ij} - R_{ij}) \geq t\right\} + \mathbb{P}\left\{\sum_{i=1}^n (R_{ij} - A_{ij}) \geq t\right\} \\ &\leq 2 \exp\left(-\frac{2t^2}{n}\right). \end{aligned}$$

Setting $t = \sqrt{2n \log n}$, we obtain

$$\mathbb{P}\{|d_j(\mathbf{A}) - d_j(\mathbf{R})| \geq \sqrt{2n \log n}\} \leq 2e^{-4 \log n} = \frac{2}{n^4}, \quad (\text{O.2})$$

which establishes (B21).

For (B22), by the mean value theorem, we have

$$|\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_j(\mathbf{R}))| = |d_j(\mathbf{A}) - d_j(\mathbf{R})| \left| \frac{\partial \phi}{\partial d}(\alpha, c_j) \right|,$$

for some $c_j \in [d_j(\mathbf{R}) - |d_j(\mathbf{A}) - d_j(\mathbf{R})|, d_j(\mathbf{R}) + |d_j(\mathbf{A}) - d_j(\mathbf{R})|]$.

Using the bound from (O.2), it follows that

$$|\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_j(\mathbf{R}))| \leq \sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,1}) \right|,$$

for some $c_{j,1} \in [d_j(\mathbf{R}) - \sqrt{2n \log n}, d_j(\mathbf{R}) + \sqrt{2n \log n}]$ for each $j = 1, \dots, n$, with probability at least $1 - \frac{2}{n^4}$. $\square$

Lemma O1.1. *Under Assumptions (1)-(2),*

$$\|\mathbf{D}_\mathbf{A} - \mathbf{D}_\mathbf{R}\| \leq C\sqrt{n \log n}, \quad (\text{O.3})$$

with probability at least $1 - \frac{14}{n^2}$, where C is a fixed positive constant independent of n .

Proof of Lemma O1.1. Since both $\mathbf{D}_\mathbf{A}$ and $\mathbf{D}_\mathbf{R}$ are diagonal matrices, we first derive an entrywise bound. For any $i = 1, \dots, n$, we have

$$\begin{aligned} & \left| \frac{\sum_j A_{ij} \phi(\alpha, d_j(\mathbf{A}))}{\phi(\alpha, d_i(\mathbf{A}))} - \frac{\sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R}))}{\phi(\alpha, d_i(\mathbf{R}))} \right| \\ & \leq \left| \frac{\sum_j A_{ij} \phi(\alpha, d_j(\mathbf{A}))}{\phi(\alpha, d_i(\mathbf{A}))} - \frac{\sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R}))}{\phi(\alpha, d_i(\mathbf{A}))} \right| + \left| \frac{\sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R}))}{\phi(\alpha, d_i(\mathbf{A}))} - \frac{\sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R}))}{\phi(\alpha, d_i(\mathbf{R}))} \right| \\ & = \textcircled{1} + \textcircled{2}. \end{aligned} \quad (\text{O.4})$$

Define $d_{j,-i}(\mathbf{A}) = d_j(\mathbf{A}) - A_{ij}$, noting that $d_{j,-i}(\mathbf{A})$ is independent of A_{ij} . Then, the numerator of the first term $\textcircled{1}$ in (O.4) can be decomposed as:

$$\begin{aligned}
& \left| \sum_j A_{ij} \phi(\alpha, d_j(\mathbf{A})) - \sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R})) \right| \\
&= \left| \sum_j A_{ij} (\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{A}))) + \sum_j (A_{ij} - R_{ij}) \phi(\alpha, d_{j,-i}(\mathbf{A})) \right. \\
&\quad \left. + \sum_j R_{ij} [\phi(\alpha, d_{j,-i}(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))] + \sum_j R_{ij} [\phi(\alpha, d_j(\mathbf{R})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))] \right| \\
&\leq \left| \sum_j A_{ij} (\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{A}))) \right| + \left| \sum_j (A_{ij} - R_{ij}) \phi(\alpha, d_{j,-i}(\mathbf{A})) \right| \\
&\quad + \left| \sum_j R_{ij} [\phi(\alpha, d_{j,-i}(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))] \right| + \left| \sum_j R_{ij} [\phi(\alpha, d_j(\mathbf{R})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))] \right| \\
&= \mathbf{G} + \mathbf{H} + \mathbf{J} + \mathbf{K}. \tag{O.5}
\end{aligned}$$

Bound for G in (O.5). By the mean value theorem and Lemma B.5, we have

$$\begin{aligned}
& |\{\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{A}))\} - \{\phi(\alpha, d_j(\mathbf{R})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))\}| \\
&\leq |\{\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_j(\mathbf{R}))\} - \{\phi(\alpha, d_{j,-i}(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))\}| \\
&\leq 2 \max(|\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_j(\mathbf{R}))|, |\phi(\alpha, d_{j,-i}(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))|) \\
&\leq 2\sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,2}) \right|,
\end{aligned}$$

for some $c_{j,2} \in [d_j(\mathbf{R}) - \sqrt{2n \log n} - 1, d_j(\mathbf{R}) + \sqrt{2n \log n}]$ for each $j = 1, \dots, n$, with probability at least $1 - \frac{2}{n^4}$.

Applying the triangle inequality,

$$\begin{aligned}
|\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{A}))| &\leq |\phi(\alpha, d_j(\mathbf{R})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))| + 2\sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,2}) \right| \\
&\leq \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,3}) \right| + 2\sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,2}) \right|,
\end{aligned}$$

for some $c_{j,3} \in [d_{j,-i}(\mathbf{R}), d_j(\mathbf{R})]$ for each $j = 1, \dots, n$, with probability at least $1 - \frac{2}{n^4}$.

Note that this bound is not optimal but is sufficient for the proof of Lemma O1.1. Consider

$$\Lambda = \bigcap_j \left\{ |\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{A}))| \leq \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,3}) \right| + 2\sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,2}) \right| \right\}.$$

By the union bound³¹, we obtain $\mathbb{P}\{\Lambda\} \geq 1 - \frac{2}{n^3}$.

Denote $\mathbb{P}_S\{T\} = \mathbb{P}\{S \cap T\}$ for events S and T . Define

$$\begin{aligned} t = & 2 \left(\log n \sum_j \left(\left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,3}) \right| + 2\sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,2}) \right| \right)^2 \right)^{1/2} \\ & + E \sum_j |A_{ij} - R_{ij}| \left(\left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,3}) \right| + 2\sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,2}) \right| \right). \end{aligned} \quad (\text{O.6})$$

Applying Hoeffding's inequality (see Hoeffding, 1963), we obtain

$$\begin{aligned} & \mathbb{P} \left\{ \left| \sum_j (A_{ij} - R_{ij}) (\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{A}))) \right| \geq t \right\} \\ & \leq \mathbb{P}_\Lambda \left\{ \left| \sum_j (A_{ij} - R_{ij}) (\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{A}))) \right| \geq t \right\} + \mathbb{P}\{\Lambda^C\} \\ & \leq \mathbb{P}_\Lambda \left\{ \sum_j |A_{ij} - R_{ij}| \left(\left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,3}) \right| + 2\sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,2}) \right| \right) \geq t \right\} + \mathbb{P}\{\Lambda^C\} \\ & \leq 2 \exp \left(- \frac{4 \log n \sum_j \left(\left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,3}) \right| + 2\sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,2}) \right| \right)^2}{\sum_j \left(\left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,3}) \right| + 2\sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,2}) \right| \right)^2} \right) + \frac{2}{n^3} \\ & \leq 2 \exp(-4 \log n) + \frac{2}{n^3} \\ & \leq \frac{4}{n^3}. \end{aligned} \quad (\text{O.7})$$

Similarly, with probability at least $1 - 2n^{-3}$, the remaining term $\sum_j R_{ij} (\phi(\alpha, d_j(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{A})))$

³¹For events $E_1, \dots, E_n$ satisfying $\mathbb{P}(E_j) \geq 1 - c_j$ for all $j = 1, \dots, n$, it holds that $\mathbb{P}(\cap_j E_j) = \mathbb{P}((\cup_j E_j^C)^C) = 1 - \mathbb{P}(\cup_j E_j^C) \geq 1 - \sum_j \mathbb{P}(E_j^C) \geq 1 - \sum_j c_j$.

can be bounded by

$$\sum_j R_{ij} \left(\left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,3}) \right| + 2\sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,2}) \right| \right).$$

Bound for H in (O.5). By the mean value theorem and Lemma B.5, we obtain

$$\begin{aligned} |\phi(\alpha, d_{j,-i}(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))| &\leq \sqrt{2(n-1) \log(n-1)} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,4}) \right| \\ &\leq \sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,4}) \right|, \end{aligned}$$

for some $c_{j,4} \in [d_j(\mathbf{R}) - \sqrt{2n \log n} - 1, d_j(\mathbf{R}) + \sqrt{2n \log n}]$ for each $j = 1, \dots, n$, with probability at least $1 - \frac{2}{n^4}$.

Consider the event:

$$\Gamma = \bigcap_j \left\{ \phi(\alpha, d_{j,-i}(\mathbf{A})) \leq \phi(\alpha, d_j(\mathbf{R})) + \sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,4}) \right| \right\}. \quad (\text{O.8})$$

Using the same proof technique as in Lemma B.5 and applying the union bound, we obtain

$$\mathbb{P}\{\Gamma\} \geq 1 - \frac{2}{n^3}.$$

Define

$$t = 2 \left(\log n \sum_j \left(\phi(\alpha, d_j(\mathbf{R})) + \sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,4}) \right| \right)^2 \right)^{1/2}. \quad (\text{O.9})$$

Applying Hoeffding's inequality, we obtain

$$\begin{aligned}
& \mathbb{P} \left\{ \left| \sum_j (A_{ij} - R_{ij}) \phi(\alpha, d_{j,-i}(\mathbf{A})) \right| \geq t \right\} \\
& \leq \mathbb{P}_\Gamma \left\{ \left| \sum_j (A_{ij} - R_{ij}) \phi(\alpha, d_{j,-i}(\mathbf{A})) \right| \geq t \right\} + \mathbb{P}\{\Gamma^C\} \\
& \leq 2 \exp \left(- \frac{8 \log n \sum_j (\phi(\alpha, d_j(\mathbf{R})) + \sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,4}) \right|)^2}{\sum_j (\phi(\alpha, d_j(\mathbf{R})) + \sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,4}) \right|)^2} \right) + \frac{2}{n^3} \\
& \leq 2 \exp(-8 \log n) + \frac{2}{n^3} \\
& \leq \frac{4}{n^3}.
\end{aligned} \tag{O.10}$$

Bound for J in (O.5). Conditioning on the event $\bigcap_j \{|\phi(\alpha, d_{j,-i}(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))| \leq \sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,4}) \right|\}$, we obtain

$$\left| \sum_j R_{ij} [\phi(\alpha, d_{j,-i}(\mathbf{A})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))] \right| \leq \sqrt{2n \log n} \sum_j R_{ij} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,4}) \right|. \tag{O.11}$$

Thus, this inequality holds with probability at least $1 - \frac{2}{n^3}$ by the argument above (O.8).

Bound for K in (O.5). By the mean value theorem, we have

$$\left| \sum_j R_{ij} [\phi(\alpha, d_j(\mathbf{R})) - \phi(\alpha, d_{j,-i}(\mathbf{R}))] \right| \leq \sum_j R_{ij} R_{ij} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,5}) \right| \leq \sum_j R_{ij} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,5}) \right|, \tag{O.12}$$

for some $c_{j,5} \in [d_{j,-i}(\mathbf{R}), d_j(\mathbf{R})]$ for each $j = 1, \dots, n$.

Bound for ① in (O.4). Assumptions 1 and 2 imply that $d_j(\mathbf{R})$ is of the same order for all j . By Lemma B.4, this also holds for $\phi(\alpha, d_j(\mathbf{R}))$ and $\frac{\partial \phi}{\partial d}(\alpha, d_j(\mathbf{R}))$. Using assumption 1 and Property 3 (B3) of ϕ , we combine the four bounds (O.7) – (O.12) to obtain a bound for the

numerator of ①:

$$\left| \sum_j A_{ij} \phi(\alpha, d_j(\mathbf{A})) - \sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R})) \right| \leq C \sqrt{n \log n} \phi(\alpha, d_i(\mathbf{R})), \quad (\text{O.13})$$

with probability at least $1 - \frac{10}{n^3}$, where C is a fixed constant independent of n .

By Lemma B.5 and the triangle inequality, we obtain

$$\begin{aligned} \frac{1}{\phi(\alpha, d_i(\mathbf{A}))} &\leq \frac{1}{\phi(\alpha, d_i(\mathbf{R})) - |\phi(\alpha, d_i(\mathbf{A})) - \phi(\alpha, d_i(\mathbf{R}))|} \\ &\leq \frac{1}{\phi(\alpha, d_i(\mathbf{R})) - \sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{j,1}) \right|}, \end{aligned}$$

with probability at least $1 - \frac{2}{n^4}$. Therefore, we obtain

$$\textcircled{1} = \left| \frac{\sum_j A_{ij} \phi(\alpha, d_j(\mathbf{A}))}{\phi(\alpha, d_i(\mathbf{A}))} - \frac{\sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R}))}{\phi(\alpha, d_i(\mathbf{A}))} \right| \leq C \sqrt{n \log n},$$

with probability at least $1 - \frac{12}{n^3}$.

Bound for ② in (O.4). By Lemma B.5 and the bound for $\frac{1}{\phi(\alpha, d_i(\mathbf{A}))}$, we have

$$\begin{aligned} \textcircled{2} &= \left| \frac{\sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R}))}{\phi(\alpha, d_i(\mathbf{A}))} - \frac{\sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R}))}{\phi(\alpha, d_i(\mathbf{R}))} \right| \\ &= \sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R})) \frac{|\phi(\alpha, d_i(\mathbf{A})) - \phi(\alpha, d_i(\mathbf{R}))|}{\phi(\alpha, d_i(\mathbf{A})) \phi(\alpha, d_i(\mathbf{R}))} \\ &\leq \sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R})) \frac{\sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{i,0}) \right|}{\phi(\alpha, d_i(\mathbf{R})) (\phi(\alpha, d_i(\mathbf{R})) - \sqrt{2n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{i,0}) \right|)} \\ &\leq C d_i(\mathbf{R}) \frac{\sqrt{n \log n} \left| \frac{\partial \phi}{\partial d}(\alpha, c_{i,0}) \right|}{\phi(\alpha, d_i(\mathbf{R}))} \\ &\leq C \sqrt{n \log n}, \end{aligned} \quad (\text{O.14})$$

with probability at least $1 - \frac{2}{n^4}$.

Bound for ① + ② in (O.4). Combining (O.13) and (O.14), we obtain the entrywise bound.

For any i ,

$$\left| \frac{\sum_j A_{ij} \phi(\alpha, d_j(\mathbf{A}))}{\phi(\alpha, d_i(\mathbf{A}))} - \frac{\sum_j R_{ij} \phi(\alpha, d_j(\mathbf{R}))}{\phi(\alpha, d_j(\mathbf{R}))} \right| \leq C \sqrt{n \log n}, \quad (\text{O.15})$$

with probability at least $1 - \frac{14}{n^3}$. Finally, applying the union bound,

$$\|\mathbf{D}_\mathbf{A} - \mathbf{D}_\mathbf{R}\| \leq C \sqrt{n \log n}, \quad (\text{O.16})$$

with probability at least $1 - \frac{14}{n^2}$. □

Proof of Proposition B.2. Rewrite the left hand side as:

$$\begin{aligned} \|\mathbf{D}_\mathbf{A}^{-1/2} - \mathbf{D}_\mathbf{R}^{-1/2}\| &= \left\| (\mathbf{D}_\mathbf{A}^{-1} - \mathbf{D}_\mathbf{R}^{-1}) (\mathbf{D}_\mathbf{A}^{-1/2} + \mathbf{D}_\mathbf{R}^{-1/2})^{-1} \right\| \\ &= \left\| (\mathbf{D}_\mathbf{A} - \mathbf{D}_\mathbf{R}) (\mathbf{D}_\mathbf{A} \mathbf{D}_\mathbf{R})^{-1} (\mathbf{D}_\mathbf{A}^{-1/2} + \mathbf{D}_\mathbf{R}^{-1/2})^{-1} \right\| \\ &\leq \|\mathbf{D}_\mathbf{A} - \mathbf{D}_\mathbf{R}\| \cdot \|\mathbf{D}_\mathbf{A}^{-1}\| \cdot \|\mathbf{D}_\mathbf{R}^{-1}\| \cdot \left\| (\mathbf{D}_\mathbf{A}^{-1/2} + \mathbf{D}_\mathbf{R}^{-1/2})^{-1} \right\|. \end{aligned}$$

By Assumption 2, we have $\frac{\max_i d_i(\mathbf{R})}{\min_i d_i(\mathbf{R})} < \infty$. Combining this with Property 2' and the fact that $\phi(0, d) = 1$, it follows that $\frac{\max_i \phi(\alpha, d_i(\mathbf{R}))}{\min_i \phi(\alpha, d_i(\mathbf{R}))} < \infty$. Conditioning on the event $\|\mathbf{D}_\mathbf{A} - \mathbf{D}_\mathbf{R}\| \leq C \sqrt{n \log n}$,

$$\begin{aligned} \left\| (\mathbf{D}_\mathbf{A}^{-1/2} + \mathbf{D}_\mathbf{R}^{-1/2})^{-1} \right\| &= \max_i \left((\mathbf{D}_\mathbf{A})_{ii}^{-1/2} + (\mathbf{D}_\mathbf{R})_{ii}^{-1/2} \right)^{-1} \\ &\leq \left(\min_i (\mathbf{D}_\mathbf{A})_{ii}^{-1/2} + \min_i (\mathbf{D}_\mathbf{R})_{ii}^{-1/2} \right)^{-1} \\ &\leq \left(\min_i (\mathbf{D}_\mathbf{R})_{ii}^{-1/2} + \min_i ((\mathbf{D}_\mathbf{R})_{ii} + |(\mathbf{D}_\mathbf{A})_{ii} - (\mathbf{D}_\mathbf{R})_{ii}|)^{-1/2} \right)^{-1} \\ &\leq C \left(\min_i (\mathbf{D}_\mathbf{R})_{ii}^{-1/2} \right)^{-1} \\ &\leq C \min_i \sqrt{d_i(\mathbf{R})}. \end{aligned}$$

Similarly,

$$\|\mathbf{D}_\mathbf{A}^{-1}\| = \left(\min_i (\mathbf{D}_\mathbf{A})_{ii} \right)^{-1} \leq C \left(\min_i d_i(\mathbf{R}) \right)^{-1}.$$

Combining the results and applying Lemma O1.1, we have

$$\left\| \mathbf{D}_{\mathbf{A}}^{-1/2} - \mathbf{D}_{\mathbf{R}}^{-1/2} \right\| \leq C \sqrt{n \log n} \left(\min_i d_i(\mathbf{R}) \right)^{-3/2},$$

with probability at least $1 - \frac{14}{n^2}$. □

O2 Additional Proofs for Section C.2: Characterization of Convergence Speed

Proof of Lemma C.1. Before proving Lemma C.1, we establish the following lemma.

Lemma O2.1. *Let the $m \times m$ matrix $\mathbf{F}_{\mathbf{T}^*}$ be defined as:*

$$(\mathbf{F}_{\mathbf{T}^*})_{kl} = \frac{n_l p_{kl} \phi(\alpha, \sum_h n_h p_{lh})}{\sum_{l'} n_{l'} p_{kl'} \phi(\alpha, \sum_h n_h p_{l'h})}. \quad (\text{O.17})$$

Then, $\mathbf{F}_{\mathbf{T}^*}$ has the same nonzero eigenvalues as $\mathbf{T}^*$.

Proof of Lemma O2.1. First, note that $\mathbf{R}$ is a matrix with m^2 blocks. Within each block, the entries are identical, as is $\mathbf{T}^*$. From the definition of $\mathbf{T}^*$, if i and j are in groups k and l , respectively, we have

$$T_{ij}^* = \frac{R_{ij} \phi(\alpha, d_j(\mathbf{R}))}{\sum_{j'} R_{ij'} \phi(\alpha, d_j(\mathbf{R}))} = \frac{p_{kl} \phi(\alpha, \sum_h n_h p_{lh})}{\sum_{l'} n_{l'} p_{kl'} \phi(\alpha, \sum_h n_h p_{l'h})}.$$

After a suitable rearrangement of the vertices, the matrix $\mathbf{T}^*$ takes the following block form:

$$\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1m} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \cdots & \mathbf{B}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{m1} & \mathbf{B}_{m2} & \cdots & \mathbf{B}_{mm} \end{bmatrix},$$

where each $\mathbf{B}_{kl}$ is a block matrix in which all entries are identical. Denote the entries in block

$\mathbf{B}_{kl}$ by B_{kl} , which takes the value of T_{ij}^* in (O.17) if i and j are in groups k and l , respectively. Consider eigenvectors in the form of $\mathbf{v} = (v_1, v_1, \dots, v_2, v_2, \dots, v_m, v_m \dots)^\top$. The equation $\mathbf{T}^* \mathbf{v} = \lambda \mathbf{v}$ implies: $n_1 B_{11} v_1 + n_2 B_{12} v_2 + \dots + n_m B_{1m} v_m = \lambda v_1, \dots$ and $n_1 B_{m1} v_1 + n_2 B_{m2} v_2 + \dots + n_m B_{mm} v_m = \lambda v_m$, which completes the proof of Lemma O2.1. $\square$

We focus on the case where $d_1^* > d_2^*$ in this proof of Lemma C.1. The proof for $d_1^* < d_2^*$ is almost identical and thus omitted. By applying Lemma O2.1, we reduce the $n \times n$ matrix $\mathbf{T}^*$ to an $m \times m$ matrix $\mathbf{F}_{\mathbf{T}^*}$. Recall that for vertices in group 1 (the group of size n_1), $d_1^* = n_1 p + (m-1)n_2 q$, and for vertices in the remaining groups, $d_2^* = n_1 q + n_2 p + (m-2)n_2 q$. Then, the matrix $\mathbf{F}_{\mathbf{T}^*}$ takes the following form:

$$\mathbf{F}_{\mathbf{T}^*} = \begin{bmatrix} a & b & b & b & \dots & b \\ e & c & d & d & \dots & d \\ e & d & c & d & \dots & d \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ e & d & d & \dots & c & d \\ e & d & d & \dots & d & c \end{bmatrix}, \quad (\text{O.18})$$

where

$$\begin{aligned} a &= \frac{n_1 p \phi(\alpha, d_1^*)}{n_1 p \phi(\alpha, d_1^*) + (m-1)n_2 q \phi(\alpha, d_2^*)}, \\ b &= \frac{n_2 q \phi(\alpha, d_2^*)}{n_1 p \phi(\alpha, d_1^*) + (m-1)n_2 q \phi(\alpha, d_2^*)}, \\ c &= \frac{n_2 p \phi(\alpha, d_2^*)}{n_1 q \phi(\alpha, d_1^*) + n_2 p \phi(\alpha, d_2^*) + (m-2)n_2 q \phi(\alpha, d_2^*)}, \\ d &= \frac{n_2 q \phi(\alpha, d_2^*)}{n_1 q \phi(\alpha, d_1^*) + n_2 p \phi(\alpha, d_2^*) + (m-2)n_2 q \phi(\alpha, d_2^*)}, \\ e &= \frac{n_1 q \phi(\alpha, d_1^*)}{n_1 q \phi(\alpha, d_1^*) + n_2 p \phi(\alpha, d_2^*) + (m-2)n_2 q \phi(\alpha, d_2^*)}. \end{aligned}$$

We perform row operations on the matrix $\mathbf{F}_{\mathbf{T}^*} - \lambda \mathbf{I}_m$ before finding the zeros of the character-

istic polynomial:

$$\det(\mathbf{F}_{\mathbf{T}^*} - \lambda \mathbf{I}_m) = 0. \quad (\text{O.19})$$

Subtracting the last row from rows $2, 3, \dots, m-1$, equation (O.19) becomes:

$$\det \begin{bmatrix} a - \lambda & b & b & b & \dots & b \\ 0 & c - d - \lambda & 0 & 0 & \dots & -(c - d - \lambda) \\ 0 & 0 & c - d - \lambda & 0 & \dots & -(c - d - \lambda) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c - d - \lambda & -(c - d - \lambda) \\ e & d & d & \dots & d & c - \lambda \end{bmatrix} = 0.$$

We now see that one eigenvalue is $\lambda = c - d$, with algebraic multiplicity $m - 2$.

Suppose $\lambda \neq c - d$. Multiplying rows $2, 3, \dots, m-1$ by $1/(c - d - \lambda)$, we obtain:

$$\det \begin{bmatrix} a - \lambda & b & b & b & \dots & b \\ 0 & 1 & 0 & 0 & \dots & -1 \\ 0 & 0 & 1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ e & d & d & \dots & d & c - \lambda \end{bmatrix} = 0.$$

Then, subtracting multiples of rows $2, 3, \dots, m-1$ from the first and last row, we have:

$$\det \begin{bmatrix} a - \lambda & 0 & 0 & 0 & \dots & (m-1)b \\ 0 & 1 & 0 & 0 & \dots & -1 \\ 0 & 0 & 1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ e & 0 & 0 & \dots & 0 & c - \lambda + (m-2)d \end{bmatrix} = 0. \quad (\text{O.20})$$

Computing the determinant in (O.20), we see that the eigenvalues different from $c - d$ should

satisfy the equation:

$$(a - \lambda)(c - \lambda + (m - 2)d) - e(m - 1)b = 0.$$

Note that whether m is even or odd does not affect the equation. By solving the quadratic equation, we find that the other two eigenvalues are $\lambda = 1$ and $\lambda = a - e$. To determine the second-largest eigenvalue, we compare $a - e$ and $c - d$, both of which are positive. Thus, in the computation below, $|\lambda_2| = \lambda_2$, and the absolute value is omitted. Comparing the two candidates $a - e$ and $c - d$, denoting the denominators of a and c as D_1 and D_2 respectively, we obtain

$$\begin{aligned} & D_1 D_2 (a - e - c + d) \\ &= n_1 p \phi(\alpha, d_1^*) [n_1 q \phi(\alpha, d_1^*) + n_2 p \phi(\alpha, d_2^*) + (m - 2) n_2 q \phi(\alpha, d_2^*)] \\ &\quad - (n_2 (p - q) \phi(\alpha, d_2^*) + n_1 q \phi(\alpha, d_1^*)) [n_1 p \phi(\alpha, d_1^*) + (m - 1) n_2 q \phi(\alpha, d_2^*)] \\ &= (m - 1) n_2 q \phi(\alpha, d_2^*) (p - q) (n_1 \phi(\alpha, d_1^*) - n_2 \phi(\alpha, d_2^*)). \end{aligned} \tag{O.21}$$

This proves (C3) in Lemma C.1, since $n_1 \phi(\alpha, d_1^*) \geq n_2 \phi(\alpha, d_2^*)$ is equivalent to $\alpha \geq g^{-1}(n_1/n_2)$. We are left with the special case of $m = 2$. If $m = 2$, there is no eigenvalue equal to $c - d$. This completes the computations in Lemma C.1. $\square$

Proof of Proposition C.1 In this proof, we focus on the case where $d_1^* > d_2^*$ and $p > q$. The case $d_1^* < d_2^*$ or $p < q$ can be handled analogously. We therefore provide only brief arguments for that case at the end of the proof and omit the remaining details. For $m \geq 3$, to show monotonicity, we first consider $\alpha \geq g^{-1}(n_1/n_2)$, for which the second-largest eigenvalue is:

$$\lambda_2 = a - e = \frac{n_1 p \phi(\alpha, d_1^*)}{n_1 p \phi(\alpha, d_1^*) + (m - 1) n_2 q \phi(\alpha, d_2^*)} - \frac{n_1 q \phi(\alpha, d_1^*)}{n_1 q \phi(\alpha, d_1^*) + (n_2 p + (m - 2) n_2 q) \phi(\alpha, d_2^*)}, \tag{O.22}$$

From this expression, we have $\lim_{\alpha \rightarrow \infty} \lambda_2 = 0$. We rewrite the above expression of λ_2 as:

$$\lambda_2 = \frac{1}{1 + C_1 g(\alpha)} - \frac{1}{1 + C_2 g(\alpha)}, \quad (\text{O.23})$$

where $C_1 = \frac{(m-1)n_2 q}{n_1 p}$, $C_2 = \frac{n_2 p + (m-2)n_2 q}{n_1 q}$, and $g(\alpha, d_1^*, d_2^*) = \frac{\phi(\alpha, d_2^*)}{\phi(\alpha, d_1^*)}$. Since d_1^* and d_2^* are fixed here, we omit them in the notation below and write $g(\alpha)$.

Note that $0 < C_1 < C_2$ for $p > q$, hence $\lambda_2 > 0$. Then, the derivative with respect to α is:

$$\frac{\partial \lambda_2}{\partial \alpha} = -\frac{C_1 \frac{dg}{d\alpha}}{(1 + C_1 g(\alpha))^2} + \frac{C_2 \frac{dg}{d\alpha}}{(1 + C_2 g(\alpha))^2} = \frac{dg}{d\alpha} \left[\frac{(C_2 - C_1)(1 - C_1 C_2 g^2(\alpha))}{(1 + C_1 g(\alpha))^2 (1 + C_2 g(\alpha))^2} \right]. \quad (\text{O.24})$$

Property 2 in (B1) implies that $\frac{dg}{d\alpha} < 0$. Thus, $\frac{\partial \lambda_2}{\partial \alpha} \leq 0$ for $1 - C_1 C_2 g^2(\alpha) \geq 0$, or:

$$\alpha \geq g^{-1} \left(\frac{1}{\sqrt{C_1 C_2}} \right) = g^{-1} \left(\frac{n_1}{n_2} \left(\frac{p}{(m-1)(p + (m-2)q)} \right)^{1/2} \right). \quad (\text{O.25})$$

Similarly, for $\alpha < g^{-1}(n_1/n_2)$,

$$\lambda_2 = c - d = \frac{n_2(p - q)g(\alpha)}{n_1 q + n_2(p + (m-2)q)g(\alpha)}. \quad (\text{O.26})$$

Then, taking the derivative and simplifying, we get:

$$\frac{\partial \lambda_2}{\partial \alpha} = \frac{\partial g}{\partial \alpha} \frac{n_1 n_2 (p - q) q}{(n_1 q + n_2(p + (m-2)q)g(\alpha))^2} < 0, \quad (\text{O.27})$$

since $\frac{\partial g}{\partial \alpha} < 0$. For $m = 2$, noting that $\lambda_2 = a - e$, the proof follows similarly from (O.22)–(O.25).

We now consider several additional cases under the condition $m \geq 3$.

If $d_1^* > d_2^*$ and $p < q$, then $C_1 > C_2 > 0$, $\lambda_2 < 0$, and Equation (O.24) becomes

$$\frac{\partial |\lambda_2|}{\partial \alpha} = -\frac{\partial \lambda_2}{\partial \alpha} = \frac{C_1 \frac{dg}{d\alpha}}{(1 + C_1 g(\alpha))^2} - \frac{C_2 \frac{dg}{d\alpha}}{(1 + C_2 g(\alpha))^2} = \frac{dg}{d\alpha} \left[\frac{(C_1 - C_2)(1 - C_1 C_2 g^2(\alpha))}{(1 + C_1 g(\alpha))^2 (1 + C_2 g(\alpha))^2} \right]. \quad (\text{O.28})$$

For $d_1^* < d_2^*$ and $p > q$, when $\alpha \geq g^{-1}(n_1/n_2)$,

$$\lambda_2 = c - d = \frac{n_2(p - q)g(\alpha)}{n_1q + n_2(p + (m - 2)q)g(\alpha)} > 0. \quad (\text{O.29})$$

Therefore, when $\alpha \geq g^{-1}(n_1/n_2)$, we have

$$\frac{\partial|\lambda_2|}{\partial\alpha} = \frac{\partial\lambda_2}{\partial\alpha} = \frac{\partial g}{\partial\alpha} \frac{n_1n_2(p - q)q}{(n_1q + n_2(p + (m - 2)q)g(\alpha))^2} > 0, \quad (\text{O.30})$$

where we use the fact that when $d_1^* < d_2^*$, Property 2' implies $\frac{\partial g}{\partial\alpha} > 0$.

The remaining cases are analogous and are therefore omitted. $\square$

Finally, we determine how large an increase in α needs to be in order to observe a change in convergence speed with high probability. We answer this question by applying Theorem 1 in the following proposition.

Proposition O2.1. *We assume that $|\alpha_0| + |\alpha_1| < C$ for some positive constant C and $\alpha_0, \alpha_1 \in \mathcal{D}_\alpha$ as defined in Proposition C.1. There exists an integer $n_0 \in \mathbb{N}$, such that for all $n > n_0$, if*

$$\alpha_1 - \alpha_0 > \hat{C} \frac{\sqrt{\log n}}{\tau_n \sqrt{n}} \max_{\alpha \in [\alpha_0, \alpha_1]} \left\{ \left[\frac{\partial\phi/\partial\alpha(\alpha, d_1)}{\phi(\alpha, d_1)} - \frac{\partial\phi/\partial\alpha(\alpha, d_2)}{\phi(\alpha, d_2)} \right]^{-1} \right\}, \quad (\text{O.31})$$

we have, for some constant $\hat{C} > 0$ independent of n ,³²

$$\mathbb{P} \{ |\lambda_2(\mathbf{T}(\alpha_0))| - |\lambda_2(\mathbf{T}(\alpha_1))| > 0 \} \geq 1 - \frac{16}{n^2}.$$

Remark 1. *The condition in (O.31) may seem complicated but is not restrictive: the right-hand side of the inequality approaches zero as n goes to ∞ . This condition simply describes the minimum required size of $\alpha_1 - \alpha_0$.*

This proposition determines the required increase from α_0 to α_1 so that the change in the

³² $\hat{C}$ depends on $\limsup_n p_n/q_n$, $\limsup_n n_2/n_1$, $\limsup_n \phi(\alpha_1, d_2)/\phi(\alpha_1, d_1)$ and the constant $\tilde{C}$ in Theorem 1.

magnitude of the eigenvalue exceeds the error, whose size is given by Theorem 1. Therefore, Proposition O2.1 states that if α increases from α_0 to α_1 satisfying the conditions in Proposition O2.1, then we will observe a decrease in $|\lambda_2(\mathbf{T})|$ and thus an increase in convergence speed with a probability tending to one as n goes to ∞ .

Proof of Proposition O2.1. In this proof we focus on the case $d_1^* > d_2^*$ and $p > q$, the other cases can be proved similarly and we omit the details. By (O.23) we have $|\lambda_2| = \lambda_2$. Recall that in Lemma C.1 and Proposition C.1, we computed $|\lambda_2(\mathbf{T}^*(\alpha))|$ and $\frac{\partial |\lambda_2(\mathbf{T}^*(\alpha))|}{\partial \alpha}$. Since $|\lambda_2(\mathbf{T}^*(\alpha))|$ is positive and monotonically decreasing on $\mathcal{D}_\alpha$, for $\alpha_1 > \alpha_0$,

$$|\lambda_2(\mathbf{T}^*(\alpha_0))| - |\lambda_2(\mathbf{T}^*(\alpha_1))| = \lambda_2(\mathbf{T}^*(\alpha_0)) - \lambda_2(\mathbf{T}^*(\alpha_1)). \quad (\text{O.32})$$

By the mean value theorem,

$$\lambda_2(\mathbf{T}^*(\alpha_0)) - \lambda_2(\mathbf{T}^*(\alpha_1)) = (\alpha_0 - \alpha_1) \left. \frac{\partial \lambda_2(\mathbf{T}^*(\alpha))}{\partial \alpha} \right|_{\alpha=\alpha_2}, \quad (\text{O.33})$$

where $\alpha_2 \in [\alpha_0, \alpha_1]$. We want to determine the size of the derivative. Let

$$C_1 = \frac{(m-1)n_2q}{n_1p}, \quad C_2 = \frac{n_2p + (m-2)n_2q}{n_1q}.$$

Then, for $\alpha \geq g^{-1}(n_1/n_2)$, using the expression (O.24) in the proof of Proposition C.1, we have

$$\left. \frac{\partial \lambda_2(\mathbf{T}^*(\alpha))}{\partial \alpha} \right|_{\alpha=\alpha_2} = -C_3 \left(\frac{n_2}{n_1}, \frac{q}{p}, \frac{\phi(\alpha_2, d_2^*)}{\phi(\alpha_2, d_1^*)} \right) \left[\frac{\partial \phi / \partial \alpha(\alpha_2, d_1^*)}{\phi(\alpha_2, d_1^*)} - \frac{\partial \phi / \partial \alpha(\alpha_2, d_2^*)}{\phi(\alpha_2, d_2^*)} \right], \quad (\text{O.34})$$

where C_3 is some positive constant that depends on the fractions $\frac{n_2}{n_1}$, $\frac{q}{p}$, and $\frac{\phi(\alpha_2, d_2^*)}{\phi(\alpha_2, d_1^*)}$, but does not

depend on n when n is large. By Assumptions 1-2, we imply that $\max_{\alpha_2 \in [\alpha_0, \alpha_1]} C_3 \left(\frac{n_2}{n_1}, \frac{q}{p}, \frac{\phi(\alpha_2, d_2^*)}{\phi(\alpha_2, d_1^*)} \right) \leq \tilde{C}_3$ for some positive constant $\tilde{C}_3$. By assumption,

$$\alpha_1 - \alpha_0 > \hat{C} \left(\tilde{C}_3, \tilde{C} \right) \frac{\sqrt{\log n}}{\tau_n \sqrt{n}} \max_{\alpha \in [\alpha_0, \alpha_1]} \left\{ \left[\frac{\partial \phi / \partial \alpha(\alpha, d_1)}{\phi(\alpha, d_1)} - \frac{\partial \phi / \partial \alpha(\alpha, d_2)}{\phi(\alpha, d_2)} \right]^{-1} \right\},$$

where $\tilde{C}$ is the constant in Theorem 1. Combining (O.32) – (O.34), we obtain

$$|\lambda_2(\mathbf{T}^*(\alpha_0))| - |\lambda_2(\mathbf{T}^*(\alpha_1))| > \hat{C}C_3 \frac{\sqrt{\log n}}{\tau_n \sqrt{n}}. \quad (\text{O.35})$$

On the other hand, for $\alpha < g^{-1}(n_1/n_2)$, using expression (O.27) in the proof of Proposition C.1, we obtain equation (O.34) for a positive constant C_4 possibly different from C_3 . Similar to (O.35), we then get

$$|\lambda_2(\mathbf{T}^*(\alpha_0))| - |\lambda_2(\mathbf{T}^*(\alpha_1))| > \hat{C}C_4 \frac{\sqrt{\log n}}{\tau_n \sqrt{n}}. \quad (\text{O.36})$$

We then observe that

$$\begin{aligned} |\lambda_2(\mathbf{T}(\alpha_0))| - |\lambda_2(\mathbf{T}(\alpha_1))| &= |\lambda_2(\mathbf{T}^*(\alpha_0))| - |\lambda_2(\mathbf{T}^*(\alpha_1))| + (|\lambda_2(\mathbf{T}(\alpha_0))| - |\lambda_2(\mathbf{T}^*(\alpha_0))|) \\ &\quad + (|\lambda_2(\mathbf{T}^*(\alpha_1))| - |\lambda_2(\mathbf{T}(\alpha_1))|) \\ &> |\lambda_2(\mathbf{T}^*(\alpha_0))| - |\lambda_2(\mathbf{T}^*(\alpha_1))| + C_5 \frac{\sqrt{\log n}}{\tau_n \sqrt{n}}, \end{aligned}$$

with probability at least $1 - \frac{16}{n^2}$ for some constant C_5 , by Theorem 1. Letting $\hat{C} = \frac{(1-C_5)}{\min(C_3, C_4)}$, we see that the result of Proposition O2.1 follows. $\square$

O3 Extension: Perturbation of Adjacency Matrix

In this section, we examine the effects of perturbations in the network structure. Specifically, we introduce a perturbation term to each entry of the adjacency matrix $\mathbf{A}$ and analyze its impact on our main results.³³

More formally, for $i, j = 1, 2, \dots, n$, let ϵ_{ij} be independent and identically distributed random variables supported on $[0, 1]$. The perturbed adjacency matrix $\tilde{\mathbf{A}}$ is then defined entrywise

³³Ideally, the perturbation would be applied directly to the matrix $\mathbf{T}$. However, this could easily disrupt the row-stochastic structure of the matrix. For theoretical simplicity, we instead introduce the perturbation to the adjacency matrix $\mathbf{A}$.

as:

$$\tilde{A}_{ij} = (1 - \delta)A_{ij} + \delta\epsilon_{ij},$$

for some small enough $\delta \in [0, 1]$. Replacing $\mathbf{A}$ with $\tilde{\mathbf{A}}$ in (3), we define the perturbed weight matrix $\tilde{\mathbf{T}}$ by:

$$\tilde{T}_{ij} = \frac{\tilde{A}_{ij}\phi(\alpha, d_j(\tilde{\mathbf{A}}))}{\sum_{j'} \tilde{A}_{ij'}\phi(\alpha, d_{j'}(\tilde{\mathbf{A}}))}.$$

Similarly, let $\tilde{\mathbf{R}} = \mathbb{E}\tilde{\mathbf{A}}$ and define the corresponding deterministic matrix $\tilde{\mathbf{T}}^*$ as:

$$\tilde{T}_{ij}^* = \frac{\tilde{R}_{ij}\phi(\alpha, d_j(\tilde{\mathbf{R}}))}{\sum_{j'} \tilde{R}_{ij'}\phi(\alpha, d_{j'}(\tilde{\mathbf{R}}))}.$$

By a similar argument as in the proof of Theorem 1 and using the Bauer–Fike theorem Bauer and Fike, 1960, we obtain the following result:

Proposition O3.1. *Given any degree-dependence parameter $\alpha \in \mathbb{R}$ and a random network $\mathbf{A}(\mathbf{P}, \mathbf{n})$, there exists a positive constant $\tilde{C}$, independent of n , such that for all $n > 0$, the following results hold for the realized learning matrices $\mathbf{T}(\mathbf{A}, \alpha)$ and the perturbation learning matrix $\tilde{\mathbf{T}} = \mathbf{T}(\tilde{\mathbf{A}}, \alpha)$:*

$$\mathbb{P}\left\{|\lambda_2(\mathbf{T}) - \lambda_2(\tilde{\mathbf{T}})| \geq \tilde{C} \frac{\sqrt{n \log n}}{\min_i d_i(\mathbf{R})}\right\} \leq \frac{16}{n^2}.$$

The additional perturbation term does not alter the structural assumptions imposed in the paper. Moreover, the concentration result of Theorem 1 continues to hold, as the perturbation ϵ_{ij} is bounded. Consequently, the concentration inequalities used in the proof of Theorem 1 remain unaffected, ensuring that the results that rely on this theorem follow.

O4 Alternatives to Assumption 3 Based on Model Primitives

Recall that Assumption 3 is stated in terms of eigenvalues rather than model primitives. This section provides alternative assumptions based solely on model primitives.

Assumption 3 plays two key roles. First, it guarantees that the realized adjacency matrix $\mathbf{A}$ (with high probability) and the expected matrix $\mathbf{R}$ are primitive (Proposition A.1), providing the basis for our discussion of convergence speed and influence. Second, for the asymptotic result in Theorem 1, it ensures that the spectral gap of the expected matrix remains strictly bounded away from zero. This provides the structural stability required for the eigenvectors to be robust to noise (i.e., deviations between the realized and expected matrices).

O4.1 Alternative Assumption for the Primitivity of $\mathbf{A}$ with High Probability

First, we provide an alternative condition for the first role of Assumption 3 and establish an analogue to Proposition A.1.

Assumption 4 (Block connectivity). *Let $\frac{\max_{1 \leq i \leq m} \{n_i\}}{\min_{1 \leq i \leq m} \{n_i\}} = c_n$. We assume $c_n \ll \log n$. Moreover, for any $1 \leq i, j \leq m$, there exist some $1 \leq k, l \leq m$ such that $p_{ik}p_{kj} \geq \frac{\sqrt{7mc_n} \log n}{n}$ and $p_{ik}p_{kl}p_{li} \geq \frac{\sqrt{\log n}}{n^{3/2}}$.*

Proposition O4.1. *Under Assumptions 1, 2, and 4, there exists $n_0 \in \mathbb{N}$ such that $\mathbf{A}$ is strongly connected and aperiodic with probability at least $1 - n^{-4}$ for all $n \geq n_0$. In other words, $\mathbf{A}$ is primitive with high probability. Moreover, $\mathbf{R}$ is primitive.*

Proof of Proposition O4.1. Without loss of generality, suppose that $\frac{\max_{1 \leq i \leq m} n_i}{\min_{1 \leq i \leq m} n_i} = c_n$.

To establish that $\mathbf{A}$ is strongly connected with probability at least $1 - n^{-4}$, it suffices to show that the diameter of the network is at most 2. Let A_n be the event that there exists a pair (i, j)

not connected by a path of length 2:

$$P(A_n) = P(\exists 1 \leq i, j \leq n \text{ s.t. there is no length-2 path connecting } i \text{ and } j).$$

By Assumption 4, there exists $n_a \in \mathbb{N}$ such that for all $n \geq n_a$, we can bound this probability using the union bound and the independence of edges:

$$\begin{aligned} P(A_n) &\leq \sum_{i,j=1}^n P(\text{no length-2 path connects } i \text{ and } j) \\ &= \sum_{i,j=1}^n P((\mathbf{A}^2)_{ij} = 0) \leq \sum_{i,j=1}^n P\left(\sum_{k \neq i,j} A_{ik} A_{kj} = 0\right) \\ &\leq n^2 \left(1 - \frac{7mc_n \log n}{n}\right)^{\min_{1 \leq t \leq m} n_t} \leq n^2 \left(1 - \frac{7mc_n \log n}{n}\right)^{\frac{n}{mc_n}} \\ &\leq n^2 e^{-7 \log n} = n^{-5} \leq n^{-4}. \end{aligned} \tag{O.37}$$

Thus, $\mathbf{A}$ is strongly connected with high probability.

Next, consider aperiodicity. Recall that a strongly connected graph is aperiodic (primitive) if the greatest common divisor (GCD) of the lengths of its cycles is 1. Therefore, it suffices to show that the network contains cycles of lengths 2 and 3, as $\text{GCD}(2, 3) = 1$.

Define B_n as the event that some node i is not part of a length-2 cycle:

$$P(B_n) = P(\exists 1 \leq i \leq n \text{ s.t. there is no length-2 cycle at } i).$$

Then there exists $n_b \in \mathbb{N}$ such that for all $n \geq n_b$,

$$\begin{aligned} P(B_n) &\leq \sum_{i=1}^n P(\text{no length-2 cycle at } i) \\ &= \sum_{i=1}^n P((\mathbf{A}^2)_{ii} = 0) \leq P(A_n) \leq n^{-4}. \end{aligned} \tag{O.38}$$

Similarly, define C_n as the event that some node i is not part of a length-3 cycle:

$$P(C_n) = P(\exists 1 \leq i \leq n \text{ s.t. there is no length-3 cycle at } i).$$

Then there exists $n_c \in \mathbb{N}$ such that for all $n \geq n_c$,

$$\begin{aligned} P(C_n) &\leq \sum_{i=1}^n P(\text{no length-3 cycle at } i) \\ &= \sum_{i=1}^n P((\mathbf{A}^3)_{ii} = 0) \leq \sum_{i,j=1}^n P\left(\sum_{k,l \neq i,j} A_{ik} A_{kl} A_{lj} = 0\right) \\ &\leq n^2 \left(1 - \frac{\sqrt{\log n}}{n^{3/2}}\right)^{\min_{1 \leq t \leq m} n_t^2} \leq n^2 e^{-\sqrt{\log n} \sqrt{n}} \leq n^{-4}. \end{aligned} \quad (\text{O.39})$$

Combining these results, with high probability the network is strongly connected and contains cycles of lengths 2 and 3. Since $\text{GCD}(2, 3) = 1$, the network is aperiodic. Finally, for $\mathbf{R}$, note that under these assumptions all entries of $\mathbf{R}^2$ are positive, and therefore $\mathbf{R}$ is primitive. $\square$

O4.2 Alternative Assumption for Bounded Spectral Gap

Let $\gamma_n = 1 - |\lambda_2(\mathbf{T}^*)|$. Proposition O4.1 (with Assumption 4 replacing Assumption 3) implies that $\gamma_n > 0$, allowing us to establish the following alternative statement for Theorem 1 (the influence part):

$$\mathbb{P}\left\{\|\mathbf{s}(\mathbf{T}) - \mathbf{s}(\mathbf{T}^*)\|_2 \geq \frac{\tilde{C}}{\gamma_n} \frac{\sqrt{\log n}}{\tau_n \sqrt{n}}\right\} \leq \frac{16}{n^2}.$$

The second role of Assumption 3 is to provide a uniform lower bound, $c > 0$, for γ_n in the denominator. This subsection provides alternative sufficient conditions for this bound.

We first show in the next proposition that a sufficient condition is that the quotient matrix $\mathbf{F}_{\mathbf{T}^*}$ (defined in Equation (O.17)) converges to a primitive limit. We then provide a further sufficient condition based solely on model primitives.

Proposition O4.2. *If there exists a primitive matrix $\mathbf{C}$ such that $\lim_{n \rightarrow \infty} \mathbf{F}_{\mathbf{T}^*} = \mathbf{C}$, then the spectral gap sequence $\{\gamma_n\}$ is bounded below by some positive constant $c > 0$.*

Proof of Proposition O4.2. Since $\mathbf{C}$ is the limit of a sequence of row-stochastic matrices, $\mathbf{C}$ is also row-stochastic. Moreover, since $\mathbf{C}$ is primitive, the Perron–Frobenius theorem implies that its largest eigenvalue is simple and equal to 1, and all other eigenvalues are strictly less than 1 in magnitude. That is, $|\lambda_2(\mathbf{C})| < 1$.

Let $c_1 = \frac{1}{2}(1 - |\lambda_2(\mathbf{C})|) > 0$. By Lemma O2.1, the non-zero eigenvalues of $\mathbf{T}^*$ are exactly the eigenvalues of $\mathbf{F}_{\mathbf{T}^*}$. Since eigenvalues are continuous functions of the matrix entries (Ostrowski’s theorem), and $\lim_{n \rightarrow \infty} \mathbf{F}_{\mathbf{T}^*} = \mathbf{C}$, it follows that

$$\lim_{n \rightarrow \infty} |\lambda_2(\mathbf{T}^*)| = \lim_{n \rightarrow \infty} |\lambda_2(\mathbf{F}_{\mathbf{T}^*})| = |\lambda_2(\mathbf{C})|.$$

Therefore, there exists n_c such that for all $n > n_c$, $|\lambda_2(\mathbf{T}^*)| \leq |\lambda_2(\mathbf{C})| + c_1 = 1 - c_1$. This implies $\gamma_n \geq c_1 > 0$.

Finally, letting $c = \min\{\gamma_1, \dots, \gamma_{n_c}, c_1\} > 0$, we have $\gamma_n \geq c > 0$ for all n . □

When is the limit of $\mathbf{F}_{\mathbf{T}^*}$ primitive? We further provide a sufficient condition based on the model primitives.

Assumption 5. $\lim_{n \rightarrow \infty} \frac{n_l p_{kl}}{n \tau_n} = c_{kl} > 0$ for all $1 \leq k, l \leq m$.

Lemma O4.1. *Under Assumption 5, there exists a primitive matrix $\mathbf{C}$ such that $\lim_{n \rightarrow \infty} \mathbf{F}_{\mathbf{T}^*} = \mathbf{C}$.*

Proof of Lemma O4.1. Recall that the (k, l) -th entry of $\mathbf{F}_{\mathbf{T}^*}$ is given by:

$$(\mathbf{F}_{\mathbf{T}^*})_{kl} = \frac{n_l p_{kl} \phi(\alpha, \sum_h n_h p_{lh})}{\sum_{l'} n_{l'} p_{kl'} \phi(\alpha, \sum_h n_h p_{l'h})}.$$

Under the condition that $\lim_{n \rightarrow \infty} \frac{n_l p_{kl}}{n \tau_n} = c_{kl} > 0$, we can normalize the terms in the fraction by dividing the numerator and denominator by $\phi(\alpha, n \tau_n)$ and $n \tau_n$. Provided ϕ is a regularly varying function (e.g., $\phi(x) = x^\alpha$), the ratio of the ϕ terms converges to the ratio of their arguments. Specifically,

$$\lim_{n \rightarrow \infty} \frac{\phi(\alpha, \sum_h n_h p_{lh})}{\phi(\alpha, n \tau_n \sum_h c_{lh})} = 1.$$

Substituting the limits c_{kl} and simplifying, the matrix converges element-wise to a limit matrix $\mathbf{C}$ with entries determined by the constants c_{kl} . Since $c_{kl} > 0$ for all k, l , the limit matrix $\mathbf{C}$ is strictly positive, and thus primitive. $\square$

Collectively, Assumptions 4 and 5 constitute sufficient conditions based on model primitives to replace Assumption 3.

References

- Bauer, Friedrich L., and C. T. Fike. 1960. “Norms and exclusion theorems.” *Numerische Mathematik* 2 (1): 137–41.
- Hoeffding, Wassily. 1963. “Probability inequalities for sums of bounded random variables.” *Journal of the American Statistical Association* 58 (301): 13–30.
- Vershynin, Roman. 2018. *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge: Cambridge University Press.