Online Appendix

Free entry in a Cournot market with overlapping ownership

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A Main appendix

A.1 Additional simulation results

Figure 6 plots the socially optimal level(s) $\lambda^o \coloneqq \arg\max_{\lambda\in[0,1]} \mathrm{TS}_{n^*(\lambda)}$ of overlapping ownership under linear demand and linear-quadratic costs as a function of the entry cost f and the level c_2 of decreasing, constant, or increasing MC.[56]

A.2 Some commonly used conditions

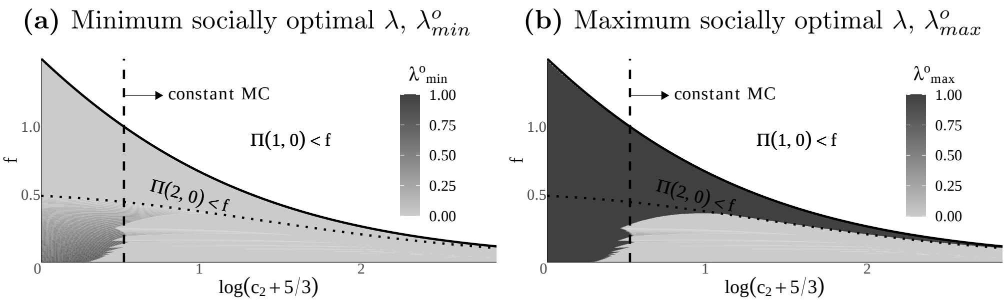
In the proofs to come, it will be useful to remember that if $\Delta > 0$ (resp. $\Delta < 0$), then

$$
\begin{aligned}
&(1+\lambda+\Delta/n)/H_n = 1 + H_n^{-1} - \Lambda_n^{-1} C''(q)/P'(Q) \\
\overset{\text{(resp. } \geq)}{\leq}\ &(1+\lambda+\Delta/\Lambda_n)/H_n = 1 + H_n^{-1} + \left[(1-\lambda)(1-H_n) - C''(q)/P'(Q)\right]/(\Lambda_n H_n) \\
\overset{\text{(resp. } \geq)}{\leq}\ &(1+\lambda+\Delta)/H_n = (2 - C''(q)/P'(Q))/H_n,
\end{aligned}
$$

where $\Lambda_n \coloneqq 1+\lambda(n-1) = nH_n$. Also, $E_{P'}(Q) < \frac{1+\lambda+\Delta(Q,Q_{-i})}{1-(1-\lambda)(1-s_i)}$ on L implies that for any $n \in [1,+\infty)$ and any $Q < \overline{Q}$, $E_{P'}(Q) < (1+\lambda+\Delta(Q,(n-1)Q/n))/H_n$. Thus, part (ii) of the maintained assumption implies that when $\Delta < 0$, $E_{P'}(Q)$ is also lower than $(1+\lambda+\Delta/\Lambda_n)/H_n$ and $(1+\lambda+\Delta/n)/H_n$ in the symmetric equilibrium.

[56] It can be checked that $\mathrm{sgn}\{\widehat{n}^*(\lambda) - \widehat{n}^o(\lambda)\}$, $\Pi(n,\lambda)$, TS_n, and $\widehat{n}^*(\lambda)$ depend on a, c_1, and f only through $(a-c_1)^2/f$ (see the discussion in the Derivation of Numerical result 2 in Appendix B). Thus, $a = 3$, $c_1 = 1$ in Figure 2 and Figure 6 in the appendix is, modulo the integer constraint on n, without loss of generality in the following sense. If plotted without the integer constraint, then for any a, c_1, f such that $(a-c_1)^2/f = 80$, Figure 2 will give the same result about when entry is excessive or insufficient. Similarly, Figure 6 without the integer constraint (*i.e.*, for $\widehat{\lambda}^o \coloneqq \arg\max_{\lambda\in[0,1]} \mathrm{TS}_{\widehat{n}^*(\lambda)}$) will give the same result for any a and c_1 with the scaling of the f-axis adjusted. For instance, if a is equal to 2 instead of 3, then the f-axis scale will be divided by $(3-1)^2/(2-1)^2 = 2$ (e.g., where 0.5 under $a = 3$ on the f-axis, we will have 0.25 under $a = 2$).

Figure 6: Socially optimal level(s) $\lambda^o \coloneqq \arg\max_{\lambda\in[0,1]} \mathrm{TS}_{n^*(\lambda)}$ of overlapping ownership under linear demand and linear-quadratic costs as a function of the entry cost f and the level c_2 of decreasing, constant, or increasing MC



Note: $a = 3$, $b = c_1 = 1$, $c_2 \in [-2/3,15]$. For better readability, an increasing transformation is applied on the x-axis (*i.e.*, c_2). On the dashed line, there is constant MC (*i.e.*, $c_2 = 0$). On the left (resp. right) of the line, marginal cost is decreasing (resp. increasing). In the white region above the solid line, the net monopoly profit is negative (*i.e.*, $\Pi(1,0) < f$), and thus no firm enters for every λ. Above the dotted line, the net duopoly profit is negative (*i.e.*, $\Pi(2,0) < f$). Thus, in the region between the solid and the dotted line, only one firm enters absent overlapping ownership (*i.e.*, for $\lambda = 0$). In the region on the left of (and including) the dashed line (and below the solid line), one firm enters if $\lambda = 1$ (to see why, look at the derivation of Remark 4.2 in the proof of Proposition 5). In some cases, λ^o is not a singleton. In these cases, every optimal $\lambda \in \lambda^o$ induces entry by only one firm, in which case the exact value of λ does not matter. Panel (a) plots the minimum among all optimal levels of overlapping ownership, while panel (b) plots the maximum one. For example, in the region between the dotted and solid lines, both $0 \in \lambda^o$ and $1 \in \lambda^o$ lead to entry by a single firm, maximizing total surplus.

A.3 Proofs of section 2

Where clear we may simplify notation (e.g., omitting the subscript n).

Proof of Proposition 1 Wlog we can constrain attention to quantity profiles $\boldsymbol{q} \in \left\{\boldsymbol{x} \in [0,\overline{q}]^n : \sum_{i\in\mathcal{F}} x_i \le \overline{Q}\right\}$. Also, the best response of firm i depends on $\boldsymbol{q}_{-i}$ only through Q_{-i}. Denote by $r(Q_{-i})$ the best response correspondence of a firm (the same for all firms). If it is a differentiable function, its slope is given by $r_i'(Q_{-i}) = -1 + \Delta(Q,Q_{-i})/[1 + \lambda + \Delta(Q,Q_{-i}) - (s_i + \lambda(1 - s_i))E_{P'}(Q)]$, for $q_i = r(Q_{-i})$. The proof is then similar to that of Theorem 2.1 in Amir and Lambson (AL; 2000).[57]

Case $\boldsymbol{\Delta} > 0$: We first prove statement (a).

Existence of symmetric equilibrium: Firm i's problem is equivalent to

[57]The proof of uniqueness under $\Delta > 0$ is not considered in AL but is also an extension of standard results.

choosing the total quantity to be given by the correspondence $R:[0,\overline{Q}]\to[0,\overline{Q}]$ defined as

$$R(Q_{-i}) \coloneqq \underset{Q\in[Q_{-i},Q_{-i}+\overline{q}]}{\arg\max}\left\{P\left(Q\right)\left[Q-(1-\lambda)Q_{-i}\right]-C(Q-Q_{-i})\right\} = r(Q_{-i})+Q_{-i}.$$

taking Q_{-i} as given. The maximand above is strictly supermodular since $\Delta>0$, so by Theorem A.1 in AL every selection from $R(Q_{-i})$ is non-decreasing in Q_{-i}. Thus, every selection of the correspondence $B:[0,(n-1)\overline{q}]\rightrightarrows[0,(n-1)\overline{q}]$ given by $B(Q_{-i})\coloneqq(n-1)R(Q_{-i})/n$ is also non-decreasing in Q_{-i}. By Tarski's intersection point theorem (Theorem A.3 in AL), B has a fixed point, which is a symmetric equilibrium.

Non-existence of asymmetric equilibria: Suppose by contradiction that an asymmetric equilibrium exists, and denote it by $\widetilde{\boldsymbol{q}}$. Then, any permutation of $\widetilde{\boldsymbol{q}}$ should also be an equilibrium, and since $\widetilde{\boldsymbol{q}}$ is asymmetric there exists a permutation $\widehat{\boldsymbol{q}}$ with a firm i such that $\widehat{q}_i>\widetilde{q}_i$. But $\widetilde{Q}=\widehat{Q}$, so $\widehat{Q}_{-i}<\widetilde{Q}_{-i}$. Thus, $R\left(\widehat{Q}_{-i}\right)=R\left(\widetilde{Q}_{-i}\right)=\widetilde{Q}\geq\widetilde{Q}_{-i}>\widehat{Q}_{-i}\implies R\left(\widehat{Q}_{-i}\right)>\widehat{Q}_{-i}$, so $\widetilde{Q}=R\left(\widehat{Q}_{-i}\right)$ makes the first derivative of the firm's objective non-negative, that is $P\left(\widetilde{Q}\right)+P'\left(\widetilde{Q}\right)\left[\widetilde{Q}-(1-\lambda)\widehat{Q}_{-i}\right]-C'(\widetilde{Q}-\widehat{Q}_{-i})\geq 0$. Also, since the firm's action space is not bounded from above, it trivially holds that $P\left(\widetilde{Q}\right)+P'\left(\widetilde{Q}\right)\left[\widetilde{Q}-(1-\lambda)\widetilde{Q}_{-i}\right]-C'(\widetilde{Q}-\widetilde{Q}_{-i})\leq 0$. The last two inequalities imply

$$-(1-\lambda)P'\left(\widetilde{Q}\right)-\frac{C'(\widetilde{Q}-\widetilde{Q}_{-i})-C'(\widetilde{Q}-\widehat{Q}_{-i})}{\widetilde{Q}_{-i}-\widehat{Q}_{-i}}\leq 0. \tag{5}$$

Last, since every selection from $R(Q_{-i})$ is non-decreasing in Q_{-i}, it follows from $R\left(\widehat{Q}_{-i}\right)=R\left(\widetilde{Q}_{-i}\right)=\widetilde{Q}$ that $R\left(Q_{-i}\right)=\widetilde{Q}$ for all $Q_{-i}\in[\widehat{Q}_{-i},\widetilde{Q}_{-i}]$. Therefore, in (5) we can let $\widehat{Q}_{-i}\to\widetilde{Q}_{-i}$, which gives $\Delta(\widetilde{Q},\widetilde{Q}_{-i})\leq 0$, a contradiction.

For part (b) it remains to show that at most one symmetric equilibrium exists. $E_{P'}<(1+\lambda+\Delta)/H_n$ on L—which holds given that $E_{P'}<(1+\lambda+\Delta/n)/H_n$ and $\Delta>0$ on L—implies that $\partial^2\left(\pi_i+\lambda\sum_{j\neq i}\pi_j\right)\Big/\left(\partial q_i\right)^2<0$, so that $r(Q_{-i})$ is a differentiable function. At a symmetric quantity profile we have $r'(Q_{-i})=-1+\Delta(Q,Q_{-i})/(1+\lambda+\Delta(Q,Q_{-i})-H_nE_{P'}(Q))$. Symmetric equilibria are solutions to $g(q)\equiv r((n-1)q)-q=0$. Thus, there will be at most one symmetric equilibrium if $g' < 0$, that is, if for any $q \in [0,\overline{Q}/n)$,

$$-\frac{1+\lambda - H_n E_{P'}(nq)}{1+\lambda+\Delta(nq,(n-1)q) - H_n E_{P'}(nq)} < \frac{1}{n-1} \iff E_{P'}(nq) < \frac{1+\lambda+\Delta(nq,(n-1)q)/n}{H_n}$$

which is true, since by assumption it is true on L.

Case $\boldsymbol{\Delta} < 0$: We first prove part (a) for $m = n$. $\Delta < 0$ and $E_{P'}(Q) < \frac{2-C''(Q-Q_{-i})/P'(Q)}{1-(1-\lambda)(1-s_i)}$ implies that the objective function of each firm is strictly concave in its quantity (in the part where $P(Q) > 0$). Thus, for Q_{-i} such that $r(Q_{-i}) > 0$, $r(Q_{-i})$ is a differentiable function with slope $r_i'(Q_{-i}) = -1 + \Delta/(2 - C''(q_i)/P'(Q) - (s_i + \lambda(1-s_i))E_{P'}(Q)) < -1$ given $\Delta < 0$. Thus, again $g' < 0$ since $r' < -1 < (n-1)^{-1}$ for every $n \geq 2$. Also, $g(0) \geq 0$ and $\lim_{q\to\infty} g(q) = -\infty$, so by continuity of g there exists a unique symmetric equilibrium.

We now prove part (a) for $m < n$. Let q_m be the symmetric equilibrium quantity produced by each firm when m firms are in the market. The m firms are clearly best-responding by producing q_m each. Also, $r'(Q_{-i}) < -1$ (when $r(Q_{-i}) > 0$) implies that $r(mq_m) = r((m-1)q_m + q_m) \leq \max\{r((m-1)q_m) - q_m, 0\} = 0$, since by definition of q_m, $r((m-1)q_m) = q_m$. Thus, the non-producing firms are also best-responding.

To show part (b) assume by contradiction that there is an equilibrium $\widetilde{\boldsymbol{q}}$ of a different type. Then there exist firms i and j such that $\widetilde{q}_i \neq \widetilde{q}_j$, $\widetilde{q}_i > 0$, $\widetilde{q}_j > 0$ in that equilibrium. Wlog let $\widetilde{q}_i > \widetilde{q}_j$. Given that $R'(Q_{-i}) = r'(Q_{-i}) + 1 < 0$ (when $R(Q_{-i}) > Q_{-i}$) it follows that $R(\widetilde{Q}_{-i}) = R(\widetilde{Q}_{-j}) \implies \widetilde{Q}_{-i} = \widetilde{Q}_{-j} \implies \widetilde{q}_i = \widetilde{q}_j$, a contradiction. **Q.E.D.**

Note: The second order of differentiability of $P(Q)$ in Proposition 1 is inessential. However, it simplifies the arguments and interpretation and emphasizes the tension between the assumption $\Delta < 0$ and the one on $E_{P'}(Q)$. The latter guarantees that π_i is strictly concave in q_i whenever $P(Q) > 0$. Decreasing MC is needed for $\Delta < 0$ but at the same time tends to violate profit concavity.[58]

[58]In the $\Delta > 0$ case, for $\lambda = 0$ we recover the condition $C'' - P' > 0$, under which AL show that a symmetric equilibrium exists and there are no asymmetric equilibria (Theorem 2.1). In the $\Delta < 0$ case, the assumption on $E_{P'}$ guarantees that the firm's objective is quasiconcave in its quantity, under which condition AL show the same result. For $\lambda = 1$, increasing MC is necessary for the uniqueness of the (symmetric) equilibrium. To see why, notice for example that with constant MC, there are infinitely many equilibria (the symmetric one included), all with the same total quantity arbitrarily distributed

Proof of Corollary 1.1 $\Delta(Q,Q_{-i}) = 1 - \lambda + c_2/b$, constant over L. $E_{P'}(Q) = 0$, also constant. Last, we have that $1 + \lambda + \Delta(Q,Q_{-i}) = 2 + c_2/b$. The result then follows from Proposition 1. Notice also that $Q_n = (a - c_1)/[b(H_n + 1) + c_2/n]$, which is positive since $a > c_1$ and $c_2 > -2bc_1/a > -2b$. $\Pi(n,\lambda) = (a - c_1)^2\,(bnH_n + c_2/2)\,/\,[bn(H_n + 1) + c_2]^2$ is also positive. Last, $C'(q_n) = [bc_1(H_n{+}1){+}ac_2/n]/[b(H_n{+}1){+}c_2/n]$ is positive given $c_2 > -2bc_1/a$, so in equilibrium marginal cost is positive. **Q.E.D.**

Proof of Proposition 2 (i) From the pricing formula (1) the Implicit Function Theorem gives $\partial Q_n/\partial\lambda = -(n-1)Q/[n + \Lambda - C''(Q/n)/P'(Q) - \Lambda E_{P'}(Q)] < 0$. For fixed n, total surplus changes with λ in the same direction as total quantity: $d\,\mathrm{TS} = P(Q)dQ - \sum_{i=1}^{n} C'(q)dq = (P(Q) - C'(q))\,dQ$. Differentiating $\Pi\,(n,\lambda)$ with respect to λ we get

$$\frac{\partial\Pi\,(n,\lambda)}{\partial\lambda} = P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial\lambda} + (P(Q_n) - C'(q_n))\,\frac{\partial Q_n}{\partial\lambda}\frac{1}{n} = P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial\lambda}\frac{n-\Lambda_n}{n},$$

which is positive for $\lambda < 1$, where the second equality follows from the pricing formula (1).

(ii) Using the pricing formula (1) we get

$$\begin{aligned}\frac{\partial\Pi\,(n,\lambda)}{\partial n} &= P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial n} - Q_nP'(Q_n)H_n\frac{n\frac{\partial Q_n}{\partial n} - Q_n}{n^2}\\ &\propto -\left[(1-\lambda)\left(H_n^{-1} - 1\right) + n + \Lambda_n - H_n^{-1}C''\,(q_n)\,/P'\,(Q_n) - \Lambda_nE_{P'}\,(Q_n)\right] < 0,\end{aligned}$$

where the inequality follows from what we have seen in section A.2.

(iii) $\partial q_n/\partial n = \partial(Q_n/n)/\partial n = n^{-1}\partial Q_n/\partial n - Q_n/n^2 \propto -(1 + \lambda - H_nE_{P'}(Q))$.

(iv) From the pricing formula (1) the Implicit Function Theorem gives $\partial Q_n/\partial n = q_n\Delta/(n(1 + \lambda + \Delta/n - H_nE_{P'}\,(Q_n))) \propto \Delta$. **Q.E.D.**

across firms, since each firm maximizes aggregate industry profits. Analogously, with $C'' < 0$ it is an equilibrium for firms to concentrate all production in one firm to take advantage of the decreasing MC, as indicated in part (ii-a) of the proposition.

A.4 Proofs of sections 3 and 4

Proof of Proposition 3 The derivative of $\Psi(n,\lambda)$ with respect to n is equal to

$$\frac{\partial\Psi(n,\lambda)}{\partial n} = \lambda\left(\Pi\left(n,\lambda\right) - \Pi\left(n-1,\lambda\right)\right) + \Lambda_n \frac{\partial\Pi\left(n,\lambda\right)}{\partial n} - \left(\Lambda_n - 1\right)\left.\frac{\partial\Pi\left(\nu,\lambda\right)}{\partial\nu}\right|_{\nu=n-1}$$
$$\propto E_{\Delta\Pi,n} - \left(\frac{\Lambda_n - 1}{\Lambda_n} + \frac{n-1}{\Lambda_n}\frac{\left.\frac{\partial\Pi(\nu,\lambda)}{\partial\nu}\right|_{\nu=n-1}}{\Pi\left(n,\lambda\right) - \Pi\left(n-1,\lambda\right)}\right) < 0,$$

and the result obtains given Proposition 1. **Q.E.D.**

Proof of Proposition 4 The derivative of $\Psi(n,\lambda)$ with respect to λ is given by

$$\frac{\partial\Psi(n,\lambda)}{\partial\lambda} = \left(n-1\right)\left(\Pi\left(n,\lambda\right) - \Pi\left(n-1,\lambda\right)\right) + \Lambda_n \frac{\partial\Pi\left(n,\lambda\right)}{\partial\lambda} - \left(\Lambda_n - 1\right)\frac{\partial\Pi\left(n-1,\lambda\right)}{\partial\lambda}$$
$$\propto -\frac{\lambda\left(\frac{\partial\Pi(n,\lambda)}{\partial\lambda} - \frac{\partial\Pi(n-1,\lambda)}{\partial\lambda}\right)}{\Pi\left(n,\lambda\right) - \Pi\left(n-1,\lambda\right)} - \frac{1}{\lambda}\frac{\lambda\frac{\partial\Pi(n,\lambda)}{\partial\lambda}\Big/\Pi(n,\lambda)}{\frac{\Pi(n,\lambda)-\Pi(n-1,\lambda)}{\Pi(n,\lambda)}(n-1)} - 1.$$

The result follows by the Implicit Function Theorem given Proposition 3. **Q.E.D.**

Proof of Proposition 5 We have $\partial\,\mathrm{TS}_n/\partial n = \Pi(n,\lambda) - f - \Lambda_n Q_n P'(Q_n)\partial q_n/\partial n$. Given $\Psi\left(\widehat{n}^*(\lambda),\lambda\right) = f$, $d\,\mathrm{TS}_n/dn\big|_{n=\widehat{n}^*(\lambda)}$ is equal to (denote $\Pi_n(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial n$)

$$-\phi\left(\widehat{n}^*(\lambda),\lambda\right)\lambda\widehat{n}^*(\lambda)\Pi_n\left(\widehat{n}^*(\lambda),\lambda\right) - \Lambda_{\widehat{n}^*(\lambda)}Q_{\widehat{n}^*(\lambda)}P'\left(Q_{\widehat{n}^*(\lambda)}\right)\left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)},$$

and the result follows from single-peakedness of total surplus in n (and given $P' < 0$), if we substitute in $\Pi(n,\lambda)/\partial n$ and $\partial q_n/\partial n$ from the proof of Proposition 2. For Remark 4.2, notice that $\Delta < 0$ on L implies $C''(q) < 0$ for every $q < \overline{Q}$. By Proposition 2, Q_n is decreasing in n, and thus, so is consumer surplus. Also, $n\Pi(n,\lambda) \equiv P(Q_n)Q_n - nC(q_n) < P(Q_n)Q_n - C(Q_n) \leq P(q_1)q_1 - C(q_1) = \Pi(1,\lambda)$, where the first inequality follows from $C'' < 0$, and the second from q_1 being the monopolist's optimal quantity. Thus, both consumer surplus and industry profits are maximized for $n = 1$, so $n^o(\lambda) = 1$. Last, $n\Pi(n,\lambda) < \Pi(1,\lambda)$ for $n = 2$ and $\lambda = 1$ implies that $\Psi(2,1) = 2\Pi(2,1) - \Pi(1,1) < 0$, so $n^*(1) = 1$. Proposition 7 and Remark B.1 in the appendix also show that $\Psi(n,1) = n\Pi(n,1) - (n-1)\Pi(n-1,1) < 0$ for every $n \geq 2$, so a single firm entering is the unique equilibrium. **Q.E.D.**

Proof of Proposition 6 We have that $d\widehat{n}^*(\lambda)/df = (\partial\Psi(n,\lambda)/\partial n)^{-1}\big|_{n=\widehat{n}^*(\lambda)}$, and part (ii) follows if we take the directional derivative of $d\widehat{n}^*(\lambda)/df$. **Q.E.D.**

B Additional material

Where clear we may simplify notation, for example omitting the subscript n,λ for equilibrium objects. We may also write for example Q_n instead of $Q_{n^*(\lambda)}$, n instead of $n^*(\lambda)$. Also, we write $\Pi_n(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial n$, $\Pi_\lambda(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial\lambda$, $\Pi_{n\lambda}(n,\lambda) \equiv \partial^2\Pi(n,\lambda)/(\partial n\partial\lambda)$, $\Pi_{nn}(n,\lambda) \equiv \partial^2\Pi(n,\lambda)/(\partial n)^2$.

B.1 Individual firm's objective function under overlapping ownership

Here we briefly describe settings of common and cross ownership which can give rise to the Cournot-Edgeworth λ oligopoly model that we study.

B.1.1 Firm objectives under common ownership

The objective function that we use can arise under common ownership. One example of an ownership structure that gives rise to it is described in section B.10.1. For additional examples, see López and Vives (2019; Table 1 and Online Appendix) and Azar and Vives (2021).

B.1.2 Firm objectives under cross ownership

Firm objectives under cross ownership are also described in Gilo et al. (2006) and López and Vives (2019). Assume that we start with each firm i being held by shareholders who do not hold shares of any of the other firms. Then, each *firm* i buys share $\alpha \in [0, 1/(N-1))$ of every other firm $k \in \mathcal{F} \setminus \{i\}$ *without control rights*. In other words, each firm i acquires a claim to share α of the *total earnings* of every other firm. The total earnings of each firm

i now include the profit directly generated by firm i and firm i's earnings from its claims over the other firms' total earnings. It can be shown that this gives rise to our symmetric model with $\lambda \coloneqq \alpha/[1-(N-2)\alpha] \in [0,1)$.

B.2 Pricing-stage equilibria under parametric assumptions

CESL demand is of the form

$$P(Q) = \begin{cases} a + bQ^{1-E} & \text{if } E > 1 \\ \max\{a - b\ln Q, 0\} & \text{if } E = 1 \\ \max\{a - bQ^{1-E}, 0\} & \text{if } E < 1 \end{cases}$$

for parameters $a \geq 0$ and $b > 0$. For $E = 0$ this reduces to linear demand, while for $a = 0$ and $E > 1$ it reduces to constantly elastic demand with elasticity $\eta = (E-1)^{-1}$.

Claim 1 provides the equilibria under parametric assumptions on the demand and cost functions. The total quantity is decreasing in the level of overlapping ownership, λ.

Claim 1. Under CESL demand and constant returns to scale the total equilibrium quantity in the pricing stage is

$$Q_n = \begin{cases} \left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \\ \left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

where $H_n \coloneqq \Lambda_n/n$, $\Lambda_n \coloneqq 1 + \lambda(n-1)$. Under linear demand and linear-quadratic costs, it is $Q_n = \frac{a-c_1}{b(1+H_n)+c_2/n}$.

Proof of Claim 1 Under CESL demand and constant marginal costs the pricing formula $P(Q_n) - C'(q_n) = -H_n Q_n P'(Q_n)$ gives

$$\begin{aligned} a + b(Q_n)^{1-E} - c &= H_n b(E-1)(Q_n)^{1-E} && \text{if } E > 1 \\ a - b\ln Q_n - c &= H_n b && \text{if } E = 1 \\ a - b(Q_n)^{1-E} - c &= H_n b(1-E)(Q_n)^{1-E} && \text{if } E < 1 \end{aligned}$$

and the result follows. In the case $E > 1$, $E < 2$ and $c > a$ guarantee that there is an interior equilibrium. Notice that if $a > c$, then the profit per unit $P(Q) - AC(q) \geq a - c > 0$ is positive and bounded away from zero for every $Q \geq q \geq 0$, and thus there is no equilibrium. In the case $E < 1$, if $a \leq c$, then in the unique equilibrium $Q_n = 0$.

For linear demand and linear-quadratic costs the pricing formula $P(Q_n) - C'(q_n) = -H_n Q_n P'(Q_n)$ gives $a - bQ_n - c_1 - c_2(Q_n/n) = H_n b Q_n$, and the result follows. **Q.E.D.**

B.3 Additional comparative statics of pricing stage equilibrium

We now study how aggregate industry profits depend on the number of firms.

$$\mu_n := 1 - \frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\frac{1 - H_n}{\eta(Q_n) - H_n}.$$

Proposition 7. The following statements hold:

(i) if $\mu_n \leq 0$, aggregate industry profits are decreasing in n,

(ii) if $\mu_n > 0$, aggregate industry profits are decreasing (resp. increasing) in n if $E_C(q_n) \overset{(\text{resp. }>)}{<} \mu_n^{-1}$,

(iii) if $C''(q) < 0$ for every $Q \in [0,Q_n]$, then monopoly maximizes aggregate industry profits, $\Pi(1,\lambda) > n\Pi(n,\lambda)$.

Proof of Proposition 7 (i-ii) Given what we see in the proof of Proposition 8, for aggregate industry profits we have that

$$\begin{aligned}
\frac{\partial\left[n\Pi(n,\lambda)\right]}{\partial n} &= P(Q_n)\frac{Q_n}{n} - C(q_n) + nP'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}(1 - H_n) + H_n\right] \\
&\propto -\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}(1 - H_n) - \eta(Q_n)\frac{P(Q_n) - C'(q_n) + C'(q_n)\frac{E_C(q_n)-1}{E_C(q_n)}}{P(Q_n)} + H_n\right] \\
&\overset{(1)}{=} -\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}(1 - H_n) - \eta(Q_n)\left(1 - \frac{H_n}{\eta(Q_n)}\right)\frac{E_C(q_n) - 1}{E_C(q_n)}\right] \\
&\propto E_C(q_n)\left(1 - \frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\frac{1 - H_n}{\eta(Q_n) - H_n}\right) - 1,
\end{aligned}$$

where $H_n < \eta(Q_n)$ from the pricing formula (1).

(iii) We have that

$$n\Pi(n,\lambda) \equiv P(Q_n)Q_n - nC(q_n) \overset{C''<0}{<} P(Q_n)Q_n - C(Q_n) \leq P(q_1)q_1 - C(q_1) = \Pi(1,\lambda),$$

where the last inequality follows by the definition of q_1 being the monopolist's optimal quantity.

To see why Remark B.2 holds notice that for $\lambda = 1$

$$\begin{aligned}\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n} &= P(Q_n)\frac{Q_n}{n} - C(q_n) + nP'(Q_n)\left(\frac{Q_n}{n}\right)^2 \\ &\overset{C''>0}{>} P(Q_n)\frac{Q_n}{n} - C'(q_n)q_n + P'(Q_n)\frac{Q_n^2}{n} \propto \frac{P(Q_n)-C'(q_n)}{P(Q_n)} - \frac{1}{\eta(Q_n)} = 0,\end{aligned}$$

where the equality follows from the pricing formula (1). **Q.E.D.**

Remark B.1. If $\Delta < 0$, then $\partial Q_n/\partial n > 0$, so $\mu_n \leq 1$, and thus, aggregate industry profits are decreasing in n if $E_C(q_n) < 1$. If for example $C'' < 0$ globally (consistent with $\Delta < 0$), then indeed $E_C(q_n) < 1$.

Remark B.2. If $\lambda = 1$ and $C''(q) > 0$ for every $q \in [0,q_n]$, aggregate industry profits are increasing in n.

Consider the extreme case of $\lambda = 1$ and notice the following. Condition $\Delta > 0$ requires decreasing returns to scale, so that aggregate gross profits increase with n (*i.e.*, $n\Pi(n,1) > (n-1)\Pi(n-1,1)$ for any n) due to savings in variable costs as production is distributed across more firms, even though the total quantity increases (see Proposition 2), and thus price decreases with the number of firms. Intuitively, aggregate gross profits increasing in n for $\lambda = 1$ is tied to the uniqueness of the (symmetric) equilibrium in the pricing stage. Since firms jointly maximize aggregate profits, the latter should increase with n for firms to strictly prefer to spread production evenly. On the other hand, under constant returns to scale aggregate profits are constant in n; increasing the number of firms simply changes how the firms can jointly produce the fixed level of total output that maximizes joint profits.[59] Last, under increasing returns to scale it is an equilibrium for all production to be concentrated in a single firm.

[59]As argued already, in this case, the are infinitely many equilibria of the pricing stage, all with the same total quantity.

B.4 When $\Psi(n,\lambda)$ is not (globally) decreasing in n

Our assumption that $\Psi(n,\lambda)$ is (globally) decreasing in n may fail—yet without strongly affecting our results. For example, under constant MC, (gross) aggregate industry profits are independent of the number of firms when $\lambda = 1$, so $\Psi(n,1) = 0$ for every $n \geq 2$, and thus $n^*(1) = 1$. It can also be seen that under decreasing MC, $\Psi(n,1) = n\Pi(n,1) - (n-1)\Pi(n-1,1) < 0$ for every $n \geq 2$ (see Proposition 7 and Remark B.1), so $n^*(1) = 1$. Thus, under $c_2 < 0$ and $\lambda = 1$, only one firm entering is the unique free entry equilibrium, even though $\Psi(n,1)$ need not be globally decreasing in n. In Figure 3c, where there is increasing MC, $n^*(1) = 5$.

B.5 Concavity of total surplus in the number of firms

Lemma 1. TS_n is globally strictly concave in n if for every n

$$\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1-\lambda - H_n\left(\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right)(1-E_{P'}(Q_n)) + \frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1} n - 1\right)\right] > \frac{1-\lambda}{n}.$$

Under constant marginal costs and $E_{P'}(Q_n) < 2$ for every n, this is true if $E'_{P'}(Q) \equiv \partial E_{P'}(Q)/\partial Q$ is not too high; particularly, $E'_{P'} \leq 0$ is sufficient, and thus so is CESL demand.

Proof of Lemma 1 We have seen that the first derivative of equilibrium total surplus with respect to n is given by

$$\frac{d\,\mathrm{TS}_n}{dn} = \Pi(n,\lambda) - f - \Lambda_n Q_n P'(Q_n)\frac{\frac{\partial Q_n}{\partial n} - q_n}{n},$$

so if we denote $\Pi_n(n,\lambda) \equiv \partial \Pi(n,\lambda)/\partial n$, the second derivative is given by

$$\begin{aligned}
\frac{d^2\,\mathrm{TS}_n}{(dn)^2} =& \Pi_n(n,\lambda) - \lambda Q_n P'(Q_n)\frac{\frac{\partial Q_n}{\partial n} - q_n}{n} - \Lambda_n \frac{\partial Q_n}{\partial n} P'(Q_n)\frac{\frac{\partial Q_n}{\partial n} - q_n}{n} \\
& - \Lambda_n Q_n P''(Q_n)\frac{\partial Q_n}{\partial n}\frac{\frac{\partial Q_n}{\partial n} - q_n}{n} - \Lambda_n Q_n P'(Q_n)\frac{\left(\frac{\partial^2 Q_n}{(\partial n)^2} - \frac{dq_n}{\partial n}\right)n - \frac{\partial Q_n}{\partial n} + q_n}{n^2} \\
\propto & -\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1-\lambda - H_n\left(\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right)(1-E_{P'}(Q_n)) + \frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1} n - 1\right)\right] \\
& + \frac{1-\lambda}{n}.
\end{aligned}$$

Under constant marginal costs

$$\frac{\partial Q_n}{\partial n} = \frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}(Q_n)}\frac{Q_n}{n} \implies$$

$$\frac{\partial^2 Q_n}{(\partial n)^2} = (1-\lambda)\left(-\frac{1+\lambda-\lambda E_{P'}(Q_n)-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}}{(n+\Lambda-\Lambda E_{P'}(Q_n))^2}\frac{Q_n}{n}+\frac{\frac{\frac{\partial Q_n}{\partial n}n-Q_n}{n^2}}{n+\Lambda-\Lambda E_{P'}(Q_n)}\right),$$

$$\frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1} n = -n\frac{1+\lambda-\lambda E_{P'}(Q_n)-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}}{n+\Lambda-\Lambda E_{P'}(Q_n)}+\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1$$

so that

$$\frac{d^2\,\mathrm{TS}_n}{(dn)^2} \propto -\left\{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1-\lambda-H_n\left(\begin{array}{r}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right)(2-E_{P'}(Q_n))\\ -n\frac{1+\lambda-\lambda E_{P'}(Q_n)-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}}{n+\Lambda-\Lambda E_{P'}(Q_n)}-1\end{array}\right)\right]-\frac{1-\lambda}{n}\right\}$$

$$\propto -\left\{\begin{array}{r} n(1-\lambda)(n+\Lambda_n-\Lambda_n E_{P'}(Q_n))-(n+\Lambda_n-\Lambda_n E_{P'}(Q_n))^2 \\ -\Lambda_n\left(\begin{array}{r}-(n+\Lambda_n-(1-\lambda)-\Lambda_n E_{P'}(Q_n))(2-E_{P'}(Q_n))\\ -n\left(1+\lambda-\lambda E_{P'}(Q_n)-\Lambda_n E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right)-(n+\Lambda_n-\Lambda_n E_{P'}(Q_n))\end{array}\right)\end{array}\right\}.$$

The partial derivative of the expression in the brackets with respect to $E_{P'}(Q_n)$ is given by

$$\begin{aligned}&-\Lambda n(1-\lambda)+2\Lambda(n+\Lambda-\Lambda E_{P'}(Q_n))\\ &-\Lambda(\Lambda(2-E_{P'}(Q_n))+n+\Lambda-(1-\lambda)-\Lambda E_{P'}(Q_n)+\Lambda-(1-\lambda)+\Lambda)\\ \propto&\lambda n-(3\Lambda-2(1-\lambda))=-(2\Lambda-(1-\lambda))<0,\end{aligned}$$

so that, given $E_{P'}(Q_n)<2$, for $d^2\,\mathrm{TS}_n/(dn)^2$ to be negative it is sufficient that

$$\begin{aligned}n(1-\lambda)(n+\Lambda-2\Lambda)-(n+\Lambda-2\Lambda)^2&\\ -\Lambda\left(-n\left(1+\lambda-2\lambda-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right)-(n+\Lambda-2\Lambda)\right)\geq 0 &\iff\\ 1-\lambda-\frac{\Lambda}{n}E'_{P'}(Q_n)Q_n\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\geq -\frac{(\Lambda+1-\lambda)(n-\Lambda)}{\Lambda n}&,\end{aligned}$$

which is true for $E'_{P'}$ not too high. **Q.E.D.**

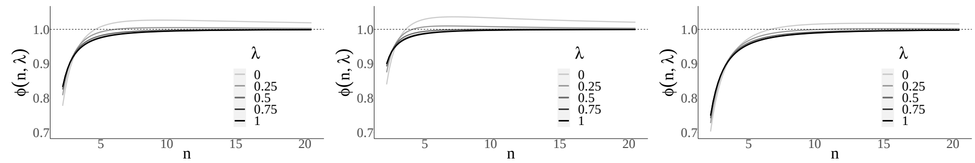
Remark B.3. More generally, all else constant, the condition of Lemma 1 is satisfied if the elasticity of the slope of Q_n with respect to n, $\frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1} n$, is not too high. Also, remember that $\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} \in (0,1)$ under the assumptions of Proposition 2(iii-a), so all else constant, in that case the condition is satisfied if $E_{P'}(Q)$ is not too high.

B.6 Numerical results showing that ϕ is close to 1

The numerical results of Figure 7 verify that $\phi(n,\lambda)$ is indeed close to 1, especially for $n \geq 3$.

Figure 7: $\phi(n,\lambda)$ under linear demand and linear-quadratic costs

(a) $b = 1$, $c_2 = 1$ **(b)** $b = 2$, $c_2 = 1$ **(c)** $b = 1$, $c_2 = 2$



Note: It can be checked that $\phi(n,\lambda)$ is invariant with respect to c_1 and the demand parameter a.

B.7 Derivation of Numerical Results

Under CESL demand and constant returns to scale, given Claim 1 we find that

$$\Pi(n,\lambda) = \begin{cases} \frac{1}{n}\left[a + b\left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1-E}{E-1}} - c\right]\left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ \frac{1}{n}\left[a - b\ln\left(e^{\frac{a-c-bH_n}{b}}\right) - c\right]e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \\ \frac{1}{n}\left[a - b\left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1-E}{1-E}} - c\right]\left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

$$= \begin{cases} \frac{H_n(E-1)b^{\frac{1}{E-1}}}{n}\left[\frac{1-H_n(E-1)}{c-a}\right]^{\frac{2-E}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ \frac{bH_n}{n}e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \\ \frac{H_n(1-E)}{nb^{\frac{1}{1-E}}}\left[\frac{a-c}{1+H_n(1-E)}\right]^{\frac{2-E}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

Derivation of Numerical Result 1 Parameters a, b and c only affect the magnitudes of $d\widehat{n}^*(\lambda)/d\lambda$ and $dQ_{\widehat{n}^*(\lambda)}/d\lambda$, and not their signs. The result then is obtained in a way analogous to the one described in the Derivation of Numerical Result 3. ■

Derivation of Numerical Result 2 Notice that, given a fixed n, $\phi(n,\lambda)$ is independent of a, c_1, and f. Also, $E_{P'}(Q_n) = 0$ and $\Delta = 1 - \lambda + c_2/b$ always (independently of a, c_1, and f). Thus, the expressions in Proposition 5 depend on a, c_1, and f only through their effect on $\widehat{n}^*(\lambda)$. Also, if we look at the expression for $\Pi(n,\lambda)$ in the proof of Corollary 1.1, it is easy to see that $\widehat{n}^*(\lambda)$ depends on a, c_1, and f only though $(a-c_1)^2/f$.[60] Further, $\Pi(2,0) \geq f$ if and only if $(a-c_1)^2/f \geq 2(3b+c_2)^2/(2b+c_2)$, so the values that b and c_2 can take that make the net monopoly profit non-negative also depend on a, c_1, and f only through $(a-c_1)^2/f$. Finally, $\Delta \geq 0$ is satisfied for every λ if and only if $c_2 \geq 0$, which does not depend on a, c_1, or f. Thus, without loss of generality, we can let $a = 1$, $c_1 = 0$ and only vary b, c_2, and f in the simulations. We then numerically check that for every $(b,c_2,f) \in \{(b,c_2,f) : \exists(t_1,t_2,t_3) \in \{0,1,\ldots,9\}^3$ such that $b = 0.01 + 1.11t_1, c_2 = 100t_2/9, f = 0.001 + [(2b+c_2)/(2(3b+c_2)^2) - 0.001]t_3/9\}$ (*i.e.*, for 1,000 parametrizations) there exists a threshold $\overline{\lambda}$ as claimed by solving for $\widehat{n}^*(\lambda)$ and $\widehat{n}^o(\lambda)$ for every $\lambda \in \{0,0.05,0.1,\ldots,0.95\}$. ■

Derivation of Numerical Result 3 It is easy to see that the signs of derivatives of $\Psi(n,\lambda)$ are independent of a, b and c. Thus, we can wlog set (i) $a = b = 1$ and $c = 2$ for the case $E \in (1,2)$, and (ii) $a = 2$, $b = c = 1$ for the case $E < 1$. We numerically find that

$$\min_{(n,\lambda,E)\in[2,7]\times[0,1]\times[1.001,1.7]} \frac{\Psi(n,\lambda)}{\partial\lambda\partial n} \approx 2.31 \cdot 10^{-6} > 0,$$

which is reached for $n = 7$, $\lambda = 0$ and $E = 1.001$. In the case of $E < 1$, we similarly find that

$$\min_{(n,\lambda,E)\in[2,8]\times[0,1]\times[-1000,0.999]} \frac{\Psi(n,\lambda)}{\partial\lambda\partial n} \approx 1.11 \cdot 10^{-7} > 0,$$

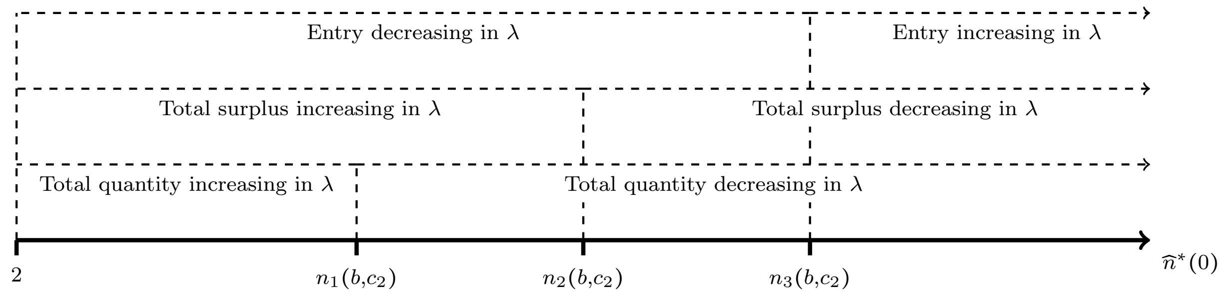
which is reached for $n = 8$, $\lambda = 0$ and $E = 0.999$. In additional simulations, allowing E to be even lower than -1000 does not change the result. ■

[60] Namely, $\widehat{n}^*(\lambda)$ is increasing in $(a-c_1)^2/f$.

B.8 Additional results on the linear-quadratic model

Claim 2 studies how entry, the total quantity and total surplus change with overlapping ownership around $\lambda = 0$. Figure 8 summarizes the results.

Figure 8: Comparative statics around $\lambda = 0$ under linear demand and linear-quadratic cost



Note: See Claim 2 for precise statement. $n_1(b,c_2) > 2$ only under significantly decreasing MC.

Claim 2. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Let demand be linear and cost be linear-quadratic with $a > c_1 \geq 0$, $-b \neq c_2 > -2bc_1/a$, and assume $\widehat{n}^*(0) \geq 2$. Then, there exist thresholds $n(b,c_2) \in \mathbb{R}^3$ (that depend on b and c_2) with $n_3(b,c_2) > n_2(b,c_2) > \max\{n_1(b,c_2), 2\}$ such that starting from $\lambda = 0$:

(i) entry is locally increasing (resp. decreasing) in λ if $\widehat{n}^*(0) \overset{\text{(resp. <)}}{>} n_3(b,c_2)$,

(ii) the total surplus is locally increasing (resp. decreasing) in λ if $\widehat{n}^*(0) \overset{\text{(resp. >)}}{<} n_2(b,c_2)$,

(iii) if $c_2 > -3b/2$, then $n_1(b,c_2) < 2$ and the total quantity is locally decreasing in λ,

(iv) if $c_2 < -3b/2$, then $n_1(b,c_2) > 2$ and the total quantity is locally increasing (resp. decreasing) in λ if $\widehat{n}^*(0) \overset{\text{(resp. >)}}{<} n_1(b,c_2)$,

Proof of Claim 2 The total derivative of $\widehat{n}^*(\lambda)$ at $\lambda = 0$ is

$$\frac{d\widehat{n}^*(\lambda)}{d\lambda} = (n-1)\frac{b\left[(n-1)(bn+c_2)^2 - (n+1+c_2/b)(b+c_2/2)(b(2n+1)+2c_2)\right]}{2(b+c_2/2)(bn+c_2)^2}\Bigg|_{n=\widehat{n}^*(0)},$$

where the denominator is positive and the numerator is a third-degree polynomial in n. In part (i), n_3 is the unique real root of the polynomial, which has a negative discriminant. In part (ii), the discriminant is positive, and the result follows with n_3 the highest of the three real roots of the polynomial equation above. Also,

$$\frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda} = \frac{\partial Q_n}{\partial \lambda} + \frac{\partial Q_n}{\partial n}\frac{d\widehat{n}^*(\lambda)}{d\lambda} = \frac{Q_{\widehat{n}^*(\lambda)}}{n+1+c_2/b}\left[(1+c_2/b)\frac{d\widehat{n}^*(\lambda)}{d\lambda}\frac{1}{n} - (n-1)\right]\Bigg|_{n=\widehat{n}^*(0)}$$

and for $n_1 \equiv (-2b^2 - 5bc_2 - c_2^2)/(2b^2) + \sqrt{(6b^3c_2 + 11b^2c_2^2 + 6bc_2^3 + c_2^4)/b^4}/2$ the corresponding results follow. For $\lambda = 0$, $\Psi(\widehat{n}^*(\lambda),\lambda) = \Pi(\widehat{n}^*(\lambda),\lambda) = f$, we get $d\,\mathrm{TS}_{\widehat{n}^*(\lambda)}/d\lambda \propto dQ_{\widehat{n}^*(\lambda)}/d\lambda - q_{\widehat{n}^*(\lambda)}d\widehat{n}^*(\lambda)/d\lambda$ and for $n_2 \equiv (2b - c_2 + \sqrt{8b^2 + 6bc_2 + c_2^2})/(2b)$ the corresponding result follows. It can be checked that $n_3 > n_2 > n_1$. **Q.E.D.**

Part (i) of the Corollary extends our finding that if without overlapping ownership many (resp. few) firms enter, then marginally increasing overlapping ownership will increase (resp. decrease) entry.

Part (ii) shows that marginally increasing λ above 0 increases total surplus if and only if entry is low. Particularly, the direct (negative) effect of an increase in λ on total surplus is dominated by the alleviation of excessive entry (since for $\lambda = 0$ entry is excessive) due to the increase in λ. We thus obtain another sufficient condition: if absent overlapping ownership, entry would be low, then a planner that regulates overlapping ownership (but not entry) should choose a positive level of it.

Parts (iii) and (iv) show that introducing a small amount of overlapping ownership may only increase the total quantity when MC is significantly decreasing (which means that the Cournot market is quasi-anticompetitive) and entry is low. In that case, the softening of pricing competition due to the increase in overlapping ownership is dominated by the concurrent decrease in entry—which tends to increase the total quantity since the market is quasi-anticompetitive. This yields a sufficient condition for consumer surplus to be maximized by some $\lambda > 0$. As shown in Figure 3d, this condition is not necessary, since with decreasing MC a positive level of overlapping ownership can be optimal under a consumer surplus standard even when overlapping ownership decreases the total quantity around $\lambda = 0$.

B.9 Free entry under pre-entry overlapping ownership and the presence of maverick firms

This section presents a model of free entry with pre-entry overlapping ownership under the presence of maverick firms.

For simplicity, model the maverick firms as a competitive fringe that in the first stage (where oligopolists enter) submit an aggregate supply schedule. Namely, there is a set $\mathcal{F}_m$ of infinitesimal firms. Firm $i \in \mathcal{F}_m$ chooses to either be inactive or produce one (infinitesimal) unit of the good at cost $\chi(i)$.[61] Thus, the aggregate supply function by the maverick firms in the third stage $S : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $S(p) \coloneqq \int_{i \in \mathcal{F}_m} I\left(\chi(i) \leq p\right) di$. $S : \mathbb{R}_+ \to \mathbb{R}_+$ with $S(p) = 0$ for every $p \in [0, \underline{p}]$ and $S'(p) > 0$ for every $p > \underline{p}$ where $\underline{p} \geq 0$. Then, the price $p > 0$ in the competitive equilibrium among the maverick firms will be implicitly given by $P^{-1}(p) = Q + S(p)$, where Q is the total quantity produced by the oligopolists.[62] This means that in the second stage, the oligopolists are essentially faced with inverse demand $\widetilde{P} : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\widetilde{P}(Q) = \begin{cases} P\left(Q + \omega^{-1}(Q)\right) \in \left(\underline{p}, P(Q)\right) & \text{if } P(Q) > \underline{p} \\ P(Q) & \text{if } P(Q) \leq \underline{p} \end{cases}$$

where $\omega : \mathbb{R}_{++} \to \mathbb{R}_+$ is given by $\omega(y) \coloneqq P^{-1} \circ S^{-1}(y) - y$.[63] $\omega^{-1}(Q)$ gives the quantity supplied in the competitive equilibrium among the maverick firms when the oligopolists produce Q. For example, in the case of (i) linear demand $P(Q) = \max\{a - bQ, 0\}$, (ii) linear maverick aggregate supply schedule $S(p) = \max\{(p - \underline{p})/b_m, 0\}$ with $b_m > 0$ and $\underline{p} \geq 0$, and (iii) constant MC (for the oligopolists), $C(q) = cq$, with $a > c \geq \underline{p}$,[64] for any $Q \in [0, (a-c)/b]$, $\widetilde{P}$ is

[61] This cost can be thought to include any applicable entry costs. Since maverick firms are infinitesimal and each supply an infinitesimal quantity, their entry cost is also infinitesimal.

[62] We assume that $S(p) > P^{-1}(p)$ for p large enough.

[63] To see this substitute $p = P\left(Q + \omega^{-1}(Q)\right)$ in $P^{-1}(p) = Q + S(p)$, which gives

$$Q + \omega^{-1}(Q) = Q + S \circ P\left(Q + \omega^{-1}(Q)\right) \iff P^{-1} \circ S^{-1} \circ \omega^{-1}(Q) - \omega^{-1}(Q) = Q,$$

which is true by definition of ω.

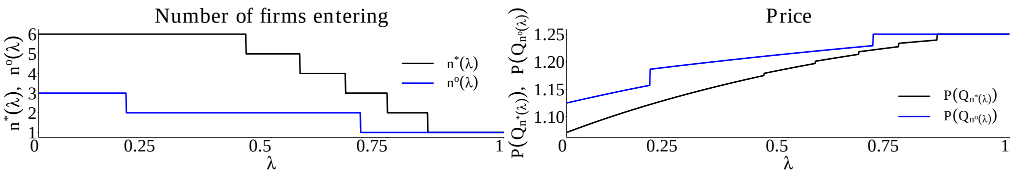
[64] For $c = \underline{p}$, the most efficient maverick firms is as efficient as the oligopolists.

given by[65]

$$\widetilde{P}(Q) = a - \overbrace{\frac{a-\underline{p}}{1+b_m/b}}^{<a} - \overbrace{\frac{b}{1+b/b_m}}^{<b} Q.$$

The (prospect of) entry by maverick firms essentially changes the demand faced by the commonly-owned firms by depressing it and making it more elastic. If in the paper wherever P we read $\widetilde{P}$, the results on the effects of overlapping ownership on entry and the price continue to hold (with the number of firms n not counting maverick firm entry). A comparison of Figure 9 with Figure 3a in the paper (the two figures use the same parametrization but in the former maverick firms are added) shows entry to be less sensitive to overlapping ownership due to the presence of the maverick firms, as argued in the paper.

Figure 9: Equilibrium with pre-entry overlapping ownership under the presence of maverick firms for varying λ



Note: Lines represent values in equilibrium; linear demand, constant MC: $a = 2$, $b = c = 1$, $f = 0.01$; linear maverick aggregate supply schedule: $b_m = \underline{p} = 1$.

Last, the total surplus $\widetilde{\mathrm{TS}}(\boldsymbol{q})$ now includes the maverick firms' surplus, where $\boldsymbol{q}$ still is the quantity profile of the oligopolists. Denote by $\widetilde{\mathrm{TS}}_n$ the pricing stage equilibrium total surplus when n commonly-owned firms enter. Equation (4) also applies in the case with maverick firms but with $\widetilde{\Xi}(n,\lambda) \coloneqq (n-1)\left(\widetilde{\Pi}(n-1,\lambda) - \widetilde{\Pi}(n,\lambda)\right)$, $\widetilde{\Pi}(n,\lambda) \coloneqq \widetilde{P}(Q_n)q_n - C(q_n)$ and $\widetilde{P}$ replacing Ξ, Π and P. Q_n, q_n are still the quantities produced by the commonly-owned firms in the pricing stage equilibrium where n of them enter. $\widehat{n}^*(\lambda)$ is now pinned down by $\widetilde{\Pi}(\widehat{n}^*(\lambda),\lambda) - \lambda\widetilde{\Xi}(\widehat{n}^*(\lambda),\lambda) = f$.

Provided $\widetilde{P}(Q) \geq \underline{p}$ or equivalently $P(Q) \geq \underline{p}$,[66] total surplus now includes

[65] The inverse demand $\widetilde{P}$ for higher Q does not play a role since the commonly-owned firms will never produce more than $(a-c)/b$. To derive $\widetilde{P}$, solve for it in $(a-\widetilde{P}(Q))/b = Q + (\widetilde{P}(Q)-\underline{p})/b_m$.

[66] Otherwise, wherever $\widetilde{P}(Q)$ substitute $\underline{p}$, and the equation reduces to $\widetilde{\mathrm{TS}}(\boldsymbol{q}) = \mathrm{TS}(\boldsymbol{q})$.

the maverick firms' surplus and is thus given by

$$\widetilde{\mathrm{TS}}(\boldsymbol{q}) \coloneqq \overbrace{\int_0^{Q+S(\widetilde{P}(Q))} \left(P(X) - \widetilde{P}(Q)\right) dX}^{\text{consumer surplus}} + \overbrace{\int_{\underline{p}}^{\widetilde{P}(Q)} S(p)dp}^{\text{maverick firms' surplus}} \overbrace{+\widetilde{P}(Q)Q - \sum_{i=1}^{n} C(q_i) - nf}^{\text{commonly-owned firms' profits}}$$
$$= \mathrm{TS}(\boldsymbol{q}) + \underbrace{\int_Q^{Q+S(\widetilde{P}(Q))} P(X)dX - S\left(\widetilde{P}(Q)\right)\widetilde{P}(Q)}_{\geq 0;\ \text{consumer surplus ``due to'' maverick firms' production}} + \underbrace{\int_{\underline{p}}^{\widetilde{P}(Q)} S(p)dp}_{\geq 0;\ \text{maverick firms' surplus}}$$

where $\boldsymbol{q}$ still the quantity profile of the oligopolists and $\mathrm{TS}(\boldsymbol{q}) \equiv \int_0^Q P(X)dX - \sum_{i=1}^n C(q_i) - nf$ the total surplus without maverick firms. For any fixed quantity profile of the oligopolists, total surplus is higher when the maverick firms are present (and produce) compared to when they are not. We have that

$$\begin{aligned}\frac{d\widetilde{\mathrm{TS}}_n}{dn} =& P(Q_n)\left(n\frac{\partial q_n}{\partial n} + q_n\right) - C(q_n) - nC'(q_n)\frac{\partial q_n}{\partial n} - f \\ &+ \left[\begin{array}{r}\left(1 + S'(\widetilde{P}(Q_n))\widetilde{P}'(Q_n)\right) P\left(Q_n + S(\widetilde{P}(Q_n))\right) - P(Q_n) \\ -S'(\widetilde{P}(Q_n))\widetilde{P}'(Q_n)\widetilde{P}(Q_n) - S(\widetilde{P}(Q_n))\widetilde{P}'(Q_n) + S(\widetilde{P}(Q_n))\widetilde{P}'(Q_n)\end{array}\right]\frac{\partial Q_n}{\partial n} \\ =& \widetilde{\Pi}(n,\lambda) - f - (1 + \lambda(n-1))\, Q_n\widetilde{P}'(Q_n)\frac{\partial q_n}{\partial n},\end{aligned}$$

where $\widetilde{\mathrm{TS}}_n$ is the pricing stage equilibrium total surplus when n commonly-owned firms enter, $\widetilde{\Pi}(n,\lambda) \coloneqq \widetilde{P}(Q_n)q_n - C(q_n)$, and Q_n, q_n are still the quantities produced by the commonly-owned firms in the pricing stage equilibrium where n of them enter.

Whether there is excessive or insufficient entry by commonly-owned firms will depend on the same forces identified in the previous section but with adjusted magnitude since P is replaced by $\widetilde{P}$. Notice that excessive or insufficient entry is based on a planner that controls the entry of oligopolists and allows them and the maverick firms to produce freely. Importantly, given the production decisions of the oligopolists, the maverick firms' production level maximizes total surplus since the maverick firms are perfect competitors.

B.10 Free entry under post-entry overlapping ownership

In the last section overlapping ownership develops before entry, thus directly affecting the incentives of firms to enter. In this section we study the case where potential entrants have no prior overlapping ownership, but after they enter the market and before they pick quantities in the second stage they develop overlapping ownership, so that they have an Edgeworth coefficient of effective sympathy $\lambda \in [0,1]$. Now, the only channel through which overlapping ownership affects entry is by increasing profits in the post-entry game. Firms expect this and therefore entry increases with overlapping ownership.

This can be interpreted as a long-run equilibrium whereby start-up firms (or already existing firms but without overlapping ownership) enter the industry and then develop overlapping ownership through time. Appendix B.1 describes explicitly how post-entry overlapping ownership can arise. Also, given that the extent to which overlapping ownership affects corporate conduct is an open empirical question, this section can also be interpreted as studying pre-entry overlapping ownership when it affects pricing but does *not* cause firms to internalize their entry externality.

The exogeneity of λ is important with post-entry overlapping ownership, since the incentives of firms to allow for ownership ties after entry are not modeled. For instance, if the number of shares that investors buy from the entrepreneurs depended on the extent of entry—since the latter affects profits, then λ would be a function of n. Although the exogeneity of λ is restrictive, if firms become publicly traded after entry (at least in the long-run), they indeed have limited control over their ownership ties, since for instance investment funds are free to buy shares of all firms.

B.10.1 An example of post-entry overlapping ownership

Post-entry overlapping ownership can for example arise in the form of common ownership as described below. There is a finite set $\mathcal{J}$ of investors. Let all firms be newly-established and the set of investors $\mathcal{J}$ be partitioned into $\{J_0\} \cup \cup_{i\in\mathcal{F}} \{J_i\}$ with $|J_i| = |J_0| = m$ for every $i \in \mathcal{F}$. Before entry each firm i is (exclusively) held by the set J_i of entrepreneurs with $\beta_{ji} = 1/m$ for every $j \in J_i$; there is no common ownership before entry, so when considering entry, the entrepreneurs of each firm unanimously agree to maximize their own firm's profit.[67] After entry, the set J_0 of investors, who previously held no shares of any firm, buy firm shares. Each investor $j \in J_0$ now holds share $\beta'_{ji} = \sigma/m$ of each firm i that has entered, and each entrepreneur $j \in J_i$ holds share $\beta'_{ji} = (1-\sigma)/m$ of her firm for some $\sigma \in [0,1]$. That is, after entry each entrepreneur sells the same amount of shares to the investors, who are now uniformly invested in all firms in the industry. Consider the O'Brien and Salop (2000) model, who assume that the manager of firm i maximizes a weighted average of the shareholders' portfolio profits (*i.e.*, she maximizes $\sum_{j\in\mathcal{J}} \gamma_{ji} u_j(\boldsymbol{q})$, where γ_{ji} captures the extent of j's control over firm i and $u_j(\boldsymbol{q}) \coloneqq \sum_{i\in\mathcal{F}} \beta_{ji}\pi_i(\boldsymbol{q})$ is j's total portfolio profit) and for every firm i that has entered let $\gamma'_{ji} = \widetilde{\gamma}/m$ be the control each investor $j \in J_0$ has over firm i for some $\widetilde{\gamma} \in [0,1]$, and $\gamma'_{ji} = (1-\widetilde{\gamma})/m$ the control each entrepreneur $j \in J_i$ has over her firm i.[68] After entry, it is easy to see that the manager of each firm i maximizes

$$\pi_i(\boldsymbol{q}) + \lambda \sum_{k\neq i} \pi_k(\boldsymbol{q}), \quad \text{where } \lambda = \frac{1}{1 + (\widetilde{\gamma}^{-1} - 1)(\sigma^{-1} - 1)} \in [0,1].$$

Here λ is increasing in the common owners' level of holdings σ and control $\widetilde{\gamma}$. Under proportional control $\sigma = \widetilde{\gamma}$, and $\lambda = \left[1 + (\sigma^{-1} - 1)^2\right]^{-1}$.

B.10.2 The entry stage

Each firm only looks at its own profit to decide whether to enter as there is no overlapping ownership when it does so.[69] $\boldsymbol{q}_n$ is a free entry equilibrium production profile if and only if

$$\Pi(n,\lambda) \geq f > \Pi(n+1,\lambda)$$

[67] This relies on the fact that a firm's entrepreneurs only hold shares of their firm both before and after entry. Common ownership develops after entry not through a firm's entrepreneurs investments in other firms but because outside investors invest in multiple firms.

[68] For every other pair of entrepreneur j and firm i, $\beta'_{ji} = \gamma'_{ji} = 0$.

[69] Formally, if a firm does not enter, its payoff is 0; if it does, it is $(1 + \lambda(n-1))(\Pi(n,\lambda) - f)$. Thus, it is optimal for an n-th firm to enter if and only if $\Pi(n,\lambda) \geq f$.

as in Mankiw and Whinston (1986). If overlapping ownership develops only after firms enter, it affects the incentives of firms to enter only through its effect on product market outcomes. We assume that there exists n such that $\Pi(n,\lambda) < f$ for any λ.

B.10.3 Existence and uniqueness of equilibrium

Proposition 8 studies the existence and uniqueness of a free entry equilibrium.

Proposition 8. $\Pi(n,\lambda)$ is decreasing in n and a unique free entry equilibrium exists.

Proof of Proposition 8 Given that $\Pi(n,\lambda)$ is decreasing in n by Proposition 2, the result follows given that $\Pi(n,\lambda) < f$ for n large. **Q.E.D.**

In equilibrium, firms enter until profits have fallen so much that if an additional firm enters, gross profit will no longer cover the entry cost. $\widehat{n}^*(\lambda)$ is uniquely pinned down by $\Pi(\widehat{n}^*(\lambda),\lambda) = f$ and $n^*(\lambda) = \max\{n \in \mathbb{N} : \Pi(n,\lambda) \geq f\} = \lfloor \widehat{n}^*(\lambda) \rfloor$.

B.10.4 Overlapping ownership effects

Proposition 9 studies the effects of overlapping ownership.

Proposition 9. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Then

(i) the number of firms entering is increasing in λ,

(ii) individual quantity, total quantity, and total surplus are decreasing in λ,

(iii) if $C'' \geq 0$, then the MHHI is increasing in λ.

Proof of Proposition 9 Given $\Pi(\widehat{n}^*(\lambda),\lambda) = f$, the Implicit Function Theorem gives

$$\frac{d\widehat{n}^*(\lambda)}{d\lambda} = \frac{(n-1)(H_n^{-1}-1)}{1 + H_n + \Lambda_n^{-1}\left[(1-\lambda)(1-H_n) - C''(q_n)/P'(Q_n)\right] - H_n E_{P'}(Q_n)} > 0,$$

where the inequality follows from what we have seen in section A.2.

(ii) The total derivative of the total quantity is then proportional to

$$\frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda} \propto \frac{\partial Q_n}{\partial \lambda}\frac{\Lambda_n(1+\lambda)+1-\lambda-C''(q_n)/P'(Q_n)-\Lambda_n^2 E_{P'}(Q_n)/n}{(n-1)(n-\Lambda_n)}+\frac{\partial Q_n}{\partial n}$$

$$= \frac{Q_n\left[\begin{array}{r}-\left(\Lambda_n(1+\lambda)+\Delta-\Lambda_n^2 E_{P'}(Q_n)/n\right)\\ +(n-\Lambda_n)\Delta/n\end{array}\right]}{(n-\Lambda_n)\left(n+\Lambda_n-C''(q_n)/P'(Q_n)-\Lambda_n E_{P'}(Q_n)\right)} = -\frac{\Lambda_n Q_n}{n(n-\Lambda_n)} < 0,$$

so total quantity decreases with λ, and thus so does individual quantity since the number of firms increases with λ. The total derivative of the total surplus is

$$\frac{d\,\mathrm{TS}_{\widehat{n}^*(\lambda)}}{d\lambda} = P(Q_n)\frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda} - \frac{d\widehat{n}^*(\lambda)}{d\lambda}C(q_n) - nC'(q_n)\left(\frac{dQ_{\widehat{n}^*(\lambda)}/d\lambda}{n} - \frac{q_n}{n}\frac{d\widehat{n}^*(\lambda)}{d\lambda}\right) - \frac{d\widehat{n}^*(\lambda)}{d\lambda}f$$

$$= \frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda}\left(P(Q_n)-C'(q_n)\right) - \left(P(Q_n)-C'(q_n)\right)q_n\frac{d\widehat{n}^*(\lambda)}{d\lambda} < 0.$$

where the second equality follows from $\Pi(\widehat{n}^*(\lambda),\lambda) = f$.

(iii) Last, the total derivative of $\mathrm{MHHI}^* = H_{n^*}$ is

$$\frac{d\,\mathrm{MHHI}\left(\boldsymbol{q}_{\widehat{n}^*(\lambda)}\right)}{d\lambda} = \frac{n-1}{n} + \left(\frac{\lambda}{n} - \frac{\Lambda_n}{n^2}\right)\frac{d\widehat{n}^*(\lambda)}{d\lambda} \propto \frac{n-1}{n}\left(\frac{d\widehat{n}^*(\lambda)}{d\lambda}\right)^{-1} + \frac{\lambda n - \Lambda_n}{n^2}$$

$$\propto \frac{\left[1+\lambda+\Delta(Q_n,(n-1)q_n)/n + \left(\frac{1}{n}-\frac{1}{\Lambda_n}\right)\frac{C''(q_n)}{P'(Q_n)}\right]/H_n - E_{P'}(Q_n)}{n(n-\Lambda_n)} > 0,$$

where the inequality is implied by $C'' \geq 0$ combined with the maintained assumption (ii) that requires $E_{P'}(Q_n) < \left(1+\lambda+\Delta(Q_n,(n-1)q_n)/n\right)/H_n$. **Q.E.D.**

Remark B.4. There exists a set of thresholds $\mathcal{L} \coloneqq \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, such that

(a) for every $\lambda \in \mathcal{L}$, $\Pi(n^*(\lambda),\lambda) = f$, and $n^*(\lambda) = \widehat{n}^*(\lambda)$,

(b) for λ between two consecutive thresholds $n^*(\lambda)$ remains constant and everything behaves as in the Cournot game with a fixed number of firms.

When we take into account the integer constraint, the number of firms is a step function of λ, and individual quantity decreases with jumps down. The total quantity has a decreasing trend with jumps up (resp. down) for the values of λ at which an extra firm enters under $\Delta > 0$ (resp. $\Delta < 0$). Also, total surplus tends to decrease with λ.[70]

Importantly, even when there is free entry of firms—so that increases in λ lead to the entry of new firms as incumbents suppress their quantities, if the entering firms develop overlapping ownership after entering (up to the level the incumbents have), consumer and total surplus tend to decrease with λ, as in the symmetric case with a fixed number of firms. Also, if one looks at HHI, it will seem as if competition rises as λ increases, which can even be the case with MHHI, although the latter will increase with λ if we slightly strengthen our assumptions. Last, for appropriate levels of λ a small increase in λ can spur the entry of an extra firm causing the total quantity to rise.

The fact that the price increases with λ is to be expected. Remember that an increase in λ is met with an increase in n so that the zero profit condition $\Pi(\widehat{n}^*(\lambda),\lambda) = f$ is satisfied. When the Cournot market is quasi-anticompetitive ($\Delta < 0$), both the increase in λ and the increase in n cause the price to increase. When the Cournot market is quasi-competitive ($\Delta > 0$), the increase in λ tends to increase the price, while the increase in n tends to decrease it. The former effect dominates. For example, assume non-increasing MC and by contradiction that after an increase in λ enough additional firms enter the market to keep the price at its level before the increase in λ (or even make it lower). Then, after the increase in λ (i) each firm has a lower share of the market, (ii) the price has not increased, and (iii) the average (variable) cost of production has not decreased (due to non-increasing MC and individual quantity has decreased). Thus, individual profit has decreased, violating the zero profit condition. The result still

[70] To compare total surplus under the integer constraint on n, $\mathrm{TS}_{n^*(\lambda)}$, to its value when we ignore the integer constraint, $\mathrm{TS}_{\widehat{n}^*(\lambda)}$, notice the following. For λ between two consecutive thresholds, $\lambda \in (\lambda_k,\lambda_{k+1})$, it holds that $\widehat{n}^*(\lambda) > n^*(\lambda)$. Thus, given that total surplus is single-peaked in n, if there is (weakly) excessive entry under the integer constraint, ignoring the integer constraint exacerbates excess entry. Therefore, between two λ thresholds $\mathrm{TS}_{\widehat{n}^*(\lambda)} < \mathrm{TS}_{n^*(\lambda)}$, and for λ equal to a thresholds $\mathrm{TS}_{n^*(\lambda)}$ has a jump down. But if under the integer constraint entry is insufficient by 1 firm (which is possible), $n^*(\lambda) = n^o(\lambda) - 1$, then the above does not follow.

holds under increasing MC, since under $\Delta > 0$,

$$\left|\frac{\overbrace{\partial\Pi(n,\lambda)/\partial\lambda}^{+}}{\underbrace{\partial\Pi(n,\lambda)/\partial n}_{-}}\right| = \left|\frac{\overbrace{(1-H_n)\frac{\partial Q_n}{\partial\lambda}}^{-}}{\underbrace{(1-H_n)\frac{\partial Q_n}{\partial n}}_{+}+\underbrace{H_n\frac{Q_n}{n}}_{+}}\right| < \left|\frac{\partial Q_n/\partial\lambda}{\partial Q_n/\partial n}\right| = \left|\frac{\overbrace{dP(Q_n)/d\lambda}^{+}}{\underbrace{dP(Q_n)/dn}_{-}}\right|.$$

This means that for individual profit to stay unchanged after an increase in λ, fewer firms need to enter compared to the number of firms that need to enter for the price to remain unchanged after the increase in λ.

The mechanism behind the effect of λ on entry is akin to the impact of collusion on entry in the dynamic stochastic oligopoly model of Fershtman and Pakes (2000), where firms freely enter, set prices, and invest in quality. In their model, for example, a potential entrant only looks at its profit to decide whether to enter foreseeing the possibility of future collusion with an incumbent monopolist. This possibility increases entry incentives (*i.e.* it increases the threshold of quality that the incumbent needs to achieve to deter entry) compared to the equilibrium without collusion. This in turn causes the incumbent monopolist to invest more in quality when future collusion is possible. Overall, the collusive equilibrium features on average higher prices but also more entry and higher qualities and consumer surplus.

B.10.5 Entry cost effect on entry

Proposition 10 studies the effect of the entry cost on entry, as well as how this effect depends on the extent of overlapping ownership. It mirrors Proposition 6 with the role of internalized profit $\Psi(n,\lambda)$ now assumed by profit $\Pi(n,\lambda)$.

Proposition 10. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Then

(i) entry is decreasing in the entry cost,

(ii) if λ increases and other parameters x (e.g., demand, cost) change infinitesimally so that $\widehat{n}^*(\lambda)$ stays fixed and $\partial^2\Pi(n,\lambda)/(\partial x\partial n) = 0$ (e.g.,

(f,λ) changes in direction $\mathbf{v} := (-(d\widehat{n}^*(\lambda)/d\lambda)/(d\widehat{n}^*(\lambda)/df),1))$, then $|d\widehat{n}^*(\lambda)/df|$ changes in direction given by $\text{sgn}\{\partial^2\Pi(n,\lambda)/(\partial\lambda\partial n)|_{n=\widehat{n}^*(\lambda)}\}$.

Proof of Proposition 10 We have that $d\widehat{n}^*(\lambda)/df = (\partial\Pi(n,\lambda)/\partial n)^{-1}\big|_{n=\widehat{n}^*(\lambda)}$, and part (ii) follows if we take the directional derivative of $d\widehat{n}^*(\lambda)/df$. **Q.E.D.**

As long as individual profit is decreasing in n, the results of Proposition 10 are not specific to Cournot competition. Part (ii) says that if an increase in λ makes individual profit in the pricing stage equilibrium more (resp. less) strongly decreasing in the number of firms, then an increase in the entry cost needs to be met with a smaller (resp. larger) increase in the number of firms for the zero profit entry condition to continue to hold.

Figure 10 explains the reasoning behind this result. There are initially $n^* = 3$ firms in equilibrium, which can be a result of $\lambda = 0$ and $f = f_1$, or $\lambda = 1/2$ and $f = f_2 > f_1$. Also, for $n \le 3$, an increase of λ from 0 to 1/2 makes profit less strongly decreasing in n (*i.e.*, $\partial^2\Pi(n,\lambda)/(\partial\lambda\partial n) > 0$). Thus, an increase in the entry cost by ε will decrease entry by more when $\lambda = 1/2$ (and initially $f = f_2$) compared to when $\lambda = 0$ (and initially $f = f_1$).

Claim 3 provides sufficient conditions for the cross derivative of $\Pi(n,\lambda)$ to be negative (resp. positive), which by Proposition 10 implies that overlapping ownership alleviates (resp. exacerbates) the negative effect of the entry cost on entry.

Claim 3. Assume constant MC.

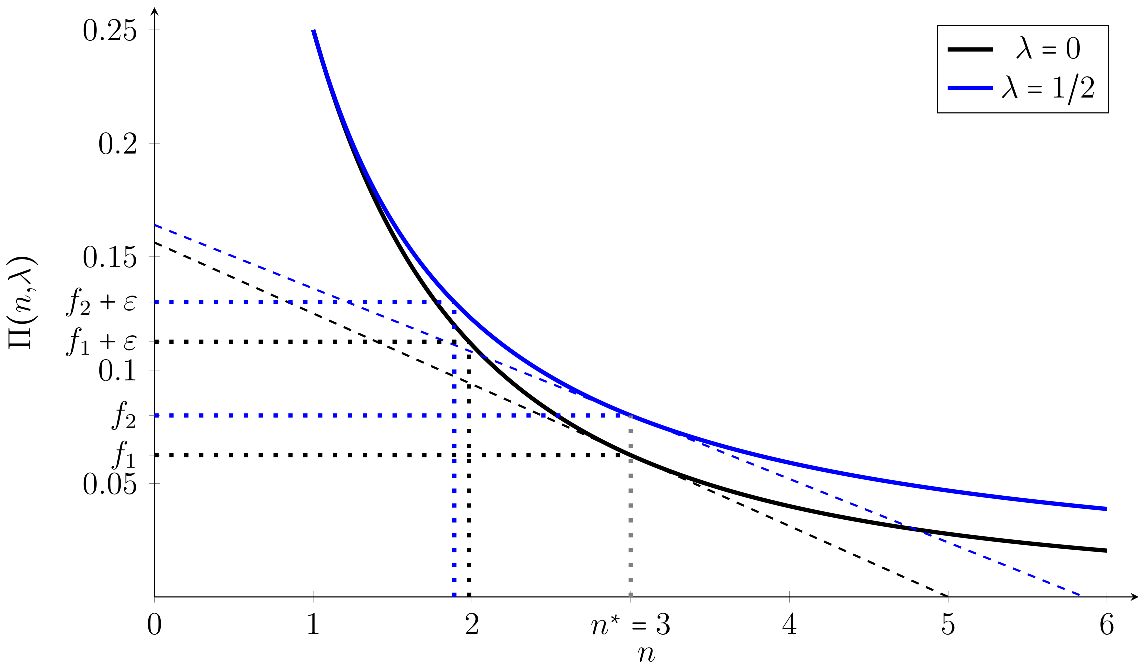
(i) If $\partial E_{P'}(Q)/\partial Q \ge 0$, $E_{P'}(Q_n) \in [0,1]$ and $n \ge 5 + E_{P'}(Q_n)$, then $\partial^2\Pi(n,\lambda)/(\partial\lambda\partial n) < 0$ for every $\lambda \in (0,1)$.

(ii) If $\partial E_{P'}(Q)/\partial Q \le 0$, $E_{P'}(Q_n) \le 0$ and $n \le 6/(2 - E_{P'}(Q_n))$, then $\partial^2\Pi(n,\lambda)/(\partial\lambda\partial n) > 0$ for every $\lambda \in (0,1)$.

Proof of Claim 3 We have $\partial\Pi(n,\lambda)/\partial n = P'(Q_n)q_n^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}(1-H_n) + H_n\right] < 0$, so

$$\frac{\partial^2\Pi(n,\lambda)}{\partial n\partial\lambda} \propto \left\{ \begin{array}{r} \left[-(1-E_{P'}(Q_n))\dfrac{\partial Q_n}{\partial n}\dfrac{n}{Q_n} - (2-E_{P'}(Q_n))\dfrac{\Lambda_n}{n-\Lambda_n}\right]\dfrac{\partial Q_n}{\partial\lambda}\dfrac{1}{Q_n} \\ -\dfrac{\partial^2 Q_n}{\partial n\partial\lambda}\dfrac{n}{Q_n} - \dfrac{n-1}{n-\Lambda_n}\left(1-\dfrac{\partial Q_n}{\partial n}\dfrac{n}{Q_n}\right) \end{array} \right\}.$$

Figure 10: Entry cost effect on entry mediated by λ under linear demand and constant MC



Note: $a = 2$, $b = 1$, $c = 1$. The black and blue solid lines represent $\Pi(n,0)$ and $\Pi(n,1/2)$, respectively. The black and blue dashed lines are tangent to the corresponding solid lines at $n = n^*$.

Denote $E'_{P'}(Q) \equiv \partial E_{P'}(Q)/\partial Q$. Under constant marginal costs

$$\frac{\partial^2 Q_n}{\partial n \partial \lambda} = \frac{\left\{ \begin{aligned} &\left(-Q_n + (1-\lambda)\frac{\partial Q_n}{\partial \lambda}\right)(n + \Lambda - \Lambda E_{P'}(Q_n)) \\ &-(1-\lambda)Q_n\left[n - 1 - (n-1)E_{P'}(Q_n) - \Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial \lambda}\right] \end{aligned} \right\}}{(n + \Lambda - \Lambda E_{P'}(Q_n))^2}\frac{1}{n}, \quad \text{so that}$$

$$\begin{aligned} \frac{\partial^2 \Pi(n,\lambda)}{\partial n \partial \lambda} &\propto \left\{ \begin{aligned} &2\Lambda^2 (E_{P'}(Q_n))^2 + \left[n\Lambda(n - \Lambda - 1) - 2n^2 - \Lambda^2\right] E_{P'}(Q_n) \\ &-n(n-\Lambda)(n + \Lambda - 6) - \frac{\Lambda(n-\Lambda)^2 Q_n E'_{P'}(Q_n)}{n + \Lambda - \Lambda E_{P'}(Q_n)} \end{aligned} \right\} \\ &< E_{P'}(Q_n)\left[2\Lambda^2 E_{P'}(Q_n) - n\Lambda - 2n^2 - \Lambda^2\right] \leq 0, \end{aligned}$$

where the first (resp. second) inequality follows from $n \geq 5 + E_{P'}(Q_n)$, $\lambda \in (0,1)$, $E_{P'}(Q_n) \leq 1$, $E'_{P'} \geq 0$ (resp. $0 \leq E_{P'}(Q_n) \leq 1$). Similarly follows part (ii). **Q.E.D.**

Claim 3 encompasses CESL demand. Therefore, under CESL demand with $E \in [0,1]$ and constant MC, in markets with not too low entry ($n \geq 6$ is sufficient), overlapping ownership makes entry less strongly decreasing in the entry cost. This means that as long as it does not induce firms to internalize the entry externality, overlapping ownership could alleviate the negative macroeconomic implications of rising entry costs documented by Gutiérrez, Jones and Philippon (2021) in the U.S. over the past 20 years. The sufficient condition of part (ii) requires $n \leq 3$, as is the case in Figure 10.

The conditions in part (i) of Claim 3 overlap with those of Numerical result 3, which deals with the case of pre-entry overlapping ownership. Thus, under the same parameterization, whether overlapping ownership exacerbates or alleviates the negative effect of the entry cost on entry will depend on the form of overlapping ownership. If overlapping ownership is present prior to entry thus making firms internalize the entry externality, then it exacerbates the effect. If it develops after entry, it alleviates the effect.

B.10.6 Equilibrium entry versus the socially optimal level of entry

The derivative of equilibrium total surplus with respect to n is given by

$$\begin{aligned}\frac{d\,\mathrm{TS}_n}{dn} &= P(Q_n)\left(n\frac{\partial q_n}{\partial n} + q_n\right) - C(q_n) - nC'(q_n)\frac{\partial q_n}{\partial n} - f \\ &= \Pi(n,\lambda) - f + n\left(P(Q_n) - C'(q_n)\right)\frac{\partial q_n}{\partial n},\end{aligned}$$

and therefore

$$\left.\frac{d\,\mathrm{TS}_n}{dn}\right|_{n=\widehat{n}^*(\lambda)} = \overbrace{\Pi\left(\widehat{n}^*(\lambda),\lambda\right) - f}^{=0} + n\left(P(Q_n) - C'(q_n)\right)\left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)} \propto \left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)},$$

so that with TS_n single-peaked in n, under business-stealing (resp. business-enhancing) competition entry is excessive (resp. insufficient). The results of Mankiw and Whinston (1986) and Amir, Castro and Koutsougeras (2014) generalize to the case of post-entry overlapping ownership. Proposition 11 shows that indeed with business-stealing competition and under the integer constraint, entry is never insufficient by more than one firm.

Proposition 11. The following statements hold:

(i) if $\Delta > 0$ and $E_{P'}(Q) < 2$ on L, then $n^*(\lambda) \geq n^o(\lambda) - 1$,

(ii) if $\Delta < 0$, then $n^*(\lambda) \geq n^o(\lambda) = 1$.

Proof of Proposition 11 Part (i): If $n^o(\lambda) \leq 2$, we are done since $n^*(\lambda) \geq 1$ given that monopoly profit is positive. For $n^o(\lambda) \geq 3$ keep in mind that $E_{P'}(Q) < 2$ on L implies that for every $n \in [2,+\infty)$, $E_{P'}(Q) < (1+\lambda)/H_n$ on L. The proof follows the proof of part (a) of Proposition 1 in Amir, Castro and Koutsougeras (ACK; 2014). By definition, $\text{TS}_{n^o(\lambda)} \geq \text{TS}_{n^o(\lambda)-1}$, which implies $\int_{Q_{n^o(\lambda)-1}}^{Q_{n^o(\lambda)}} P(X)dX - n^o(\lambda)C\left(q_{n^o(\lambda)}\right) + (n^o(\lambda)-1)\,C\left(q_{n^o(\lambda)-1}\right) \geq f$, which then gives $\Pi\left(n^o(\lambda)-1,\lambda\right) - f \geq P\left(Q_{n^o(\lambda)-1}\right)q_{n^o(\lambda)-1} - \int_{Q_{n^o(\lambda)-1}}^{Q_{n^o(\lambda)}} P(X)dX + n^o(\lambda)\left(C\left(q_{n^o(\lambda)}\right) - C\left(q_{n^o(\lambda)-1}\right)\right)$, which given $P' < 0$ and that in the Cournot game total quantity is increasing in n, implies

$$\begin{aligned}\Pi\left(n^o(\lambda)-1,\lambda\right) - f &> P\left(Q_{n^o(\lambda)-1}\right)\left(q_{n^o(\lambda)-1} + Q_{n^o(\lambda)-1} - Q_{n^o(\lambda)}\right)\\ &\quad + n^o(\lambda)\left(C\left(q_{n^o(\lambda)}\right) - C\left(q_{n^o(\lambda)-1}\right)\right) \implies\\ \Pi\left(n^o(\lambda)-1,\lambda\right) - f &> n^o(\lambda)\left(P\left(Q_{n^o(\lambda)-1}\right) - C'\left(\widetilde{q}\right)\right)\left(q_{n^o(\lambda)-1} - q_{n^o(\lambda)}\right),\end{aligned}$$

for some $\widetilde{q} \in \left[q_{n^o(\lambda)-1} - q_{n^o(\lambda)}\right]$, where the implication follows by the mean value theorem. As $R(Q_{-i})$ is non-decreasing in Q_{-i}, it follows as in the proof in ACK that there exists $\widetilde{Q}_{-i} \in \left[(n^o(\lambda)-2)q_{n^o(\lambda)-1}, (n^o(\lambda)-1)q_{n^o(\lambda)}\right]$ such that $\widetilde{q} \in r\left(\widetilde{Q}_{-i}\right)$ with $R\left(\widetilde{Q}_{-i}\right) \geq Q_{n^o(\lambda)-1}$ and $P\left(R(\widetilde{Q}_{-i})\right) \geq C'(\widetilde{q})$, so that $P\left(Q_{n^o(\lambda)-1}\right) \geq P\left(R(\widetilde{Q}_{-i})\right) \geq C'(\widetilde{q})$.

Given $E_{P'} < (1+\lambda)/H_n$, Proposition 2 implies that $q_{n^o(\lambda)-1} > q_{n^o(\lambda)}$, which combined with the above gives $\Pi\left(n^o(\lambda)-1,\lambda\right) - f \geq 0$. Also, by Proposition 2 $\Pi(n,\lambda)$ is decreasing in n, so it must be $n^*(\lambda) \geq n^o(\lambda) - 1$ for the entry condition to be satisfied.

Part (ii): Since $\Pi(1,\lambda) > f$, $n^*(\lambda) \geq 1$. Also, $\Delta < 0$ on L implies that $C''(q) < 0$ for every $q < \overline{Q}$. By Proposition 2 Q_n is decreasing in n, and thus, so is consumer surplus. Also, $n\Pi(n,\lambda) \equiv P(Q_n)Q_n - nC(q_n) < P(Q_n)Q_n - C(Q_n) \leq P(q_1)q_1 - C(q_1) = \Pi(1,\lambda)$, where the first inequality follows from $C'' < 0$. Thus, both consumer surplus and industry profits are maximized for $n = 1$, so $n^o(\lambda) = 1$. **Q.E.D.**

Remark B.5. Under a consumer surplus standard

(i) if $\Delta > 0$, then $n^o(\lambda) = \infty$ (since Q_n is increasing in n), so $n^*(\lambda) < n^o(\lambda)$,

(ii) if $\Delta < 0$, then $n^o(\lambda) = 1$ (since Q_n is decreasing in n), so $n^*(\lambda) \geq n^o(\lambda)$.

Under a consumer surplus standard, entry is insufficient (resp. excessive) when returns to scale are at most mildly increasing (resp. sufficiently increasing).

B.11 Free entry with pre-entry overlapping ownership: a more tractable framework

In this section, we make the free entry model with pre-entry overlapping ownership more tractable by ignoring the integer constraint on n. The way we do this is *not* just by letting (2) hold with equality. Instead, now each "infinitesimal" firm considers whether to enter or not examining a differential version of (3).[71] Consider firm i of "size" $\varepsilon > 0$ and let $n \in \mathbb{R}_+$ be the number of other firms entering. Firm i's payoff if it enters is $(\varepsilon + \lambda n)(\Pi(n+\varepsilon, \lambda) - f)$, while if it does not, it is $\lambda n (\Pi(n,\lambda) - f)$. The difference is

$$\varepsilon \Pi(n+\varepsilon,\lambda) + \lambda n [\Pi(n+\varepsilon,\lambda) - \Pi(n,\lambda)] - \varepsilon f.$$

Notice that for $\varepsilon = 1$ we recover the case with an integer number of firms. Dividing this expression by ε and letting $\varepsilon \to 0$ gives

$$\Pi(n,\lambda) + \lambda n \frac{\partial \Pi(n,\lambda)}{\partial n} - f.$$

Therefore, $\boldsymbol{q}_n$ is a free entry equilibrium if

$$\underbrace{\Pi(n,\lambda)}_{\substack{\text{own profit}\\ \text{from entry}}} + \underbrace{\lambda n \frac{\partial \Pi(n,\lambda)}{\partial n}}_{\substack{\text{entry externality}\\ \text{on other firms}}} = \underbrace{f}_{\substack{\text{entry}\\ \text{cost}}} \quad \text{and} \tag{6}$$

[71] Of course, the firm is infinitesimal only for the purpose of the algebra. The firm understands the (marginal) effect of its entry on market outcomes, and in the pricing stage firms still compete à la Cournot but with the symmetric equilibrium solution extended to $n \in \mathbb{R}_{++}$.

$$(7) \qquad (1+\lambda)\frac{\partial\Pi(n,\lambda)}{\partial n} + \lambda n \frac{\partial^2\Pi(n,\lambda)}{(\partial n)^2} < 0.$$

Naturally, we only consider the free entry equilibrium and planner's solution with $n \in \mathbb{R}_+$; we denote the number of firms in the two solutions by $n^*(\lambda)$ and $n^o(\lambda)$, respectively. The entry externality is now measured by $n\partial\Pi(n,\lambda)/\partial n$. (6) says that the marginal firm entering is exactly indifferent between entering or not. (7) guarantees that an extra infinitesimal firm does not want to enter, and given that $\partial\Pi(n,\lambda)/\partial n < 0$, can equivalently be written as

$$1 + \lambda - \lambda E_{\partial\Pi/\partial n,n}(n,\lambda) > 0, \qquad \text{where} \quad E_{\partial\Pi/\partial n,n}(n,\lambda) := -\frac{\frac{\partial^2\Pi(n,\lambda)}{(\partial n)^2}}{\frac{\partial\Pi(n,\lambda)}{\partial n}} n$$

is the elasticity of the slope of individual profit with respect to n. Also, given that $\partial\Pi(n,\lambda)/\partial n < 0$, $\lambda > 0$ implies through (6) that the entering firms make positive net profits in equilibrium. For $\lambda = 0$, (6) reduces to the standard zero profit condition.

Provided that (7) holds for every n, the (unique) equilibrium level of entry $n^*(\lambda)$ is pinned down by

$$\Pi\left(n^*(\lambda),\lambda\right) + \lambda n^*(\lambda)\left.\frac{\partial\Pi(n,\lambda)}{\partial n}\right|_{n=n^*(\lambda)} = f.$$

More tractable results, where discrete differences are replaced by differentials, analogous to those derived in the paper can be derived in this model. For example, Proposition 5 holds with ϕ exactly equal to 1.

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