Proof of Proposition 1. Recall that $\mu^i$ is the current matching in Round $i$ for the KTTC algorithm. We first verify that the output of the KTTC mechanism does not violate any capacity constraint; that is, for all $\ell \in L$, $\ell$ can accommodate $\mu_{\text{KTTC}}^{i}(\ell)$. This holds trivially for the null. For any non-null locality $\ell \neq \emptyset$, in any Round $i \geq 1$, $\ell$ permanently rejects any family that $\ell$ cannot accommodate alongside $\mu^i(\ell)$. Therefore, if $\ell$ can accommodate $\mu^i(\ell)$, then $\ell$ can accommodate $\mu^{i+1}(\ell)$. Since $\ell$ can accommodate $\mu^i(\ell) = \emptyset$, it follows by induction that $\ell$ can accommodate the families currently matched to $\ell$ in every Round $i \geq 1$. As $\mu_{\text{KTTC}}^i$ is the current matching in the last round, $\ell$ can accommodate $\mu_{\text{KTTC}}^i(\ell)$. We next show that the KTTC mechanism is Pareto-efficient (PE) and strategy-proof (SP).

Proof of (PE). For each Round $i = 1, \ldots, N$, let $F^i$ be the set of families that have been permanently matched by the end of Round $i - 1$. (Note that $F^1 = \emptyset$ and $F^{N+1} = F$.) If the algorithm ends in Round $N$ by matching all remaining families to the null, we use the convention that $\mu^{N+1} = \mu^N$.

The proof proceeds by induction, with the following hypothesis: there does not exist any matching $\mu$ such that $\mu(f) \succeq_f \mu^i(f)$ for all $f \in F^i$ and $\mu(f) \succ_f \mu^i(f)$ for some $f \in F^i$. Our induction hypothesis trivially holds for $i = 1$ since $F^1 = \emptyset$. We now show that if our induction hypothesis holds for some $i = 1, \ldots, N$, then it also holds for $i + 1$.

Towards a contradiction, suppose there exists a matching $\mu$ such that $\mu(f) \succeq_f \mu^{i+1}(f)$ for all $f \in F^{i+1}$ and $\mu(f) \succ_f \mu^{i+1}(f)$ for some $f \in F^{i+1}$. Note that, for all $f \in F^i$, $\mu^i(f) = \mu^{i+1}(f)$. If, for some $f \in F^i$, $\mu(f) \succ_f \mu^{i+1}(f)$, the induction hypothesis implies that there exists a family $f' \in F^i$ such that $\mu^{i+1}(f') \succ_f \mu(f')$, a contradiction. Therefore, $\mu^{i+1}(f) = \mu(f)$ for all $f \in F^i$. Note that, for all $f \in F^{i+1} \setminus F^i$, $f$ is permanently matched to $\mu^{i+1}(f)$ in Round $i$; therefore $\mu^{i+1}(f)$ is the locality $f$ prefers among those that have not permanently rejected $f$. If, for some $f \in F^{i+1} \setminus F^i$, $\mu(f) \succ_f \mu^{i+1}(f)$, then $\mu(f)$ has permanently rejected $f$ so it must be that $\mu(f)$ cannot accommodate $f$ alongside $\mu^i(\mu(f))$. Therefore, there exists a family $f' \in F^i$ such that $\mu^i(f') \neq \mu(f')$, a contradiction of the fact that for all $f \in F^i$, $\mu^i(f) = \mu^{i+1}(f)$.
Figure A.1. Counterexample for the proof of Theorem 1. $f \rightarrow \ell$: $\ell$ is $f$'s first choice. $f \rightarrow \ell$: $\ell$ is $f$'s second choice and $f$ prefers $\ell$ to its endowment. $\ell \rightarrow f$: $\ell$ is $f$'s endowment. Superscripts denote sizes and capacities, respectively.

By induction, there does not exist any matching $\mu$ such that $\mu(f) \succeq_f \mu^{N+1}(f)$ for all $f \in F^{N+1}$ and $\mu(f) \succ_f \mu^{N+1}(f)$ for some $f \in F^{N+1}$. As $\mu^{N+1} = \mu^{\text{KTTCE}}$ and $F^{N+1} = F$, this implies that $\mu^{\text{KTTCE}}$ is Pareto-efficient.

Proof of (SP). The result is implied by the fact that the KTTCE mechanism is strategy-proof (Theorem 3) and the fact that $\mu^{\text{KTTCE}} = \mu^{\text{KTTCE}}$ when every family is endowed with $\emptyset$ (Proposition 2).

Proof of Theorem 1. The proof is by counterexample. There are four families, four localities, and one dimension. The endowment is

$$\mu^E = \left( \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_2 & \ell_3 & \ell_4 \end{array} \right).$$

The preferences of families are as follows:

$$f_1 \succ f_2; \ell_3, \ell_1, \ell_2, \ldots \quad f_2; \ell_4, \ell_1, \ell_2, \ldots \quad f_3; \ell_1, \ell_3, \ldots \quad f_4; \ell_2, \ell_4, \ldots,$$

where a family’s endowment locality is denoted in boldface. The family sizes and locality capacities are

$$\nu = d_1 \left( \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \\ 1 & 1 & 2 & 2 \end{array} \right), \quad \kappa = d_1 \left( \begin{array}{cccc} \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ 1 & 2 & 2 & 2 \end{array} \right).$$

For ease of exposition, we illustrate this counterexample in Figure A.1.

We show that there are two individually rational (IR) and chain-efficient (CE) matchings:

$$\mu = \left( \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \\ \ell_1 & \ell_4 & \ell_3 & \ell_2 \end{array} \right) \quad \text{and} \quad \mu' = \left( \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \\ \ell_3 & \ell_1 & \ell_2 & \ell_4 \end{array} \right).$$

It is easy to check that matchings $\mu$ and $\mu'$ are IR and Pareto-efficient; hence they are IR and CE. It remains to show that no other IR matching is CE.

First, consider the case where $f_1$ is matched to $\ell_3$. Then, $f_3$ must be matched to $\ell_2$ and $f_4$ must be matched to $\ell_4$. Therefore, $\ell_1$ must be matched to $f_2$, yielding $\mu'$; hence $\mu'$ is the
only IR (and CE) matching where \( f_1 \) is matched to \( \ell_3 \). Analogous reasoning allows us to conclude that \( \mu \) is the only IR (and CE) matching where \( f_2 \) is matched to \( \ell_4 \).

Second, consider the case where \( f_1 \) is matched to its endowment locality \( \ell_2 \). Then, \( f_3 \) and \( f_4 \) must also be matched to their respective endowments, \( \ell_3 \) and \( \ell_4 \). Hence, \( f_2 \) must be matched to either \( \ell_2 \) or \( \ell_1 \). Matching \( f_2 \) to \( \ell_2 \) yields \( \mu^E \). Since \( f_2 \) prefers \( \ell_1 \) to \( \ell_2 \) and \( \ell_1 \) is not matched to any family, \( \mu^E \) is wasteful, hence not CE. Matching \( f_2 \) to \( \ell_1 \) yields a matching that is not CE as it contains the Pareto-improving chain \( f_1 \to \ell_3 \to f_3 \to \ell_2 \). Therefore, there is no IR and CE matching where \( f_1 \) is matched to \( \ell_2 \). Analogous reasoning allows us to reach the same conclusion for \( f_2 \).

Third, consider the case where \( f_1 \) is matched to \( \ell_1 \). Then, \( f_2 \) must be matched to either \( \ell_4 \) or \( \ell_2 \), which by our previous argument yields either \( \mu \) or \( \mu^E \). We therefore conclude that \( \mu \) and \( \mu' \) are the only two IR and CE matchings.

Suppose now that \( f_1 \) misreports its preferences by ranking \( \ell_1 \) below \( \ell_2 \), i.e., reporting \( \succ_{f_1}' : \ell_3, \ell_2, \ldots \). Using analogous reasoning as for the true preference profile, it is easy to check that \( \mu' \) is the only IR and CE matching for the manipulated preference profile \( \langle \succ_{f_1}', \succ_{-f_1} \rangle \). Similarly, \( \mu \) is the only IR and CE matching for the manipulated preference profile \( \langle \succ_{f_2}', \succ_{-f_2} \rangle \) where \( f_2 \) misreports its preferences by ranking \( \ell_1 \) below \( \ell_2 \), i.e., \( \succ_{f_2}' : \ell_4, \ell_2, \ldots \).

We can now show that no IR and CE mechanism is strategy-proof. Let \( \varphi \) be an IR and CE mechanism. If all families report their preferences truthfully, then either \( \varphi(\succ) = \mu \) or \( \varphi(\succ) = \mu' \) because \( \mu \) and \( \mu' \) are the only two IR and CE matchings. If \( f_1 \) reports \( \succ_{f_1}' \), then \( \varphi(\succ_{f_1}', \succ_{-f_1}) = \mu' \). Similarly, if \( f_2 \) reports \( \succ_{f_2}' \), then \( \varphi(\succ_{f_2}', \succ_{-f_2}) = \mu \). If \( \varphi(\succ) = \mu \), then \( \varphi(\succ_{f_1}', \succ_{-f_1})(f_1) \succ_{f_1} \varphi(\succ)(f_1) \) but if \( \varphi(\succ) = \mu' \), then \( \varphi(\succ_{f_2}', \succ_{-f_2})(f_2) \succ_{f_2} \varphi(\succ)(f_2) \). Therefore \( \varphi \) is not strategy-proof.

**Proof of Theorem 2.** The proof is by counterexample. There are four families, three localities, and two dimensions. The endowment is

\[
\mu^E = \begin{pmatrix}
    f_1 & f_2 & f_3 & f_4 \\
    \ell_2 & \ell_2 & \ell_3 & \ell_4
\end{pmatrix}.
\]

The preferences of families are as follows:

\[
\succ_{f_1} : \ell_3, \ell_2, \ldots \succ_{f_2} : \ell_4, \ell_2, \ldots \succ_{f_3} : \ell_2, \ell_4, \ell_3, \ldots \succ_{f_4} : \ell_2, \ell_3, \ell_4, \ldots ,
\]

where a family’s endowment locality is denoted in boldface. The family sizes and locality capacities are

\[
\nu = d_1 \begin{pmatrix}
    f_1 & f_2 & f_3 & f_4 \\
    2 & 1 & 2 & 3
\end{pmatrix}, \quad \kappa = d_1 \begin{pmatrix}
    \ell_2 & \ell_3 & \ell_4 \\
    4 & 3 & 3
\end{pmatrix}.
\]

For ease of exposition, we illustrate this counterexample in Figure A.2.
Observe that $\mu^E$ is non-wasteful but not chain-efficient, as it has three Pareto-improving chains:

$$(f_2, \ell_4, f_4, \ell_3, f_3, \ell_2), \quad (f_1, \ell_3, f_3, \ell_4, f_4, \ell_2), \quad \text{and} \quad (f_3, \ell_4, f_4, \ell_3).$$

Executing these chains yields the following three matchings, respectively:

$$\mu = \left(\begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_4 & \ell_2 & \ell_3 \end{array}\right), \quad \mu' = \left(\begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \\ \ell_3 & \ell_2 & \ell_4 & \ell_2 \end{array}\right), \quad \text{and} \quad \mu'' = \left(\begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_2 & \ell_4 & \ell_3 \end{array}\right).$$

We first show that $\mu$, $\mu'$, and $\mu''$ are the only matchings that Pareto dominate the endowment $\mu^E$.

Suppose towards a contradiction that there exists a matching $\tilde{\mu}$ that Pareto dominates $\mu^E$. There are three cases:

First, consider the case where $\tilde{\mu}(f_3) = \ell_3$. Then, neither $f_1$ nor $f_4$ can be matched to $\ell_3$; hence $\tilde{\mu}(f_1) = \ell_2$ and $\tilde{\mu}(f_4) = \ell_4$. Therefore, we must have that $\tilde{\mu}(f_2) = \ell_2$. It follows that $\tilde{\mu} = \mu^E$, a contradiction.

Second, consider the case where $\tilde{\mu}(f_3) = \ell_2$. Then, $f_2$ cannot be matched to $\ell_2$ so $\tilde{\mu}(f_2) = \ell_4$. As a result, $f_4$ cannot be matched to either $\ell_2$ or $\ell_4$; hence, $\tilde{\mu}(f_4) = \ell_3$. It follows that $f_1$ cannot be matched to $\ell_3$; therefore $\tilde{\mu}(f_3) = \ell_2$ and $\tilde{\mu} = \mu$, a contradiction.

Third, consider the case where $\tilde{\mu}(f_3) = \ell_4$. Neither $f_2$ nor $f_4$ can be matched to $\ell_4$; therefore, $\tilde{\mu}(f_2) = \ell_2$ and we must have that either $\tilde{\mu}(f_4) = \ell_2$ or $\tilde{\mu}(f_4) = \ell_3$. If $\tilde{\mu}(f_4) = \ell_2$, then $\tilde{\mu}(f_1) = \ell_3$ and $\tilde{\mu} = \mu'$. If $\tilde{\mu}(f_4) = \ell_3$, then $\tilde{\mu}(f_1) = \ell_2$ and $\tilde{\mu} = \mu''$, a contradiction.

We have established that $\mu$, $\mu'$, and $\mu''$ are the only matchings that Pareto dominate $\mu^E$. Suppose that $f_3$ misreports its preferences by ranking $\ell_4$ below $\ell_3$, i.e., by reporting $\succ f_3$; $\ell_2, \ell_3, \ldots$; then $\mu$ is the unique matching that Pareto dominates $\mu^E$ for the manipulated preference profile $(\succ f_3, \succ f_3)$. Similarly, suppose that $f_4$ misreports its preferences by ranking $\ell_3$ below $\ell_4$, i.e., by reporting $\succ f_4$; $\ell_2, \ell_4, \ldots$, then $\mu'$ is the unique matching that Pareto dominates $\mu^E$ for the manipulated preference profile $(\succ f_4, \succ f_4)$.
We can now show that there is no strategy-proof mechanism that Pareto improves upon \( \mu^E \). Let \( \varphi \) be a mechanism that Pareto improves upon \( \mu^E \). If all families report their preferences truthfully, then \( \varphi(\succ) \in \{ \mu, \mu', \mu'' \} \) because \( \mu, \mu' \), and \( \mu'' \) are the only three matchings that Pareto dominate \( \mu^E \) for the true preference profile. If \( f_3 \) reports \( \succ'_{f_3} \), then \( \varphi(\succ'_{f_3}, \succ_{-f_3}) = \mu \). If \( f_4 \) reports \( \succ'_{f_4} \), then \( \varphi(\succ'_{f_4}, \succ_{-f_4}) = \mu' \). If \( \varphi(\succ) \in \{ \mu, \mu'' \} \), then \( \varphi(\succ'_{f_3}, \succ_{-f_3})(f_3) \succ_{f_3} \varphi(\succ)(f_3) \). Therefore, \( \varphi \) is not strategy-proof.

**Proof of Theorem 3.** We first verify that the output of the KTTCE mechanism does not violate any capacity constraint; that is, for all \( \ell \in L \), \( \ell \) can accommodate \( \mu^{\text{KTTCE}}(\ell) \). This holds trivially for the null. For a non-null locality \( \ell \neq \emptyset \), in any Round \( i \), if there exists a family \( f \in \mu^{i+1}(\ell) \setminus \mu^i(\ell) \), then \( f \) moves to \( \ell \) as part of a feasible cycle, and therefore \( \ell \) can accommodate \( f \) alongside the families that remain at \( \ell \) after the cycle is carried out. If follows that if \( \ell \) can accommodate \( \mu^i(\ell) \), then \( \ell \) can accommodate \( \mu^{i+1}(\ell) \). By assumption, the endowment does not violate any capacity constraint; hence, by induction, \( \ell \) can accommodate the families currently matched to \( \ell \) in every Round \( i \geq 1 \). As \( \mu^{\text{KTTCE}} \) is obtained by carrying out feasible cycles from the current matching in the last round of the KTTCE algorithm, \( \ell \) can accommodate \( \mu^{\text{KTTCE}}(\ell) \).

We next show that the KTTCE mechanism is individually rational (IR) and strategy-proof (SP).

**Proof of (IR).** Consider a family \( f \), its endowment \( \ell \), and let \( i \) be the round of the KTTCE algorithm in which \( f \) is permanently matched. We need to show that \( \mu^{\text{KTTCE}}(f) \succeq_f \ell \). By construction, any family that is not permanently matched is currently matched to its endowment; thus \( f \) is currently matched to \( \ell \) at the start of Round \( i \), i.e., \( f \in \mu^i(\ell) \). Recall that in any round, locality \( \ell \) can accommodate families currently matched to \( \ell \), in particular \( \ell \) can accommodate \( \mu^i(\ell) \). As a result, \( \ell \) has not permanently rejected \( f \). By definition, \( f \) points at its most preferred locality that has not permanently rejected \( f \) yet. Therefore, \( f \) points at a locality that \( f \) weakly prefers to \( \ell \). As \( f \) is permanently matched in Round \( i \), \( f \) points at \( \mu^{\text{KTTCE}}(f) \). Therefore, \( \mu^{\text{KTTCE}}(f) \succeq_f \ell \), which is what IR requires.

**Proof of (SP).** Consider a family \( f \) with true preferences \( \succ_f \), let \( \succ'_f \) be an alternative report, and fix the reports of all other families to \( \succ_{-f} \). Denote by \( \ell = \varphi^{\text{KTTCE}}(\succ_f, \succ_{-f})(f) \), respectively \( \ell' = \varphi^{\text{KTTCE}}(\succ'_f, \succ_{-f})(f) \), the locality with which \( f \) is matched if it reports \( \succ_f \), respectively \( \succ'_f \). We need to show that \( \ell \succeq_f \ell' \). Let \( m \) and \( m' \) be the rounds in which \( f \) gets permanently matched with reports \( \succ_f \) and \( \succ'_f \), respectively.

**Case 1:** \( m \leq m' \). Whether it reports \( \succ_f \) or \( \succ'_f \), \( f \) is not permanently matched at the start of Round \( m \). Let \( L^m \) be the set of localities that have not permanently rejected \( f \) at the start of Round \( m \) (this set is nonempty as we have already shown that a family is never
by construction, we have that \( f \)'s report does not impact whether or not a given locality permanently rejects \( f \); therefore \( \hat{L}_f^m \) is the same whether \( f \) reports \( \succ_f \) or \( \succ'_f \). In addition, permanent rejections are irreversible; therefore, with either report, \( f \) will be matched to a locality in \( \hat{L}_f^m \), i.e., \( \ell, \ell' \in \hat{L}_f^m \). If \( f \) reports truthfully, it points at its most preferred locality in \( \hat{L}_f^m \) and is permanently matched to it. Hence, \( \ell \) is \( f \)'s most preferred locality in \( \hat{L}_f^m \), which implies that \( \ell \succeq_f \ell' \).

Case 2: \( m > m' \). If \( f \) reports \( \succ'_f \), \( f \) points at and is permanently matched to \( \ell' \) in Round \( m' \). If \( \ell' = \emptyset \), then \( \ell \succeq_f \ell' \) since every family prefers every locality to the null. The remainder of the proof assumes that \( \ell' \neq \emptyset \). In that case, if \( f \) reports \( \succ'_f \), a feasible cycle \( f \to \ell' \to f_2 \to \ell_2 \to \ldots \to f_n \to \ell_n \to f \) appears in Round \( m' \) (with \( n = 1, 2, \ldots \)).

Suppose now that \( f \) reports \( \succ_f \). For any \( i = m', \ldots , m \), denote by \( L_f^i \) the set of localities that have not permanently rejected \( f \) at the start of Round \( i \). We proceed by induction with the following hypothesis: In Round \( i = m', \ldots , m, \ell' \in L_f^i \), \( \ell' \) points at \( f_2 \), each \( f_j \) with \( j = 2, \ldots , n \) points at \( \ell_j \), each \( \ell_j \) with \( j = 2, \ldots , n - 1 \) points at \( f_{j+1} \), and \( \ell_n \) points at \( f \).

We first show that our inductive hypothesis holds for \( i = m' \). By construction, how family \( f \) points does not affect how other families point and which families each locality rejects until after \( f \) is permanently matched. Therefore, irrespective of whether \( f \) reports \( \succ_f \) or \( \succ'_f \), \( \ell' \) has not rejected \( f \) at the start of Round \( m' \) (hence \( \ell' \in \hat{L}_f^m \)) and, in Round \( m' \), \( \ell' \) points at \( f_2 \), each \( f_j \) with \( j = 2, \ldots , n \) points at \( \ell_j \), each \( \ell_j \) with \( j = 2, \ldots , n - 1 \) points at \( f_{j+1} \), and \( \ell_n \) points at \( f \).

We now suppose that our induction hypothesis holds for some \( i = m', \ldots , m - 1 \) and show that it holds for \( i + 1 \). As \( f \) is not matched until Round \( m > i \) and each family and locality can be part of at most one cycle, none of \( f_2, \ldots , f_n \) and \( \ell', \ell_2, \ldots , \ell_n \) are in a cycle in Round \( m' \), hence none of them are permanently matched in Round \( i \). Moreover, as the cycle \( f \to \ell' \to f_2 \to \ell_2 \to \ldots \to f_n \to \ell_n \to f \) is feasible, each \( \ell_j \) with \( j = 2, \ldots , n - 1 \) can accommodate \( f_j \) alongside \( \mu^i(\ell_j) \setminus \{f_{j+1}\} \). Therefore, \( \ell_j \) does not permanently reject \( f_j \) in Round \( i \). Analogously, \( \ell' \) does not permanently reject \( f \) in Round \( i \) (hence \( \ell' \in \hat{L}_f^{i+1} \)) and \( \ell_n \) does not permanently reject \( f_n \) in Round \( i \). It follows that, in Round \( i + 1 \), \( \ell' \) continues to point at \( f_2 \), each \( f_j \) with \( j = 2, \ldots , n \) continues to point at \( \ell_j \), each \( \ell_j \) with \( j = 2, \ldots , n - 1 \) continues to point at \( f_{j+1} \), and \( \ell_n \) continues to point at \( f \).

By induction, we conclude that \( \ell' \in \hat{L}_f^m \). Hence, since \( \ell \) is \( f \)'s most preferred family in \( \hat{L}_f^m \) by construction, we have that \( \ell \succeq_f \ell' \).

**Proof of Proposition 2.** We first show that all cycles that appear in the KTTTCE algorithm are feasible. Towards a contradiction, suppose that an infeasible cycle \( f_1 \to \ell_1 \to f_2 \to \ell_2 \to \ldots \to f_n \to \ell_n \to f_1 \) appears in some Round \( i \). Then, there exists \( j = 1, \ldots , n \) such that \( \ell_j \) cannot accommodate \( f_j \) alongside \( \mu^i(\ell_j) \setminus \{f_{j+1}\} \) (letting \( f_{n+1} = f_1 \)). As the null does not point (and therefore is not in any cycles), where \( \ell_j \neq \emptyset \), so \( \mu^E(\ell_j) = \emptyset \); hence \( f_{j+1} \) is not
in $\ell_j$’s endowment. Moreover, $f_{j+1}$ is not permanently matched at the start of Round $i$ as otherwise $\ell_j$ would not point at $f_{j+1}$; therefore $f \in \mu^i(\emptyset)$, which implies $f \notin \mu^i(\ell_j)$. Then, $\mu^i(\ell_j) \setminus \{f_{j+1}\} = \mu^i(\ell_j)$; hence $\ell_j$ cannot accommodate $f_j$ alongside $\mu^i(\ell_j)$. As $\mu^E(\ell_j) = \emptyset$, by construction all families in $\mu^i(\ell_j)$ have been permanently matched to $\ell_j$ before the start of Round $i$. Therefore, at the start of Round $i$, $\ell_j$ permanently rejects $f_j$ as $\ell_j$ cannot accommodate $f_j$ alongside all the families permanently matched to $\ell_j$. We conclude that $f_j$ does not point to $\ell_j$ in Round $i$, a contradiction.

We have shown that all cycles that appear in the KTTCE algorithm are feasible, which implies that the KTTCE algorithm does not enter the Rejection Stage in any round. As a result, each round of the KTTCE algorithm coincides with the corresponding round of the KTTC algorithm.

\[\Box\]

**Proof of Theorem 4.** Consider an instance in which the sizes of families are monotonic, the priorities of localities are lexicographic, and the endowment is $\mu^E$. Suppose that for this instance the KTTCE mechanism produces the endowment, i.e., $\mu^{\text{KTTCE}} = \mu^E$. Since the KTTCE mechanism does not Pareto improve upon this endowment, we need to show that $\mu^E$ is chain-efficient. Let $N$ be the total number of rounds of the KTTCE algorithm.

The fact that the KTTCE algorithm produces $\mu^E$ implies that the current matching is the same in every round: $\mu^E = \mu^1 = \mu^2 = \ldots = \mu^N = \mu^{N+1} = \mu^{\text{KTTCE}}$. (If the algorithm ends in Round $N$ by matching all remaining families to the null, we use the convention that $\mu^{N+1} = \mu^N$.) Let $M^*$ be the set of matchings that can be obtained by starting from $\mu^E$ and carrying out exactly one Pareto-improving chain. Now $\mu^E$ is chain-efficient if and only if $M^* = \emptyset$; therefore it remains to show that $M^* = \emptyset$.

We proceed by induction with the following hypothesis: if a locality $\ell$ has permanently rejected a family $f$ by the start of Round $i$, then for all $\mu^* \in M^*$, $\mu^*(f) \neq \ell$. Our inductive hypothesis trivially holds for $i = 1$ since no permanent rejection occurs before the start of Round 1. Assuming that the inductive hypothesis holds for some $i = 1, \ldots, N$, we show that it holds for $i + 1$.

Consider a family $f$ that has been permanently matched before the start of Round $i$. Recall that $\mu^E = \mu^i$, so $f$ is permanently matched to its endowment $\mu^E(f)$; therefore, $f$ has been permanently rejected by all localities that $f$ prefers to $\mu^E(f)$. By the induction hypothesis, we have that, for all $\mu^* \in M^*$, $\mu^*(f) = \mu^E(f)$. We can therefore conclude that all families that are permanently matched before the start of Round $i$ are matched to their endowment in all matchings contained in $M^*$.

Now consider a locality $\ell$ that permanently rejects a family $f$ in Round $i$. (As the null never permanently rejects any family, $\ell \neq \emptyset$.) We need to show that $\mu^*(f) \neq \ell$ for all $\mu^* \in M^*$. The fact that $\ell$ permanently rejects $f$ in Round $i$ implies that, at the start of Round $i$, $f$ is not permanently matched and $\ell$ has not permanently rejected $f$ yet. There
are two cases in which $\ell$ can permanently reject $f$: (1) at the beginning of Round $i$, or (2) in the Rejection Stage of Round $i$.

**Case 1:** $\ell$ permanently rejects $f$ at the beginning of Round $i$. By definition, $\ell$ cannot accommodate $f$ alongside all the families permanently matched to $\ell$. Since all families that are permanently matched before the start of Round $i$ are matched to their endowment in all matchings contained in $M^*$, we have that all the families that are permanently matched to $\ell$ at the start of Round $i$ are also matched to $\ell$ in all matchings contained in $M^*$. Therefore, $\mu^*(f) \neq \ell$ for all $\mu^* \in M^*$.

**Case 2:** $\ell$ permanently rejects $f$ in the Rejection Stage of Round $i$. By definition, $\ell$ cannot accommodate $f$ alongside $\mu^E(\ell) \setminus \{f'\}$ (where $f'$ is the family at which $\ell$ is pointing). Since a family is never permanently rejected by its endowment, we have that $\ell \neq \mu^E(f)$. By construction, $f'$ has the highest priority at $\ell$ among all families that are not permanently matched. We consider two sub-cases: (2.1) $f' \notin \mu^E(\ell)$, and (2.2) $f \in \mu^E(\ell)$.

**Sub-case 2.1:** $f' \notin \mu^E(\ell)$. As priorities are lexicographic, all families in $\mu^E(\ell)$ have a higher priority than $f'$. The fact that $\ell$ points at $f'$ implies that all families in $\mu^E(\ell)$ have been permanently matched to $\ell$ at the start of Round $i$. Recall that $\ell$ cannot accommodate $f$ alongside $\mu^i(\ell) \setminus \{f'\}$. Since $\mu^i(\ell) = \mu^E(\ell)$ and $f' \notin \mu^E(\ell)$, $\ell$ cannot accommodate $f$ alongside $\mu^E(\ell)$. As all families in $\mu^E(\ell)$ are permanently matched to $\ell$ at the start of Round $i$, $\ell$ permanently rejects $f$ at the beginning of Round $i$, contradicting our assumption that the permanent rejection occurs in the Rejection Stage.

**Sub-case 2.2:** $f' \in \mu^E(\ell)$. Towards a contradiction, suppose there exists a matching $\mu^* \in M^*$ such that $\mu^*(f) = \ell$. Recall that $\ell$ cannot accommodate $f$ alongside $\mu^E(\ell) \setminus \{f'\}$; therefore $\ell$ cannot accommodate $f$ alongside $\mu^E(\ell)$. This implies the existence of a family $\tilde{f} \in \mu^E(\ell)$ such that $\mu^*(\tilde{f}) \neq \ell$. In fact, because $\mu^*$ is obtained by carrying out exactly one Pareto-improving chain, there exists exactly one such family $\tilde{f}$. If $\tilde{f} \triangleright \ell f'$, then the fact that $\ell$ points at $f'$ implies that $\tilde{f}$ is permanently matched to $\ell$ at the start of Round $i$. Following the argument in Sub-case 2.1, we must therefore have that $\mu^*(\tilde{f}) = \mu^E(\tilde{f}) = \ell$, a contradiction. Therefore, we have that $f' \triangleright \ell \tilde{f}$ or $f' = f$. Since $f', \tilde{f} \in \mu^E(\ell)$, sizes are monotonic and priorities are lexicographic, by definition we have that $v''_{d} \geq v''_{d}$ for all $d \in D$. Therefore, the fact that $\ell$ cannot accommodate $f$ alongside $\mu^E(\ell) \setminus \{f'\}$ implies that $\ell$ cannot accommodate $f$ alongside $\mu^E(\ell) \setminus \{f\}$, a contradiction. We conclude that $\mu^*(f) \neq \ell$ for all $\mu^* \in M^*$.

We can now conclude that our induction hypothesis holds at the beginning of Round $i+1$. By induction, if a locality $\ell$ permanently rejects a family $f$ at some point in the KTTCE algorithm, then, for all $\mu^* \in M^*$, $\mu^*(f) \neq \ell$. Therefore, every family matched to its endowment at the end of the algorithm is matched to its endowment under any matching $\mu^* \in M^*$. By assumption, all families are matched to their respective endowments at the end of the
KTTCE algorithm, meaning that $M^*$ does not contain any matching other than $\mu^E$. As $\mu^E \notin M^*$ by definition, we conclude that $M^* = \emptyset$, as desired. \qed

Proof of Theorem 5. Denote by $N$ the last round of the KDA algorithm. Now, for any family $f \in F$, any locality $\ell \in L$, and any Round $i = 1, \ldots, N$, let $\hat{R}_i^f(\ell)$ be the set of families that have a higher priority than $f$ at $\ell$ and propose in Round $i$ to either $\ell$ or a less preferred locality (because they have already been permanently rejected by $\ell$ in a previous round).

We first verify that the output of the KDA algorithm does not violate any capacity constraints; that is, we show that every locality $\ell \in L$ can accommodate $\mu^{KDA}(\ell)$. By construction, all families in $\mu^{KDA}(\ell)$ propose to and are tentatively accepted by $\ell$ in Round $N$. Therefore, $\ell$ can weakly accommodate every family $f \in \mu^{KDA}(\ell)$ alongside $\mu^{KDA}(\ell) \cap \hat{R}_i^f$. Towards a contradiction, suppose that $\ell$ cannot accommodate $\mu^{KDA}(\ell)$. Then, there exists $d \in D$ such that $\sum_{f \in \mu^{KDA}(\ell)} \nu_d^f > \kappa_d^\ell$. Let $g \in F$ be the lowest-priority family such that $g \in \mu^{KDA}(\ell)$ and $\nu_d^g > 0$. (Such a family exists since $\sum_{f \in \mu^{KDA}(\ell)} \nu_d^f > \kappa_d^\ell \geq 0$.) Then,

$$\nu_d^g > 0 \quad \text{and} \quad \nu_d^g + \sum_{f \in \mu^{KDA}(\ell) \cap \hat{R}_i^f} \nu_d^f > \kappa_d^\ell;$$

therefore, $\ell$ cannot weakly accommodate $g$ alongside $\mu^{KDA}(\ell) \cap \hat{R}_i^g$, a contradiction.

We next show that $\mu^{KDA}$ is interference-free. Consider any family $f$; we need to show that $f$ does not interfere with $\mu$. By construction, $f$ proposes to and is tentatively accepted by $\mu^{KDA}(f)$ in the last round; therefore, $\mu(f)$ can weakly accommodate $f$ alongside $\hat{R}_N^f(\mu^{KDA}(f))$. Again by construction, the families in $\hat{R}_N^f(\mu^{KDA}(f))$ are exactly those that end up matched to either $\mu^{KDA}(f)$ or a less preferred locality. It follows that $\hat{R}_N^f(\mu^{KDA}(f)) = \hat{R}_\mu^{KDA}$, which implies that $f$ does not interfere with $\mu^{KDA}$.

We finally show that $\mu^{KDA}$ dominates all other interference-free matchings. Towards a contradiction, suppose that there exists an interference-free matching $\mu$ such that $\mu(f_1) \succ_f \mu^{KDA}(f_1)$ for some $f_1 \in F$. We proceed by induction. Our assumption implies that $\mu(f_1)$ permanently rejects $f_1$ in some Round $i_1$ of the KDA algorithm. For the induction argument, suppose that for some $n \in \mathbb{Z}_{>0}$, there exists a family $f_n$ such that $\mu(f_n)$ permanently rejects $f_n$ in some Round $i_n$ of the KDA algorithm. We want to show that there exists a family $f_{n+1}$ such that $\mu(f_{n+1})$ permanently rejects $f_{n+1}$ in some Round $i_{n+1} < i_n$. By construction, $\mu(f_n)$ cannot weakly accommodate $f_n$ alongside $\hat{R}_n^f(\mu(f_n))$. If all families in $\hat{R}_n^f(\mu(f_n))$ were matched at $\mu$ to either $\mu(f_n)$ or a less preferred locality, then $f_n$ would interfere with $\mu$, a contradiction. Therefore, there exists a family $f_{n+1} \in \hat{R}_n^f(\mu(f_n))$ such that $\mu(f_{n+1}) \succ_{f_{n+1}} \mu(f_n)$. By construction, as $f_{n+1}$ proposes in Round $i_n$ of the KDA algorithm to either $\mu(f_n)$ or a less preferred locality, $\mu(f_{n+1})$ permanently rejects $f_{n+1}$ in some Round $i_{n+1} < i_n$. Iterating this argument inductively, we find that some family $f_m$ is permanently rejected by $\mu(f_m)$ in Round $i_m$ such that $i_m < 1$, which is impossible. \qed
Proof of Theorem 6. We first verify that the output of the TKDA mechanism does not violate any capacity constraint; that is, for all $\ell \in L$, $\ell$ can accommodate $\mu^{\text{TKDA}}(\ell)$. Consider a locality $\ell \in L$. By construction, all families in $\mu^{\text{TKDA}}(\ell)$ propose to and are tentatively accepted by $\ell$ in the last round of the TKDA algorithm. Hence, for any $f \in \mu^{\text{TKDA}}(\ell)$,

$$\theta^f_\ell \geq |\mu^{\text{TKDA}}(\ell) \cap \widehat{F}^f_\ell| + 1 \geq 1.$$ 

If $\ell$ cannot weakly accommodate $f$ alongside $\mu^{\text{TKDA}}(\ell) \cap \widehat{F}^f_\ell$, then by construction (see Algorithm 5) $\theta^f_\ell = 0$, which contradicts the fact that $\theta^f_\ell \geq 1$. Therefore, $\ell$ can weakly accommodate every family $f \in \mu^{\text{TKDA}}(\ell)$ alongside $\mu^{\text{TKDA}}(\ell) \cap \widehat{F}^f_\ell$. As we showed in the proof of Theorem 5, this implies that $\ell$ can accommodate $\mu^{\text{TKDA}}(\ell)$.

We now prove that the TKDA mechanism is strategy-proof and interference-free.

Proof that TKDA is strategy-proof. Consider a locality $\ell \in L$ and a subset of families $G \subseteq F$. To simplify notation, let us define, for each $f \in F$, $\widehat{G}^f_\ell = G \cap \widehat{F}^f_\ell$ to be the families in $G$ that have a higher priority at $\ell$ than $f$. We also denote by $\theta^f_\ell(G)$ the threshold of family $f$ at locality $\ell$ if, in some round of the TKDA algorithm, families in $G$ propose to $\ell$. (That is, $\theta^f_\ell(G)$ is obtained by running the Threshold Calculator defined in Algorithm 5 with $\Pi_\ell = G$.)

We define the choice function of locality $\ell$, $C_\ell : 2^F \rightarrow 2^F$, as follows: for every $G \subseteq F$,

$$C_\ell(G) = \{ f \in G : |\widehat{G}^f_\ell| + 1 \leq \theta^f_\ell(G) \}.$$ 

That is, $C_\ell(G)$ is a function that selects the families that $\ell$ does not permanently reject if families in $G$ propose to $\ell$ in some round of the TKDA algorithm. One way to interpret the choice function is that locality $\ell$ “chooses” the families in $C_\ell(G)$ when it receives proposals from all families in $G$.

We now fix a locality $\ell \in L$ and two subsets of families $G \subseteq H \subseteq F$ and define two properties of the choice function:

- **Substitutability (S)**: $C_\ell(H) \cap G \subseteq C_\ell(G)$, and
- **Cardinal Monotonicity (CM)**: $|C_\ell(G)| \leq |C_\ell(H)|$.

Hatfield and Milgrom (2005) analyze properties of the Deferred Acceptance (DA) algorithm, in which localities (“hospitals” in their terminology) have choice functions. In each round of the DA algorithm, families propose to their most preferred locality that has not yet permanently rejected them. Localities tentatively accept or permanently reject proposals based on their choice functions, i.e., if locality $\ell$ receives proposals from families in $G$, then families in $C_\ell(G)$ are tentatively accepted and families in $G \setminus C_\ell(G)$ are permanently rejected.

Theorems 3 and 11 of Hatfield and Milgrom (2005) imply that the DA mechanism is strategy-proof for families if the choice function of every locality satisfies the (S) and (CM) conditions. By construction, the TKDA algorithm in our setting corresponds to deferred
proposing to

Claim A.1.

\[ \theta \]

\[ \ell \]

by family \( g \) that were removed. Part (iii) states that if the threshold of family \( \ell \) remains so no matter which families are proposing to

Lemma A.1. For every \( G \subseteq H \subseteq F \); \( f, g \in F \); and \( \ell \in L \):

(i) \( \theta^f_\ell(G) = \infty \) if and only if \( \theta^f_\ell(H) = \infty \);

(ii) if \( \theta^f_\ell(H) \in \mathbb{Z}_{\geq 0} \), then \( \theta^f_\ell(G) \leq \theta^f_\ell(H) \leq \theta^f_\ell(G) + |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell| \); and

(iii) if \( g \triangleright_{\ell} f \) and \( \theta^f_\ell(G) \neq \infty \), then

\[ \circ \theta^f_\ell(G) \geq \theta^f_\ell(G), \]

\[ \circ f \in C_\ell(G) \text{ implies that } g \in G \iff g \in C_\ell(G). \]

Part (i) of Lemma A.1 states that if the threshold of family \( f \) at \( \ell \) is infinite, then it remains so no matter which families are proposing to \( \ell \). Part (ii) states that if the threshold of family \( f \) at \( \ell \) is non-zero and finite, then removing some families from the set of families proposing to \( \ell \) may reduce \( f \)'s threshold at \( \ell \) by at most the number of proposing families that were removed. Part (iii) states that if the threshold of family \( f \) at \( \ell \) is finite, then any family \( g \) with a higher priority than \( f \) at \( \ell \) has a weakly larger threshold than \( f \) and is chosen by \( \ell \) whenever \( f \) is chosen by \( \ell \).

**Proof of (S).** Consider any family \( f \in C_\ell(H) \cap G \); we need to show that \( f \in C_\ell(G) \). The fact that \( f \in C_\ell(H) \) implies that \( \theta^f_\ell(H) \neq 0 \) and, if \( \theta^f_\ell(H) = \infty \), Lemma A.1(i) implies that \( \theta^f_\ell(G) = \infty \), hence \( f \in C_\ell(G) \). It remains to show that \( f \in C_\ell(H) \) whenever \( \theta^f_\ell(H) \in \mathbb{Z}_{>0} \).

In that case, using Lemma A.1(ii), we have that

\[ \theta^f_\ell(H) \leq \theta^f_\ell(G) + |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell|. \]

Since \( f \in C_\ell(G) \), the definition of a choice function implies that

\[ \theta^f_\ell(H) \geq |\widehat{H}^f_\ell| + 1. \]

Combining the two inequalities yields \( \theta^f_\ell(G) \geq |\widehat{G}^f_\ell| + 1 \) so \( f \in C_\ell(G) \), as required.

**Proof of (CM).** We need to show that \( |C_\ell(G)| \leq |C_\ell(H)| \). Let \( m = |H \setminus G| \) and arbitrarily label the families in \( H \setminus G \) such that \( H = G \cup \{ f_1, \ldots, f_m \} \).

**Claim A.1.** For each \( i = 1, \ldots, m \), \( |C_\ell(G \cup \{ f_1, \ldots, f_{i-1} \})| \leq |C_\ell(G \cup \{ f_1, \ldots, f_i \})| \).

Claim A.1 implies that

\[ |C_\ell(G)| \leq |C_\ell(G \cup \{ f_1 \})| \leq |C_\ell(G \cup \{ f_1, f_2 \})| \leq \ldots \leq |C_\ell(G \cup \{ f_1, \ldots, f_m \})| = |C_\ell(H)|, \]

which implies the desired result. Therefore, it remains to prove Claim A.1.
Proof of Claim A.1. Fix some $i \leq m$ and define $G' = G \cup \{f_1, \ldots, f_{i-1}\}$ and $H' = \{f_1, \ldots, f_{i-1}, f_i\}$. We need to show that $|C_\ell(G')| \leq |C_\ell(H')|$. We have that

$$|C_\ell(H')| - |C_\ell(G')| = |C_\ell(H') \setminus C_\ell(G')| - |C_\ell(G') \setminus C_\ell(H')|,$$

so we need to show that

$$|C_\ell(H') \setminus C_\ell(G')| \geq |C_\ell(G') \setminus C_\ell(H')|. \tag{1}$$

By definition, $H' = G' \cup \{f_i\}$ and, by (S), $C_\ell(H') \cap G'' \subseteq C_\ell(G')$; therefore,

$$C_\ell(H') \setminus C_\ell(G') \subseteq \{f_i\}.$$

There are two cases: $f_i \notin C_\ell(H')$ and $f_i \in C_\ell(H')$.

Case 1: $f_i \notin C_\ell(H')$. In this case, $C_\ell(H') \setminus C_\ell(G') = \emptyset$; hence, by inequality (1), we need to show that $C_\ell(G') \setminus C_\ell(H') = \emptyset$. Towards a contradiction, suppose to the contrary that there exists a family $g \in C_\ell(G') \setminus C_\ell(H')$. Since $g \in C_\ell(G')$, by the definition of the choice function $C_\ell$, we have that

$$\theta_\ell^g(G') \geq |\widehat{G}_\ell^g| + 1; \tag{2}$$

but, since $g \notin C_\ell(H')$, we also have that

$$\theta_\ell^g(H') < |\widehat{H}_\ell^g| + 1. \tag{3}$$

We now consider two subcases: $g \triangleright_\ell f_i$ and $f_i \triangleright_\ell g$.

Sub-case 1.1: $g \triangleright_\ell f_i$. In this case, as $H' = G' \cup \{f_i\}$, $\widehat{G}_\ell^{f_i} = \widehat{H}_\ell^{f_i}$ so $|\widehat{G}_\ell^{f_i}| = |\widehat{H}_\ell^{f_i}|$ and, as a family’s threshold only depends on higher-priority families (Algorithm 5), $\theta_\ell^g(G') = \theta_\ell^g(H')$. Combining these observations with inequalities (2) and (3) yields

$$\theta_\ell^g(G') \geq |\widehat{G}_\ell^{f_i}| + 1 = |\widehat{H}_\ell^{f_i}| + 1 > \theta_\ell^g(H') = \theta_\ell^g(G'),$$

a contradiction.

Sub-case 1.2: $f_i \triangleright_\ell g$. On the one hand, inequality (3) implies that $\theta_\ell^g(H') \neq \infty$, so we can apply Lemma A.1(i) to obtain that $\theta_\ell^g(G') \neq \infty$ and Lemma A.1(iii) to obtain that

$$\theta_\ell^{f_i}(G') \geq \theta_\ell^g(G'). \tag{4}$$

On the other hand, by the assumption that $f_i \notin C_\ell(H')$ in Case 1, we have that

$$\theta_\ell^{f_i}(H') < |\widehat{H}_\ell^{f_i}| + 1. \tag{5}$$

As $H' = G' \cup \{f_i\}$, $\widehat{G}_\ell^{f_i} = \widehat{H}_\ell^{f_i}$ so $|\widehat{G}_\ell^{f_i}| = |\widehat{H}_\ell^{f_i}|$ and, as a family’s threshold only depends on higher-priority families (Algorithm 5), $\theta_\ell^{f_i}(G') = \theta_\ell^{f_i}(H')$. Combining these observations with inequality (5) yields

$$\theta_\ell^{f_i}(G') = \theta_\ell^{f_i}(H') < |\widehat{H}_\ell^{f_i}| + 1 = |\widehat{G}_\ell^{f_i}| + 1. \tag{6}$$
We therefore conclude that \( \theta^g_\ell(G') < \theta^g_\ell(G') \), which contradicts inequality (4).

**Case 2:** \( f_i \in C_\ell(H') \). In this case, we have that \( C_\ell(H') \setminus C_\ell(G') = \{ f_i \} \); hence, by inequality (4), we need to show that \( |C_\ell(G') \setminus C_\ell(H')| \leq 1 \). Towards a contradiction, suppose that there exist two distinct families \( g_1, g_2 \in C_\ell(G') \setminus C_\ell(H') \). Without loss of generality, we assume that \( g_1 \vartriangleright_{\ell} g_2 \). Since \( g_1, g_2 \in C_\ell(G') \), by the definition of the choice function, we have that

\[
\theta^{g_1}_\ell(G') \geq |\hat{G}^{g_1}_\ell| + 1 \quad \text{and} \quad \theta^{g_2}_\ell(G') \geq |\hat{G}^{g_2}_\ell| + 1,
\]

but, since \( g_1, g_2 \notin C_\ell(H') \), we also have that

\[
\theta^{g_1}_\ell(H') < |\hat{H}^{g_1}_\ell| + 1 \quad \text{and} \quad \theta^{g_2}_\ell(H') < |\hat{H}^{g_2}_\ell| + 1.
\]

We now consider two subcases: \( g_1 \vartriangleright_{\ell} f_i \) and \( f_i \vartriangleright_{\ell} g_1 \).

**Sub-case 2.1:** \( g_1 \vartriangleright_{\ell} f_i \). In this case, as \( H' = G' \cup \{ f_i \} \), \( \hat{G}^{g_1}_\ell = \hat{H}^{g_1}_\ell \) so \( |\hat{G}^{g_1}_\ell| = |\hat{H}^{g_1}_\ell| \) and, as a family’s threshold only depends on higher-priority families (Algorithm 5), \( \theta^{g_1}_\ell(G') = \theta^{g_1}_\ell(H') \). Combining these observations with inequalities (7) and (8) yields

\[
\theta^{g_1}_\ell(G') \geq |\hat{G}^{g_1}_\ell| + 1 = |\hat{H}^{g_1}_\ell| + 1 > \theta^{g_1}_\ell(H') = \theta^{g_1}_\ell(G'),
\]

a contradiction.

**Sub-case 2.2:** \( f_i \vartriangleright_{\ell} g_1 \). Inequality (8) implies that \( \theta^{g_2}_\ell(H') \neq \infty \) so we can apply Lemma A.1(i) to obtain that \( \theta^{g_2}_\ell(G') \neq \infty \) and Lemma A.1(iii) to obtain that \( \theta^{g_1}_\ell(G') \geq \theta^{g_2}_\ell(G') \). Moreover, as \( g_1 \vartriangleright_{\ell} g_2 \) and \( g_1 \in G \), we have that \( \hat{C}^{g_1}_\ell \subset \hat{G}^{g_2}_\ell \) so \( |\hat{C}^{g_1}_\ell| < |\hat{G}^{g_2}_\ell| \). Combining these observations with inequality (7) yields

\[
\theta^{g_1}_\ell(G') \geq \theta^{g_2}_\ell(G') \geq |\hat{G}^{g_2}_\ell| + 1 > |\hat{G}^{g_1}_\ell| + 1.
\]

We therefore conclude that

\[
\theta^{g_1}_\ell(G') \geq |\hat{G}^{g_1}_\ell| + 2.
\]

Since \( \theta^{g_1}_\ell(H') \neq \infty \), we have two cases to consider: \( \theta^{g_1}_\ell(H') = 0 \) and \( \theta^{g_1}_\ell(H') \in \mathbb{Z}_{>0} \).

**Sub-case 2.2.1:** \( \theta^{g_1}_\ell(H') = 0 \). In this case, by the definition of thresholds (Algorithm 5), there exists a family \( h \in F \) such that (i) either \( h = g_1 \) or \( h \vartriangleright_{\ell} g_1 \) and (ii) \( \ell \) cannot weakly accommodate \( h \) alongside \( \hat{H}^{g_1}_\ell \). First, as \( h \) has a weakly higher priority than \( g_1 \) and \( \theta^{g_1}_\ell(G') \neq \infty \) (by inequality (8) and Lemma A.1(i)), we can apply Lemma A.1(iii) to obtain that \( \theta^{g_1}_\ell(G') \leq \theta^{g_1}_\ell(H') \). Second, as \( G' \subseteq H' \), we have that \( \hat{G}^{g_1}_\ell \subseteq \hat{H}^{g_1}_\ell \), therefore, \( \hat{H}^{g_1}_\ell \) is a subset of \( \hat{F}^{g_1}_\ell \) that contains all families in \( \hat{G}^{g_1}_\ell \) and alongside which \( \ell \) cannot weakly accommodate \( h \). By the definition of thresholds (Algorithm 5), it follows that \( \theta^{g_1}_\ell(G') \leq |\hat{H}^{g_1}_\ell| \). Third, as
\[ H' = G' \cup \{ f_i \}, \] we have that \( |\hat{H}_{i}^n| \leq |\hat{G}_{i}^n| + 1 \). Fourth, as \( h \) has a weakly higher priority than \( g_1 \), we have that \( |\hat{G}_{i}^n| \leq |\hat{G}_{i}^{g_1}| \). Combining these four observations yields
\[
\theta_{i}^g(G') \leq \theta_{i}^g(G') \leq |\hat{H}_{i}^n| \leq |\hat{G}_{i}^n| + 1 \leq |\hat{G}_{i}^{g_1}| + 1,
\]
which contradicts inequality (9).

Sub-sub-case 2.2.2: \( \theta_{i}^g(H') \in \mathbb{Z}_{\geq 0} \). In this case, we can apply Lemma A.1(ii) to obtain that \( \theta_{i}^g(H') \geq \theta_{i}^g(G') \). Moreover, as \( H' = G' \cup \{ f_i \} \) and \( f_i \in \hat{H}_{i}^{g_1} \) (by the assumption of Sub-case 2.2), \( f_i \triangleright g_1 \), we have that \( |\hat{H}_{i}^{g_1}| = |\hat{G}_{i}^{g_1}| + 1 \). Combining these observations with inequality (9) yields
\[
\theta_{i}^g(H') \geq \theta_{i}^g(G') \geq |\hat{G}_{i}^{g_1}| + 2 = |\hat{G}_{i}^{g_1}| + 1,
\]
which contradicts inequality (8) and completes the proof of Claim A.1. \( \square \)

Proof that TKDA is interference-free.

Let \( N \) be the number of rounds after which the TKDA algorithm terminates. For each family \( f \in F \), each locality \( \ell \in L \), and each Round \( i = 1, \ldots, N \), denote by \( \Pi_{i}^{f} \) the set of families that propose to \( \ell \) and by \( \theta_{i}^{f}(\Pi_{i}^{f}) \) the threshold of \( f \) for \( \ell \) in Round \( i \).

Fix a family \( f \in F \) and, for notational simplicity, let \( \ell = \mu_{TKDA}(f) \). We need to show that \( f \) does not interfere with \( \mu_{TKDA} \), i.e., that \( \ell \) can weakly accommodate \( f \) alongside \( \widehat{F}_{\ell}^{f}_{TKDA} \).

If \( \theta_{i}^{f}(\Pi_{i}^{N}) = \infty \), then by definition \( \ell \) can weakly accommodate \( f \) alongside \( \widehat{F}_{\ell}^{f} \), hence alongside \( \widehat{F}_{\ell}^{f}_{TKDA} \). The remainder of the proof considers the case in which \( \theta_{i}^{f}(\Pi_{i}^{N}) \neq \infty \).

By construction, \( \ell \) tentatively accepts \( f \) in Round \( N \). By definition, it follows that \( \theta_{i}^{f}(\Pi_{i}^{N}) > 0 \); hence, \( \ell \) can weakly accommodate \( f \) alongside \( \Pi_{i}^{N} \cap \widehat{F}_{\ell}^{f} \). We complete the proof by showing that \( (\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{f}) = \widehat{F}_{\ell}^{f}_{\mu_{TKDA}} \). Observing that, by construction, \( (\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{f}) \subseteq \widehat{F}_{\ell}^{f}_{\mu_{TKDA}} \), it remains to show that the case where \( (\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{f}) \subseteq \widehat{F}_{\ell}^{f}_{\mu_{TKDA}} \) leads to a contradiction.

Toward a contradiction, suppose that \( (\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{f}) \subset \widehat{F}_{\ell}^{f}_{\mu_{TKDA}} \). Let \( g \) be the highest-priority family in \( \widehat{F}_{\ell}^{f}_{\mu_{TKDA}} \setminus (\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{f}) \); \( g \) is then the highest-priority family to be permanently rejected by \( \ell \) in the TKDA algorithm. Let \( i \) be the round in which that permanent rejection occurs. Then, \( g \)'s threshold in Round \( i \) is smaller than its priority rank. As a family’s threshold only depends on higher-priority families, we have that
\[
\theta_{i}^{g}(\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{g}) = \theta_{i}^{g}(\Pi_{i}^{N}) < |\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{g}| + 1.
\]
As \( g \) is the highest-priority family to be permanently rejected by \( \ell \), every family with a higher priority than \( g \) that proposes to \( \ell \) in Round \( i \) also proposes to \( \ell \) in Round \( N \), hence \( (\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{g}) \subseteq (\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{g}) \). Moreover, (10) implies that \( \theta_{i}^{g}(\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{g}) \neq \infty \), which by Lemma A.1(i) yields \( \theta_{i}^{g}(\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{g}) \neq \infty \). Therefore, Lemma A.1(ii) applies and we have that
\[
\theta_{i}^{g}(\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{g}) \leq \theta_{i}^{g}(\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{g}) + |\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{g}| - |\Pi_{i}^{N} \cap \widehat{F}_{\ell}^{g}|.
\]
Combining (10) and (11) yields

\[
\theta^f_\ell(\Pi^N_\ell \cap \widehat{F}^g_\ell) < |\Pi^N_\ell \cap \widehat{F}^g_\ell| + 1.
\]

By assumption (and as a family’s threshold only depends on higher-priority families), we have that \( \theta^f_\ell(\Pi^N_\ell \cap \widehat{F}^g_\ell) = \theta^0_\ell(\Pi^N_\ell) \neq \infty \). Then, as \( g \triangleright \ell \), Lemma A.1(iii) applies and yields \( \theta^f_\ell(\Pi^N_\ell \cap \widehat{F}^g_\ell) \leq \theta^0_\ell(\Pi^N_\ell \cap \widehat{F}^g_\ell) \). Combining that inequality with (12), using again the property that a family’s threshold only depends on higher-priority families, and observing that, because \( g \triangleright \ell \), \( (\Pi^N_\ell \cap \widehat{F}^g_\ell) \subseteq (\Pi^N_\ell \cap \widehat{F}^0_\ell) \), we obtain that

\[
\theta^f_\ell(\Pi^N_\ell) = \theta^f_\ell(\Pi^N_\ell \cap \widehat{F}^g_\ell) \leq \theta^0_\ell(\Pi^N_\ell \cap \widehat{F}^g_\ell) < |\Pi^N_\ell \cap \widehat{F}^g_\ell| + 1 \leq |\Pi^N_\ell \cap \widehat{F}^f_\ell| + 1.
\]

If follows that \( \ell \) rejects \( f \) in Round \( N \), a contradiction. \( \square \)

It therefore remains only to prove Lemma A.1 in order to complete the proof of Theorem 6.

**Proof of Lemma A.1.** We prove each case in turn.

**Proof of (i).** If \( \theta^f_\ell(G) = \infty \), then \( \ell \) can weakly accommodate \( f \) alongside \( \widehat{F}^f_\ell \), which implies \( \theta^f_\ell(H) = \infty \). The converse is proved analogously.

**Proof of (ii).** We have assumed that \( \theta^f_\ell(H) \in \mathbb{Z}_{\geq 0} \). We first show that \( \tilde{\theta}^f_\ell(H), \tilde{\theta}^f_\ell(G), \theta^f_\ell(G) \in \mathbb{Z}_{\geq 0} \). If either \( \tilde{\theta}^f_\ell(H) = \infty \), \( \tilde{\theta}^f_\ell(G) = \infty \), or \( \theta^f_\ell(G) = \infty \), then \( \ell \) can weakly accommodate \( f \) alongside \( \widehat{F}^f_\ell \), which implies that \( \theta^f_\ell(H) = \infty \), a contradiction. It remains to show that none of \( \tilde{\theta}^f_\ell(H) \), \( \tilde{\theta}^f_\ell(G) \), or \( \theta^f_\ell(G) \) are equal to 0. Since \( \tilde{\theta}^f_\ell(H) \neq \infty \), \( \theta^f_\ell(H) \leq \tilde{\theta}^f_\ell(H) \) by definition (Algorithm 5); therefore \( \tilde{\theta}^f_\ell(H) = 0 \) implies \( \theta^f_\ell(H) = 0 \), a contradiction. If \( \tilde{\theta}^f_\ell(G) = 0 \), then \( \ell \) cannot weakly accommodate \( f \) alongside \( \widehat{F}^f_\ell \). Since \( \widehat{G}^f_\ell \subset \widehat{H}^f_\ell \), \( \ell \) cannot weakly accommodate \( f \) alongside \( \widehat{H}^f_\ell \) so \( \tilde{\theta}^f_\ell(H) = 0 \), a contradiction. If \( \theta^f_\ell(G) = 0 \) and \( \tilde{\theta}^f_\ell(G) \neq 0 \), then there exists \( g \in \widehat{F}^g_\ell \) such that \( \tilde{\theta}^g_\ell(G) = 0 \). Since \( G \subseteq H \), we have that \( \tilde{\theta}^g_\ell(H) = 0 \). Hence, as \( g \in \widehat{F}^g_\ell \), we have that \( \theta^f_\ell(H) = 0 \), a contradiction.

Having established that \( \tilde{\theta}^f_\ell(H), \tilde{\theta}^f_\ell(G), \theta^f_\ell(G) \in \mathbb{Z}_{\geq 0} \), we next show that \( \tilde{\theta}^f_\ell(G) \leq \tilde{\theta}^f_\ell(H) \). As \( \tilde{\theta}^f_\ell(H) \in \mathbb{Z}_{\geq 0} \), there exists a subset of families \( \widehat{F} \) such that \( \widehat{H}^f_\ell \subseteq \widehat{F} \subseteq \widehat{F}^f_\ell \) and \( |\widehat{F}| = \tilde{\theta}^f_\ell(H) \) alongside which \( \ell \) cannot weakly accommodate \( f \). Since \( G \subseteq H \), we have that \( \widehat{G}^f_\ell \subseteq \widehat{H}^f_\ell \) and therefore \( \widehat{G}^f_\ell \subseteq \widehat{H}^f_\ell \subseteq \widehat{F} \). Hence, we obtain that \( \tilde{\theta}^f_\ell(G) \leq |\widehat{F}| = \tilde{\theta}^f_\ell(H) \) as required.

We next show that \( \tilde{\theta}^f_\ell(H) \leq \tilde{\theta}^f_\ell(G) + |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell| \). As \( \tilde{\theta}^f_\ell(G) \in \mathbb{Z}_{\geq 0} \), there exists a subset of families \( \widehat{F} \) such that \( \widehat{G}^f_\ell \subseteq \widehat{F} \subseteq \widehat{F}^f_\ell \) and \( |\widehat{F}| = \tilde{\theta}^f_\ell(G) \) alongside which \( \ell \) cannot weakly accommodate \( f \). Therefore, \( \ell \) cannot weakly accommodate \( f \) alongside \( \widehat{F} \cup \widehat{H}^f_\ell \) so \( \tilde{\theta}^f_\ell(H) \leq |\widehat{F} \cup \widehat{H}^f_\ell| \). By construction, \( |\widehat{F} \cup \widehat{H}^f_\ell| = |\widehat{F}| + |\widehat{H}^f_\ell| - |\widehat{F} \cap \widehat{H}^f_\ell| \) and \( \widehat{G}^f_\ell \subseteq (\widehat{F} \cap \widehat{H}^f_\ell) \) so \( |\widehat{G}^f_\ell| \leq |\widehat{F} \cap \widehat{H}^f_\ell| \). Combining the preceding results with the fact that \( |\widehat{F}| = \tilde{\theta}^f_\ell(G) \) yields

\[
\tilde{\theta}^f_\ell(H) \leq |\widehat{F} \cup \widehat{H}^f_\ell| = |\widehat{F}| + |\widehat{H}^f_\ell| - |\widehat{F} \cap \widehat{H}^f_\ell| \leq |\widehat{F}| + |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell| = \tilde{\theta}^f_\ell(G) + |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell|.
\]
We have now established that $\tilde{\theta}_\ell^f(G) \leq \tilde{\theta}_\ell^f(H) \leq \tilde{\theta}_\ell^f(G) + |\widehat{H}_\ell^f| - |\widehat{G}_\ell^f|$. Since $f$ was chosen arbitrarily, we have

$$
(13) \quad \min_{g \in \widehat{F}_\ell^f \cup \{f\}} \left\{ \tilde{\theta}_\ell^g(G) \right\} \geq \min_{g \in \widehat{F}_\ell^f \cup \{f\}} \left\{ \tilde{\theta}_\ell^g(H) \right\} \leq \min_{g \in \widehat{F}_\ell^f \cup \{f\}} \left\{ \tilde{\theta}_\ell^g(G) + |\widehat{H}_\ell^f| - |\widehat{G}_\ell^f| \right\}.
$$

Let $h = \arg \min_{g \in \widehat{F}_\ell^f \cup \{f\}} \left\{ \tilde{\theta}_\ell^g(G) \right\}$. Then, we have that

$$
(14) \quad \min_{g \in \widehat{F}_\ell^f \cup \{f\}} \left\{ \tilde{\theta}_\ell^g(G) + |\widehat{H}_\ell^f| - |\widehat{G}_\ell^f| \right\} \leq \tilde{\theta}_\ell^h(G) + |\widehat{H}_\ell^f| - |\widehat{G}_\ell^f|.
$$

In addition, we have that

$$
(15) \quad |\widehat{H}_\ell^h| - |\widehat{G}_\ell^h| = |\widehat{H}_\ell^f \setminus \widehat{G}_\ell^f| \leq |\widehat{H}_\ell^f \setminus \widehat{G}_\ell^f| = |\widehat{H}_\ell^f| - |\widehat{G}_\ell^f|,
$$

where the two equalities follow from the fact that $\widehat{G}_\ell^h \subseteq \widehat{H}_\ell^h$ and $\widehat{G}_\ell^f \subseteq \widehat{H}_\ell^f$ while the fact that $(\widehat{H}_\ell^h \setminus \widehat{G}_\ell^h) \subseteq (\widehat{H}_\ell^f \setminus \widehat{G}_\ell^f)$ (as $h \in \widehat{F}_\ell^f$) implies the inequality. Combining (13), (14), and (15), we obtain

$$
(16) \quad \min_{g \in \widehat{F}_\ell^f \cup \{f\}} \left\{ \tilde{\theta}_\ell^g(G) \right\} \leq \min_{g \in \widehat{F}_\ell^f \cup \{f\}} \left\{ \tilde{\theta}_\ell^g(H) \right\} \leq \tilde{\theta}_\ell^h(G) + |\widehat{H}_\ell^f| - |\widehat{G}_\ell^f|.
$$

Finally, using the fact that $\theta_\ell^f(G), \theta_\ell^f(H) \in \mathbb{Z}_{\geq 0}$ and the definition of $\theta_\ell^f$ from Algorithm 5, we obtain that

$$
(17) \quad \theta_\ell^f(G) = \min_{g \in \widehat{F}_\ell^f \cup \{f\}} \left\{ \tilde{\theta}_\ell^g(G) \right\} \tilde{\theta}_\ell^f(G) \quad \text{and} \quad \theta_\ell^f(H) = \min_{g \in \widehat{F}_\ell^f \cup \{f\}} \left\{ \tilde{\theta}_\ell^g(H) \right\}.
$$

Combining (16) and (17) yields

$$
\theta_\ell^f(G) \leq \theta_\ell^f(H) \leq \theta_\ell^f(G) + |\widehat{H}_\ell^f| - |\widehat{G}_\ell^f|,
$$

as required.

Proof of (iii). We have assumed that $g \succ_\ell f$ and $\theta_\ell^f(G) \neq \infty$. To prove the first part of the statement, we need to show that $\theta_\ell^f(G) \geq \theta_\ell^f(G)$. If $\theta_\ell^f(G) = \infty$, the result is immediate since $\theta_\ell^f(G)$ is finite. If $\theta_\ell^f(G) \neq \infty$, the definition of thresholds in Algorithm 5 implies that

$$
\theta_\ell^f(G) = \min_{h \in \widehat{G}_\ell^f} \left\{ \tilde{\theta}_\ell^h \right\} \quad \text{and} \quad \theta_\ell^f(G) = \min_{h \in \widehat{G}_\ell^f} \left\{ \tilde{\theta}_\ell^h \right\}.
$$

Then, the fact that $g \succ_\ell f$ implies $\widehat{G}_\ell^f \subseteq \widehat{G}_\ell^f$, therefore

$$
\theta_\ell^f(G) = \min_{h \in \widehat{G}_\ell^f} \left\{ \tilde{\theta}_\ell^h \right\} \geq \min_{h \in \widehat{G}_\ell^f} \left\{ \tilde{\theta}_\ell^h \right\} = \theta_\ell^f(G),
$$

as required.

To prove the second part of the statement, we need to show that $g \in G \iff g \in C_\ell(G)$ under the additional assumption that $f \in C_\ell(G)$. By definition of a choice function, if $g \in C_\ell(G)$, then $g \in G$. It remains to show that $f \in C_\ell(G)$ and $g \in G$ imply that $g \in C_\ell(G)$. 
By definition of a choice function, $g \in C_\ell(G)$ whenever $\theta^g_\ell(G) \geq |\hat{G}^g_\ell| + 1$ or, equivalently, whenever $\theta^g_\ell(G) - |\hat{G}^g_\ell| \geq 1$. We have already established in the first part of the statement that $\theta^g_\ell(G) \geq \theta^f_\ell(G)$. Additionally, since $g \succ f$ and $g \in G$, we have that $\hat{G}^g_\ell \subset \hat{G}^f_\ell$, and therefore $|\hat{G}^g_\ell| < |\hat{G}^f_\ell|$. Taken together, these results imply that

$$\theta^g_\ell(G) - |\hat{G}^g_\ell| > \theta^f_\ell(G) - |\hat{G}^f_\ell|.$$  

Since $f \in C_\ell(G)$ by assumption, we have that $\theta^f_\ell(G) \geq |\hat{G}^f_\ell| + 1$ (by definition of a choice function). Combining $\theta^f_\ell(G) - |\hat{G}^f_\ell| \geq 1$ with (18) yields

$$\theta^g_\ell(G) - |\hat{G}^g_\ell| > \theta^f_\ell(G) - |\hat{G}^f_\ell| \geq 1.$$  

We can conclude that $\theta^g_\ell(G) > |\hat{G}^g_\ell| + 1$. By definition of a choice function, we have that $g \in C_\ell(G)$, which completes the proof of the Lemma A.1.  

Appendix B. Pareto Improvement from a Pareto-inefficient Endowment

Proposition B.1. There is no strategy-proof mechanism that Pareto improves upon every endowment that is not Pareto-efficient.

Proof. The proof is by counterexample. There are four families, three localities, and one dimension. The endowment is

\[ \mu^E = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_2 & \ell_3 & \ell_4 \end{pmatrix}. \]

The preferences of families are as follows:

\[ \succ_{f_1} : \ell_3, \ell_4, \ell_2, \ldots \quad \succ_{f_2} : \ell_4, \ell_3, \ell_2, \ldots \quad \succ_{f_3} : \ell_2, \ell_3, \ldots \quad \succ_{f_4} : \ell_2, \ell_4, \ldots, \]

where a family’s endowment locality is denoted in boldface. The family sizes and locality capacities are

\[ \nu = d_1 \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ 1 & 1 & 2 & 2 \end{pmatrix} \quad \kappa = d_1 \begin{pmatrix} \ell_2 & \ell_3 & \ell_4 \\ 2 & 2 & 2 \end{pmatrix}. \]

For ease of exposition, we illustrate this counterexample in Figure B.1.

We show that there are exactly two matchings that Pareto dominate \( \mu^E \):

\[ \mu = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_3 & \ell_3 & \ell_2 & \ell_4 \end{pmatrix} \quad \text{and} \quad \mu' = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_4 & \ell_4 & \ell_3 & \ell_2 \end{pmatrix}. \]

Matching \( \mu \) (resp. \( \mu' \)) Pareto dominates \( \mu^E \) since it matches family \( f_4 \) (resp. \( f_3 \)) to its endowment and makes the other three families better off. We now show that no other matching Pareto dominates \( \mu^E \). Towards a contradiction, suppose there exists a matching \( \tilde{\mu} \) that Pareto dominates \( \mu^E \) and is different from both \( \mu \) and \( \mu' \). Consider the case where \( \tilde{\mu}(f_3) \succ_{f_3} \mu^E(f_3) \). Then we must have that \( \tilde{\mu}(f_3) = \ell_2 \). Therefore, locality \( \ell_2 \) cannot
accommodate any other family so \( \tilde{\mu}(f_1) = \ell_4 \). In turn, locality \( \ell_4 \) cannot accommodate any other family so \( \tilde{\mu}(f_1) = \tilde{\mu}(f_2) = \ell_3 \) and \( \tilde{\mu} = \mu \). Analogously, we have that \( \tilde{\mu}(f_4) \succ_{f_4} \mu^E(f_4) \) implies \( \tilde{\mu} = \mu' \). Therefore, \( \tilde{\mu}(f_3) = \ell_3 \) and \( \tilde{\mu}(f_4) = \ell_4 \). Hence, neither \( \ell_3 \) nor \( \ell_4 \) can accommodate another family so \( \tilde{\mu}(f_1) = \tilde{\mu}(f_2) = \ell_2 \) and \( \tilde{\mu} = \mu^E \), a contradiction.

We have established that \( \mu \) and \( \mu' \) are the only two matchings that Pareto dominate \( \mu^E \) with respect to the true preferences. We now consider a preference manipulation by \( f_1 \). Suppose that \( f_1 \) misreports its true preferences by ranking \( \ell_4 \) below \( \ell_2 \), i.e., \( f_1 \) reports \( \succ'_{f_1} : \ell_3, \ell_2, \ldots \). We claim that if all other families report truthfully, then \( \mu \) is the only matching that Pareto dominates \( \mu^E \) with respect to the manipulated preference profile \( (\succ'_{f_1}, \succ_{-f_1}) \). Suppose again towards a contradiction that there exists a matching \( \tilde{\mu} \) that Pareto dominates \( \mu^E \) and is different from \( \mu \). There are three cases: First, if \( \tilde{\mu}(f_3) \succ_{f_3} \mu^E(f_3) \), then \( \tilde{\mu}(f_3) = \ell_2 \) and \( \tilde{\mu}(f_4) = \ell_4 \). Therefore, we must have that \( \tilde{\mu}(f_1) = \tilde{\mu}(f_2) = \ell_3 \) so \( \tilde{\mu} = \mu \), a contradiction. Second, if \( \tilde{\mu}(f_4) \succ_{f_4} \mu^E(f_4) \), then \( \tilde{\mu}(f_4) = \ell_2 \) and \( \tilde{\mu}(f_3) = \ell_3 \). Since neither \( \ell_2 \) nor \( \ell_3 \) can accommodate another family, \( f_1 \) must be matched to a less preferred locality than \( \mu^E(f_1) = \ell_2 \) (according to \( f_1 \)'s manipulated report) hence \( \tilde{\mu} \) does not Pareto dominate \( \mu^E \), a contradiction. Third, if \( \tilde{\mu}(f_3) = \ell_3 \) and \( \tilde{\mu}(f_4) = \ell_4 \), then \( \tilde{\mu} = \mu^E \), a contradiction. We therefore conclude that there is no matching that Pareto dominates \( \mu^E \) for the preference profile \( (\succ'_{f_1}, \succ_{-f_1}) \) and that is different from \( \mu \). By analogous reasoning, one can verify that if \( f_2 \) manipulates its preferences by reporting \( \succ'_{f_2} : \ell_4, \ell_2, \ldots \) and all other families report truthfully, then \( \mu' \) is the only matching that Pareto dominates \( \mu^E \) for the preference profile \( (\succ'_{f_2}, \succ_{-f_2}) \).

We can now show that there is no strategy-proof mechanism that Pareto improves upon \( \mu^E \). Let \( \varphi \) be a mechanism that Pareto improves upon \( \mu^E \). If all families report their preferences truthfully, then either \( \varphi(\succ) = \mu \) or \( \varphi(\succ) = \mu' \) because \( \mu \) and \( \mu' \) are the only two matchings that Pareto dominate \( \mu^E \) for the true preference profile. If \( f_1 \) reports \( \succ'_{f_1} \), then \( \varphi(\succ'_{f_1}, \succ_{-f_1}) = \mu \). Similarly, if \( f_2 \) reports \( \succ'_{f_2} \), then \( \varphi(\succ'_{f_2}, \succ_{-f_2}) = \mu' \). If \( \varphi(\succ) = \mu \), then \( \varphi(\succ'_{f_2}, \succ_{-f_2})(f_2) \succ_{f_2} \varphi(\succ)(f_2) \) but if \( \varphi(\succ) = \mu' \), then \( \varphi(\succ'_{f_1}, \succ_{-f_1})(f_1) \succ_{f_1} \varphi(\succ)(f_1) \). Therefore, \( \varphi \) is not strategy-proof. \( \square \)
In this online appendix, we expand on Section 5.1 and detail how interference-freeness relates to existing solution concepts for respecting priorities: envy-freeness and stability. We discuss the logical relationship among the three solution concepts in Online Appendix C.1; the relationship between interference-freeness and envy-freeness in the special case where there is only one dimension in Online Appendix C.2; and the existence of stable matchings in Online Appendix C.3. Online Appendix C.4 contains the proofs of the results presented.

C.1. Logical relationships between solution concepts. We begin by formally defining envy-freeness (in the sense of elimination of justified envy).

Definition C.1. Given a matching $\mu$, $f \in F$ envies family $f' \neq f$ if

\begin{enumerate}[(i)]
  \item $f$ prefers $f'$’s locality to its current match, i.e., $\mu(f') \succeq_f \mu(f)$; and
  \item $f$ has a higher priority at $f'$’s locality, i.e., $f \triangleright_{\mu(f')} f'$.
\end{enumerate}

A matching $\mu$ is envy-free if, under $\mu$, no family envies another family.

Like interference-free matchings, envy-free matchings respect priorities but may be wasteful. However, the criterion is more extreme because it precludes any priority violation: if a family prefers a locality to its own, then only lower-priority family may be matched to that locality. In contrast, interference-freeness allows priority violations that are innocuous in the sense that the envying family does not have a claim to any unit used by the envied family. Therefore, if a family misses out on a more preferred locality, lower-priority families are not to blame. Therefore, interference-freeness still constitutes a strong criterion when it comes to respecting priorities; however, the relaxation it provides allows making families better off and reducing waste. We illustrate with an example.

Example C.1 (Inadequacy of envy-freeness in matching with multidimensional constraints). There are three families $f_1$, $f_2$, and $f_3$ and one locality $\ell_1$. The priority list of $\ell_1$ is $f_1 \triangleright_{\ell_1} f_2 \triangleright_{\ell_1} f_3$. There are two dimensions and the sizes and capacities are displayed below:

\[
\nu = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \kappa = \begin{pmatrix} \ell_1 \\ \ell_1 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

The most efficient envy-free matching assigns $f_1$ to $\ell_1$ and leaves the other two families unmatched.\(^{20}\) Envy-freeness precludes assigning $f_3$ to $\ell_1$ because $f_2$ would then envy $f_3$. However, $f_2$ and $f_3$ are not really in competition since the units they require are in different dimensions. It is $f_1$ that prevents matching $f_2$ to $\ell_1$, not $f_3$. Therefore, the matching in which

\(^{20}\)If $f_1$ is unmatched, matching one or both of the other families to $\ell_1$ would violate $f_1$’s priority; therefore, the only other envy-free matching leaves all three families unmatched.
both \( f_1 \) and \( f_3 \) are matched to \( \ell_1 \) appears to be a natural solution: it is more efficient and has no meaningful priority violations. That matching is interference-free since \( \ell_1 \) can weakly accommodate \( f_3 \) alongside \( f_1 \) and \( f_2 \); in other words, the unit required by \( f_3 \) is claimed by neither \( f_1 \) nor \( f_2 \).

As interference-freeness allows for (innocuous) priority violation and envy-freeness does not, it is natural to think of interference-freeness as a relaxation of envy-freeness. Our first result formalizes this intuition.

**Proposition C.1.** Every envy-free matching is interference-free.

We next turn to stability and begin by formally defining it in our model.

**Definition C.2.** A matching \( \mu \) is (pairwise) stable if there are no \( f \in F \) and \( \ell \in L \) such that

1. \( f \) prefers \( \ell \) to its current match, i.e., \( \ell \succ_f \mu(f) \); and
2. \( \ell \) can accommodate \( f \) alongside \( \hat{F}_\ell \cap \mu(\ell) \), i.e., all families matched to \( \ell \) with a higher priority than \( f \).

In words, family \( f \) and locality \( \ell \) are a blocking pair under a matching \( \mu \) if \( f \) prefers \( \ell \) to its current match and it is possible to accommodate \( f \) in \( \ell \) without removing any higher-priority family. A matching is stable if it does not have any blocking pairs. Our definition extends the concept of stability to a setting with possibly multidimensional knapsack constraints.\(^{21}\) If \(|D| = 1\) and \( \nu_{d_1}^f = 1 \) for all \( f \in F \), Definition C.2 collapses to the “elimination of justified envy” used in school choice and other object allocation settings (Abdulkadiroğlu and Sönmez, 2003).

Envy-freeness and stability are logically independent: Envy-freeness allows waste, which stability precludes; however, stability allows for some waste-eliminating priority violations. By Proposition C.1, it follows that stability does not imply interference-freeness. As interference-freeness allows waste but stability does not, we conclude that interference-freeness and stability are also logically independent. In contrast, stability only fails due to the presence of waste or via a priority violation so non-wasteful and envy-free matchings are always stable.\(^{22}\)

Perhaps surprisingly, although interference-freeness allows for some priority violations, we can establish an analogous relationship among stability, non-wastefulness, and interference-freeness.

**Proposition C.2.** If a matching is non-wasteful and interference-free, then it is stable.

\(^{21}\)Our definition is in line with the way stability is defined in similar models (see, e.g., McDermid and Manlove (2010); Biró and McDermid (2014); Delacrétaz (2019)).

\(^{22}\)Kamada and Kojima (forthcoming) define stability to be the combination of envy-freeness and non-wastefulness, making their definition more restrictive than ours.
Proposition C.2, combined with the possible nonexistence of a stable matching (see Online Appendix C.3), formally establishes that an interference-free and non-wasteful matching may not exist. Figure C.2 summarizes the relationships among the solution concepts.

We present a small example to illustrate how stability and interference-freeness differ.

**Example C.2.** There are three families $f_1$, $f_2$, and $f_3$, one locality $\ell$ such that $f_1 \triangleright_{\ell} f_2 \triangleright_{\ell} f_3$, and one dimension $d$ such that $\nu_{f_1}^d = 1$, $\nu_{f_2}^d = 2$, $\nu_{f_3}^d = 1$, and $\kappa_{\ell}^d = 2$.

The family-optimal interference-free matching assigns $f_1$ to $\ell$ and the other two families to the null. That matching is not stable since $f_3$ would prefer to be matched to $\ell$ and one unit of $\ell$ is not used. In fact, the unique stable matching assigns both $f_1$ and $f_3$ to $\ell$ (and $f_2$ to the null), but $f_3$ interferes with this matching so it is not interference-free. The example illustrates that a family can interfere even when higher-priority families could not make use of the capacity it uses: $f_3$ interferes with the unique stable matching because the unit it uses can be claimed by $f_2$, even though $f_2$ could not actually use that unit since $\ell$ cannot accommodate $f_2$ alongside $f_1$.

**C.2. Relationship between interference-freeness and envy-freeness with one dimension.** Example C.1 shows that the family-optimal interference-free matching—which assigns $f_1$ and $f_3$ to $\ell_1$—may not be envy-free, and may therefore strictly dominate the family-optimal envy-free matching—which only assigns $f_1$ to $\ell_1$. However, because Example C.1 has two dimensions, it leaves open the question of whether there exists a tighter relationship when $|D| = 1$. The following example shows that interference-freeness constitutes a relaxation of envy-freeness even when there is only one dimension.

**Example C.3.** There are two families $f_1$ and $f_2$, one locality $\ell$ such that $f_1 \triangleright_{\ell} f_2$, and one dimension $d$ such that $\nu_{f_1}^d = \nu_{f_2}^d = 1$ and $\kappa_{\ell}^d = 2$. There are four possible matchings:

\[
\mu_1 = \begin{pmatrix} f_1 & f_2 \\ \ell_1 & \ell_2 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} f_1 & f_2 \\ \ell_1 & \emptyset \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} f_1 & f_2 \\ \emptyset & \ell_2 \end{pmatrix}, \quad \mu_4 = \begin{pmatrix} f_1 & f_2 \\ \emptyset & \emptyset \end{pmatrix}.
\]
Matchings $\mu_1$, $\mu_2$, and $\mu_4$ are envy-free but matching $\mu_3$ is not because $f_1$ envies $f_2$. However, $\mu_3$ is interference-free because $\ell$ can accommodate both $f_1$ and $f_2$; hence $f_1$ does not have a claim to the unit of capacity used by $f_2$.

In Example C.3, the family-optimal envy-free and interference-free matchings are identical and match both $f_1$ and $f_2$ to $\ell$. As our next result formalizes, this is not a coincidence.

**Proposition C.3.** If $|D| = 1$, then the KDA and TKDA mechanisms are envy-free.

As the KDA mechanism outputs the family-optimal interference-free matching (Theorem 5) and every interference-free matching is envy-free (Proposition C.1), Proposition C.3 implies the following corollary.

**Corollary C.1.** If $|D| = 1$, the family-optimal envy-free and interference-free matchings coincide.

Taken together, our results show that interference-freeness constitutes a relaxation of envy-freeness even when there is only one dimension; however, that relaxation does not have practical consequences since in that case, both our mechanisms output envy-free matchings. In contrast, using interference-freeness rather than envy-freeness matters when there are multiple dimensions. In Example C.1, both the KDA and TKDA algorithm match $f_1$ and $f_3$ to $\ell_1$, which envy-freeness would preclude. More generally, our simulation results suggest that the efficiency gains associated with using interference-freeness instead of envy-freeness as a criterion for respecting priorities can be large (see Figure F.2).

C.3. (Non-)existence of stable matchings. While stable matchings always exist in school choice models, they do not exist in ours (even when $|D| = 1$). In fact, determining whether a stable matching exists in our model (even when $|D| = 1$) is a computationally intractable problem (McDermid and Manlove, 2010). We first reproduce an example from McDermid and Manlove (2010) in which no stable matching exists and then provide a condition on the priority profile for the existence of a stable matching.

**Example C.4** (McDermid and Manlove (2010)). There are three families, two localities, and one dimension. The family sizes and locality capacities are

\[
\nu = d_1 \begin{pmatrix} f_1 & f_2 & f_3 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \kappa = d_1 \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix}.
\]

The preferences and priorities are:

\[
\succ_{f_1}: \ell_2, \ell_1, \emptyset \quad \succ_{f_2}: \ell_1, \ell_2, \emptyset \quad \succ_{f_3}: \ell_1, \emptyset, \ell_2 \quad \succ_{\ell_1}: f_1, f_3, f_2 \quad \succ_{\ell_2}: f_2, f_1, f_3.
\]

Suppose, towards a contradiction, that there exists a stable matching $\mu$ in this example. Since $\ell_2$ cannot accommodate $f_3$, either $\mu(f_3) = \ell_1$ or $\mu(f_3) = \emptyset$. If $\mu(f_3) = \ell_1$, then
\(\mu(f_2) = \ell_2\) as otherwise \(f_2\) and \(\ell_2\) form a blocking pair. Then, \(\mu(f_1) = \emptyset\) so \(f_1\) and \(\ell_1\) form a blocking pair and \(\mu\) is not stable. If \(\mu(f_3) = \emptyset\), then \(\mu(f_1) = \ell_1\) as otherwise \(f_3\) and \(\ell_1\) form a blocking pair. In turn, \(f_2\) and \(\ell_1\) form a blocking pair unless \(\mu(f_2) = \ell_1\). However, this matching is not stable since \(f_2\) and \(\ell_2\) form a blocking pair.

One way to understand why stable matchings might not exist is that the choice function induced by \(\ell_1\)'s priorities is not substitutable (Roth, 1984) because of a complementarity between \(f_1\) and \(f_2\). If only \(f_2\) and \(f_3\) compete for \(\ell_1\), stability dictates that \(f_3\) be accepted and \(f_2\) rejected. If \(f_1\) is also competing, stability dictates that \(f_1\) and \(f_2\) be accepted and \(f_3\) rejected. Thus, there is a complementarity between \(f_1\) and \(f_2\) in the sense that \(f_2\) is accepted when \(f_1\) is also considered but not otherwise.

We next provide a condition on the priority profile for the existence of a stable matching.

**Definition C.3.** A priority profile \(\triangleright\) is aligned if for any \(f, g \in F\) such that \(\nu^f \neq \nu^g\) and any \(\ell, \ell' \in L \setminus \{\emptyset\}\), \(f \triangleright_{\ell} g\) if and only if \(f \triangleright_{\ell'} g\).

The aligned priorities condition generalizes the second part in the definition of lexicographic priorities (Definition 4). Under the aligned priorities condition, any two families with different sizes are ranked identically by all localities, but families with the same size can be ranked arbitrarily. The case of identical priorities is therefore a special case of the aligned priorities condition. If sizes are monotonic, aligned priorities also include the case where all localities give a higher priority to larger families and, symmetrically, the case where all localities give a higher priority to smaller families.

**Proposition C.4.** If sizes are monotonic and the priority profile is aligned, then a stable matching exists.

The monotonicity of sizes and alignment of the priority profile ensures that the set of families can be partitioned into \(\{F_1, F_2, \ldots, F_n\}\) such that for any \(i = 1, \ldots, n\), all families in \(F_i\) have the same size and for all \(j < i\), all families in \(F_j\) have a higher priority at all localities than all families in \(F_i\). A stable matching then can be obtained in polynomial time by running sequentially the (family-proposing) Deferred Acceptance algorithm for each subset, starting with \(F_1\). In a school choice setting, all families have the same size (\(\nu^f = 1\) for all \(f \in F\)) and priority profile is (trivially) aligned; therefore the existence of stable matchings follows immediately from Proposition C.4. As in school choice, stable matchings under size monotonicity and priority alignment can be found in polynomial time in our model.

C.4. Proofs of results in Online Appendix C.

*Proof of Proposition C.1.* We prove the contrapositive: Consider a matching \(\mu\) that is not interference-free, we need to show that \(\mu\) is not envy-free. By definition, there exists a family
f such that $\mu(f)$ cannot weakly accommodate f alongside the set

$$G = \{g \in F : g \triangleright_{\mu(f)} f \text{ and } \mu(f) \succeq_g \mu(g)\}. $$

It follows that there exists a dimension $d$ such that $\nu_d^f + \sum_{g \in G} \nu_d^g > \kappa_d^f$. If $G \setminus \mu(f) = \emptyset$, then $\mu(f)$ cannot accommodate all families matched to it, a contradiction. Therefore, there exists a family $f' \in G \setminus \mu(f)$ and by definition $f' \triangleright_{\mu(f)} f$ and $\mu(f) \succeq_{f'} \mu(f')$. We conclude that $f'$ envies $f$, so $\mu$ is not envy-free.

Proof of Proposition C.2. We prove the contrapositive. Let $\mu$ be a matching that is non-wasteful and not stable. We need to show that $\mu$ is not interference-free. As $\mu$ is not stable, there exists a family $f$ and a locality $\ell$ such that $\ell \succ_f \mu(f)$ and $\ell$ can accommodate $f$ alongside $\widehat{F}_\ell^f \cap \mu(\ell)$; that is, for all $d \in D$,

$$\nu_d^f + \sum_{g \in (\widehat{F}_\ell^f \cap \mu(\ell))} \nu_d^g \leq \kappa_d^f. \tag{19}$$

However, as $\mu$ is non-wasteful, there exists a dimension $d' \in D$ such that

$$\nu_{d'}^f + \sum_{g \in \mu(\ell)} \nu_{d'}^g > \kappa_{d'}^f. \tag{20}$$

Consider now the families in $\mu(\ell) \setminus \widehat{F}_\ell^f$ whose size in dimension $d'$ is at least 1. Inequalities (19) and (20) imply that the set of such families is nonempty. Let $f'$ be the lowest-priority family in $\mu(\ell) \setminus \widehat{F}_\ell^f$ whose size in dimension $d'$ is at least 1 (equivalently, $f'$ is the lowest-priority family in $\mu(\ell)$ such that $\nu_{d'}^f > 0$). We show that $f'$ interferes with $\mu$; that is, we show that $\ell$ cannot weakly accommodate $f'$ alongside $\widehat{F}_\mu^{f'} = \{g \in F : g \triangleright_{\ell} f' \text{ and } \ell \succeq_g \mu(g)\}$. By the definition of $f'$, $\nu_{d'}^{f'} + \sum_{g \in (\widehat{F}_\ell^{f'} \cap \mu(\ell))} \nu_{d'}^g = \sum_{g \in \mu(\ell)} \nu_{d'}^g$, which combined with (20) implies that

$$\nu_{d'}^f + \nu_{d'}^{f'} + \sum_{g \in (\widehat{F}_\ell^{f'} \cap \mu(\ell))} \nu_{d'}^g = \nu_{d'}^{f'} + \sum_{g \in \mu(\ell)} \nu_{d'}^g > \kappa_{d'}^f. \tag{21}$$

Inequality (21) and the fact that $\nu_{d'}^{f'} > 0$ imply that $\ell$ cannot weakly accommodate $f'$ alongside $\{f\} \cup (\widehat{F}_\ell^{f'} \cap \mu(\ell))$. Note that all families in $\{f\} \cup (\widehat{F}_\ell^{f'} \cap \mu(\ell))$ have a higher priority at $\ell$ than $f'$ and weakly prefer $\ell$ to the locality to which they are matched, i.e.,

$$\{f\} \cup (\widehat{F}_\ell^{f'} \cap \mu(\ell)) \subseteq \{g \in F : g \triangleright \ell f' \text{ and } \ell \succeq_g \mu(g)\} = \widehat{F}_\mu^{f'},$$

so $\ell$ cannot weakly accommodate $f'$ alongside $\widehat{F}_\mu^{f'}$ and $\mu$ is not interference-free.

Proof of Proposition C.3. We denote by $d$ the unique dimension (i.e., $D = \{d\}$), and consider each mechanism separately.
Proof that the KDA mechanism is envy-free. Let \( N \) be the number of rounds after which the KDA algorithm ends and let \( \mu \) be the outcome of the KDA algorithm. We use a piece of notation that was introduced in the proof of Theorem 5. For any family \( f \in F \), any locality \( \ell \in L \), and any Round \( i = 1, \ldots, N \), \( \tilde{R}_i^f(\ell) \) denotes the set of families that have a higher priority than \( f \) at \( \ell \) and propose in Round \( i \) either to \( \ell \) or to a less-preferred locality.

Towards a contradiction, suppose that \( \mu \) is not envy-free. Then, by definition, there exist two families \( f \) and \( f' \) such that \( f \) envies \( f' \). Letting \( \ell' = \mu(f') \), we have that \( \ell' \succ_f \mu(f) \) and \( f \succ_{\ell'} f' \). By construction, there exists a Round \( i \) in which \( \ell' \) rejects \( f \); hence \( \ell' \) cannot weakly accommodate \( f \) alongside \( \tilde{R}_i^f(\ell') \). Again by construction, we have that \( \tilde{R}_i^f(\ell') \subseteq \tilde{R}_i^N(\ell') \) so we conclude that \( \ell' \) cannot weakly accommodate \( f \) alongside \( \tilde{R}_i^N(\ell') \).

As \( d \) is the only dimension, it must be that \( \nu_d^f > 0 \), and therefore the fact that \( \ell' \) cannot weakly accommodate \( f \) alongside \( \tilde{R}_i^f(\ell') \) implies by definition that \( \nu_d^f + \sum_{g \in \tilde{R}_i^f(\ell')} \nu_d^g > \kappa_d^\ell \). As \( f \succ f' \), it follows that \( \nu_d^{f'} + \sum_{g \in \tilde{R}_i^{f'}(\ell')} \nu_d^g > \kappa_d^\ell \). Therefore, \( \ell' \) cannot weakly accommodate \( f' \) alongside \( \tilde{R}_i^{f'}(\ell') \) and we conclude that in Round \( N \), either \( f' \) does not propose to \( \ell' \) or \( \ell' \) permanently rejects \( f' \); either conclusion contradicts the assumption that \( \ell' = \mu(f') \), completing the proof.

Proof that the TKDA mechanism is envy-free. Let \( N \) be the number of rounds after which the TKDA algorithm ends and let \( \mu \) be the outcome of the TKDA algorithm. We use two pieces of notation that were introduced in the proof of Theorem 6. First, for any Round \( i = 1, \ldots, N \) and any locality \( \ell \in L \), \( \Pi_i^\ell \) denotes set of families that propose to locality \( \ell \) in Round \( i \). Second, for any family \( f \in F \), any locality \( \ell \in L \), and any set of families \( G \subseteq F \), \( \theta_i^\ell(G) \) denotes \( f \)'s threshold for \( \ell \) (calculated by Algorithm 5) when the families in \( G \) propose to \( \ell \). Moreover, we will invoke Lemma A.1, which was also introduced in the proof of Theorem 6.

Towards a contradiction, suppose that \( \mu \) is not envy-free. Then, by definition, there exist two families \( f \) and \( f' \) such that \( f \) envies \( f' \). Without loss of generality, let \( f \) be the highest-priority family among those that envy \( f' \). Formally, letting \( \ell' = \mu(f') \), we have that \( \ell' \succ_f \mu(f) \), that \( f \succ_{\ell'} f' \), and that for any \( \hat{f} \neq f \) such that \( \ell' \succ_j \mu(\hat{f}) \), \( f \succ_{\ell'} \hat{f} \). As \( \ell' \succ_f \mu(f) \), by construction there exists a Round \( i \) in which \( \ell' \) permanently rejects \( f \) so we have that

\[
\theta_i^{\ell'}(\Pi_i^{\ell'}) < |\Pi_i^{\ell'} \cap \tilde{F}_i^{\ell'}| + 1. \tag{22}
\]

The next step consists of showing that the following inequality holds:

\[
\theta_i^{\ell'}(\Pi_i^N) < |\Pi_i^{\ell'} \cap \tilde{F}_i^{\ell'}| + 1. \tag{23}
\]

If \( \theta_i^{\ell'}(\Pi_i^N) = \infty \), then by Lemma A.1(i), \( \theta_i^{\ell'}(\Pi_i^{\ell'}) = \infty \), which contradicts (22) and, if \( \theta_i^{\ell'}(\Pi_i^N) = 0 \), (23) holds trivially. Therefore, it remains to show that (23) holds in the case where \( \theta_i^{\ell'}(\Pi_i^N) \in \mathbb{Z}_{>0} \). As \( f \) has the highest-priority among the families that envy \( f' \), by
construction \( f \) is also the highest-priority family that \( \ell' \) permanently rejects throughout the TKDA algorithm; hence we have that \( \Pi_{i}^{f} \cap \widehat{F}_{\ell'}^{f} \subseteq \Pi_{i}^{N} \cap \widehat{F}_{\ell'}^{f} \). By Lemma A.1(ii), it follows that

\[
\theta_{\ell'}^{f}(\Pi_{i}^{N} \cap \widehat{F}_{\ell'}^{f}) \leq \theta_{\ell'}^{f}(\Pi_{i}^{f} \cap \widehat{F}_{\ell'}^{f}) + |\Pi_{i}^{N} \cap \widehat{F}_{\ell'}^{f}| - |\Pi_{i}^{f} \cap \widehat{F}_{\ell'}^{f}|.
\]

By construction (see Algorithm 5), \( f \)'s threshold at \( \ell' \) only depends on higher-priority families so \( \theta_{\ell'}^{f}(\Pi_{i}^{f} \cap \widehat{F}_{\ell'}^{f}) = \theta_{\ell'}^{f}(\Pi_{i}^{N}) \) and \( \theta_{\ell'}^{f}(\Pi_{i}^{f} \cap \widehat{F}_{\ell'}^{f}) = \theta_{\ell'}^{f}(\Pi_{i}^{f}) \); hence we have that

\[
\theta_{\ell'}^{f}(\Pi_{i}^{N}) \leq \theta_{\ell'}^{f}(\Pi_{i}^{f}) + |\Pi_{i}^{N} \cap \widehat{F}_{\ell'}^{f}| - |\Pi_{i}^{f} \cap \widehat{F}_{\ell'}^{f}|,
\]

which, combined with (22), implies (23).

Having established that inequality (23) holds, we now use it to reach the contradiction. We consider separately two cases: \( \theta_{\ell'}^{f}(\Pi_{i}^{N}) \neq \infty \) and \( \theta_{\ell'}^{f}(\Pi_{i}^{N}) = \infty \).

**Case 1:** \( \theta_{\ell'}^{f}(\Pi_{i}^{N}) \neq \infty \). As \( f \triangleright_{\ell'} f' \), we have that \( \Pi_{i}^{N} \cap \widehat{F}_{\ell'}^{f} \subseteq \Pi_{i}^{N} \cap \widehat{F}_{\ell'}^{f} \) and, by Lemma A.1(iii), \( \theta_{\ell'}^{f}(\Pi_{i}^{N}) \leq \theta_{\ell'}^{f}(\Pi_{i}^{N}) \). Then, (23) implies that \( \theta_{\ell'}^{f}(\Pi_{i}^{f}) < |\Pi_{i}^{N} \cap \widehat{F}_{\ell'}^{f}| + 1 \). It follows that, in Round \( N \), either \( f' \) does not propose to \( \ell' \) or \( \ell' \) permanently rejects \( f' \), both of which contradict the assumption that \( \ell' = \mu(f') \).

**Case 2:** \( \theta_{\ell'}^{f}(\Pi_{i}^{N}) = \infty \). By definition (see Algorithm 5), the case assumption implies that \( \ell' \) can weakly accommodate \( f' \) alongside \( \widehat{F}_{\ell'}^{f} \). As \( d \) is the only dimension, we have that \( \nu_{d}^{f'} > 0 \); by definition: \( \nu_{d}^{f'} + \sum_{g \in \widehat{F}_{\ell'}^{f \setminus d}} \nu_{d}^{g} \leq \kappa_{d}^{f'} \). It follows that \( \nu_{d}^{f} + \sum_{g \in \widehat{F}_{\ell'}^{f \setminus d}} \nu_{d}^{g} < \kappa_{d}^{f} \); hence \( \ell' \) can (weakly) accommodate \( f \) alongside \( \widehat{F}_{\ell'}^{f} \). By definition (see Algorithm 5), we conclude that \( \theta_{\ell'}^{f}(\Pi_{i}^{N}) = \infty \), which contradicts (23). \( \square \)

**Proof of Proposition C.4.** Suppose that sizes are monotonic and the priority profile is aligned. We introduce the Sequential Deferred Acceptance (SDA) algorithm (Algorithm C.1) and show that it produces a stable matching. We show that the alignment of the priority profile allows us to divide the families into groups such that (i) all families in the same group have the same size and (ii) any two groups can be compared in terms of priority in the sense that all families in one group have a higher priority at all localities than all families in the other group. The SDA algorithm considers one of these groups at a time in order of priority and runs the family-proposing Deferred Acceptance (DA) algorithm for the families in that group, considering the capacities that remain after families in higher-priority groups have been permanently matched.

We first show that the SDA algorithm ends after finitely many rounds. If there is a directed cycle \( (f_1, f_2, \ldots, f_n) \) in graph \( \mathbb{G}^1 \), then, for every \( \ell \in L \setminus \{\emptyset\} \), \( f_1 \triangleright_{\ell} f_2 \triangleright_{\ell} \ldots \triangleright_{\ell} f_n \triangleright_{\ell} f_1 \), a contradiction. Therefore, \( \mathbb{G}^1 \) is a directed acyclic graph. By construction, for all \( i > 1 \), \( \mathbb{G}^i \) is a directed acyclic graph since \( \mathbb{G}^i \) is constructed from \( \mathbb{G}^{i-1} \) by removing some vertices (families) and edges. Therefore, in every Round \( i \), the set of families \( \widehat{F}^i \) at which no family is pointing is nonempty. It follows that at least one family gets permanently matched in each round; hence the algorithm ends after at most \( |F| \) rounds.
Algorithm C.1. Sequential Deferred Acceptance

Construct a directed graph $G^i$ as follows. Each of the $|F|$ vertices represents a family. For each pair of families $(f, f')$, let there be a directed edge from $f$ to $f'$ if $\nu^f \neq \nu^{f'}$ and $f \succ_{\ell} f'$ for all $\ell \in L$.

For each locality $\ell$, set a counter $c^i_\ell = \kappa^\ell$.

**Round $i \geq 1$**

Let $\tilde{F}^i$ be the set of families at which no other family is pointing in graph $G^i$.

Permanently match the families in $\tilde{F}^i$ to the localities using the family-proposing Deferred Acceptance algorithm setting the capacity of each locality $\ell$ to its counter $c^i_\ell$. If all families have been permanently matched, end.

Otherwise, construct $G^{i+1}$ by removing from $G^i$ all vertices representing families in $\tilde{F}^i$ and all edges adjacent to them. For each locality $\ell \in L$, let $\tilde{F}^i_\ell$ be the set of families that have been permanently matched to $\ell$ in Round $i$. Update the counter of each locality $\ell$ as follows: $c^{i+1}_\ell = c^i_\ell - \sum_{f \in \tilde{F}^i_\ell} \nu^f$. Continue to Round $i + 1$.

We now complete the proof by showing that the matching $\mu^{\text{SDA}}$ produced by the SDA algorithm is stable. Consider any family $f$ and any locality $\ell$ such that $\ell \succ_f \mu^{\text{SDA}}(f)$. We need to show that $\ell$ cannot accommodate $f$ alongside $\tilde{F}^i_\ell \cap \mu^{\text{SDA}}(\ell)$.

Let Round $i$ be the round in which family $f$ is permanently matched to $\mu^{\text{SDA}}(f)$, i.e., $f \in \tilde{F}^i$. Consider the Round $i$ “submarket,” in which the DA algorithm permanently matches the families in $\tilde{F}^i$ to the localities and the counter of each locality $\ell$ is $c^i_\ell$. By construction, all families in the Round $i$ submarket have the same size, i.e., $\nu^{f'} = \nu^f$ for all $f' \in \tilde{F}^i$. (If this were not the case, then, as the priority profile is aligned, one of $f$ or $f'$ would have a higher priority at all localities and would point at the other, a contradiction.) Therefore, for each locality, there is a maximum number of families that the locality can accommodate, which makes the submarket isomorphic to a school choice problem. As the DA algorithm produces a stable matching in the school choice problem (Abdulkadiroğlu and Sönmez, 2003), the DA algorithm produces a stable matching in the Round $i$ submarket. Therefore, $f$ and $\ell$ do not form a blocking pair in the Round $i$ submarket. As $\ell \succ_f \mu^{\text{SDA}}(f)$ by assumption, it follows that $\ell$’s Round $i$ counter does not allow $\ell$ to accommodate $f$ alongside higher-priority families permanently matched to $\ell$ in Round $i$, i.e., alongside $\tilde{F}^i_\ell \cap \tilde{F}^i_j$. By construction, all families that have been permanently matched to $\ell$ before Round $i$, i.e., all families in $\cup_{j=1}^{i-1} \tilde{F}^j_j$, have a higher priority at $\ell$ than $f$. Hence, $\ell$ cannot accommodate $f$ alongside all families that have a higher priority than $f$ and with which $\ell$ has been permanently matched by the end of Round $i$, i.e., $\ell$ cannot accommodate $f$ alongside $\tilde{F}^i_\ell \cap (\cup_{j=1}^{i-1} \tilde{F}^j_j)$. By construction, all families that are permanently matched to $\ell$ in any round of the SDA algorithm are matched to $\ell$ at $\mu^{\text{SDA}}$, which implies that $\cup_{j=1}^{i-1} \tilde{F}^j_j \subseteq \mu^{\text{SDA}}(\ell)$. We therefore conclude that $\ell$ cannot accommodate $f$ alongside $\tilde{F}^i_\ell \cap \mu^{\text{SDA}}(\ell)$, as required. \qed
Appendix D. Efficiency of the TKDA Algorithm

In Online Appendix D.1, we show that the TKDA algorithm is more efficient than any envy-free mechanism that satisfies the cardinal monotonicity condition. A similar result replacing envy-freeness interference-freeness holds when $|D| = 1$. In Online Appendix D.2, we provide a lower bound for the efficiency of the KDA and TKDA algorithms. We introduce a solution to improve the efficiency of the TKDA algorithm in Online Appendix D.3 and show in Online Appendix D.4 that it cannot be generalized. All proofs are in Online Appendix D.5.

D.1. TKDA and cardinal monotonicity. As noted in Section 5.2, a key condition for a deferred-acceptance-type procedure to be strategy-proof is that its associated choice function satisfies the cardinal monotonicity condition. In this section, we show that the choice function of the TKDA algorithm tentatively accepts every family that is tentatively accepted by any choice function satisfying envy-freeness and cardinal monotonicity conditions. When $|D| = 1$, we show that the choice function of the TKDA algorithm maximizes the number of tentatively accepted families among choice functions that satisfy the interference-freeness and cardinal monotonicity conditions. When $|D| > 1$, we argue that the problem of maximizing the number of tentatively accepted families subject to the interference-freeness and cardinal monotonicity conditions is computationally intractable.

Fixing a locality $\ell$, recall from the proof of Theorem 6 that a choice function $C_\ell : F \rightarrow F$ determines, for any subset $G \subseteq F$ of proposition families, which families the locality tentatively accepts ($C_\ell(G)$) or permanently rejects ($G \setminus C_\ell(G)$). We now define four properties of a choice function:

1. $C_\ell$ is feasible if, for every $G \subseteq F$, $\ell$ can accommodate $C_\ell(G)$;
2. $C_\ell$ is envy-free if, for every $G \subseteq F$, every family in $C_\ell(G)$ has a higher priority than any family in $G \setminus C_\ell(G)$;
3. $C_\ell$ is interference-free if, for every $G \subseteq F$, $\ell$ can weakly accommodate any family $f \in C_\ell(G)$ alongside $G \cap \hat{F}_\ell$;
4. $C_\ell$ satisfies the cardinal monotonicity condition if $|C_\ell(G)| \leq |C_\ell(H)|$ for every $G \subseteq H \subseteq F$.

Denote by $C^T_\ell$ the choice function of any locality $\ell$ in the TKDA algorithm: For each family, a threshold is calculated by Algorithm 5 and the family is tentatively accepted if its threshold does not exceed its priority rank among proposing families, otherwise the family is permanently rejected. We first assess that choice function against the four properties previously defined.

**Proposition D.1.** The choice function $C^T_\ell$ is feasible, interference-free, and satisfies the cardinal monotonicity condition. Moreover, if $|D| = 1$, then $C^T_\ell$ is envy-free.
Proposition D.1 is closely related to the properties of the TKDA mechanism (Theorem 6 and Proposition C.3). We now compare \( C^T_\ell \) to other choice functions that have similar properties.

**Proposition D.2.** For any feasible and envy-free choice function \( C_\ell \) that satisfies the cardinal monotonicity condition, and for any subset of families \( G \subseteq F \), we have that \( C_\ell(G) \subseteq C^T_\ell(G) \).

Proposition D.2 provides a lower bound for the efficiency of the TKDA mechanism as its choice function is at least as efficient as any envy-free choice function that satisfies the cardinal monotonicity condition: any family that is accepted by such a choice function is accepted by the TKDA choice function. Moreover, the TKDA mechanism can do even better by allowing families that can be weakly accommodated alongside all higher-priority families to be tentatively accepted even when some higher-priority families are permanently rejected.

However, Proposition D.2 leaves open the question of what can be said about feasible and interference-free choice functions that satisfy the cardinal monotonicity condition. Our next result provides an answer for the case where there is only one dimension.

**Proposition D.3.** If \(|D| = 1\), then for any feasible and interference-free choice function \( C_\ell \) that satisfies the cardinal monotonicity condition, and for any subset of families \( G \subseteq F \), we have that \(|C_\ell(G)| \leq |C^T_\ell(G)|\). Moreover, every family in \( C^T_\ell(G) \setminus C_\ell(G) \) has a higher priority than any family in \( C_\ell(G) \setminus C^T_\ell(G) \).

When \(|D| = 1\), the TKDA choice function is envy-free (Proposition D.1) and accepts the highest-priority families until accepting the next family would violate either the locality’s capacity or the cardinal monotonicity condition. Other feasible interference-free choice functions satisfying the cardinal monotonicity condition might not accept a larger number of families but might accept lower-priority families. The next example illustrates this.

**Example D.1.** There are four families \( f_1, f_2, f_3, \) and \( f_4 \) and one locality \( \ell \). The priority list of \( \ell \) is \( f_1 \triangleright_\ell f_2 \triangleright_\ell f_3 \triangleright_\ell f_4 \). There is one dimension and the sizes and capacities are displayed below:

\[
\nu = d_1 \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ 2 & 1 & 1 & 1 \end{pmatrix} \quad \kappa = d_1 \begin{pmatrix} \ell \\ 3 \end{pmatrix}.
\]

One can verify that \( C^T_\ell(\{f_2, f_3, f_4\}) = \{f_2, f_3\} \): the rank of \( f_2 \) is \( \infty \) while the rank of both \( f_3 \) and \( f_4 \) is 2. Therefore, \( f_4 \) is rejected even though \( \ell \) can accommodate all three families. This occurs because of the cardinal monotonicity condition: when all four families propose, only \( f_1 \) and \( f_2 \) can be accepted; therefore, the cardinal monotonicity condition dictates that at most two families be accepted when \( f_2, f_3, \) and \( f_4 \) propose. Then, any feasible and interference-free choice function that satisfies the cardinal monotonicity condition can accept
at most two families, i.e., no more than the TKDA choice function. However, not every such choice function needs to accept the same two families: $f_4$ could be accepted instead of either $f_2$ or $f_3$. □

Proposition D.3 provides a strong efficiency result for the TKDA mechanism when there is only one dimension: its choice function cannot accept more families without violating the interference-freeness or cardinal monotonicity property. When there are multiple dimensions, Proposition D.2 continues to provide a lower bound for the efficiency of the TKDA mechanism but Proposition D.3 breaks down: there may be feasible and interference-free choice functions that satisfy the cardinal monotonicity condition and accept more families than $C^T_\ell$.

**Example D.2.** There are four families $f_1$, $f_2$, $f_3$, and $f_4$ and one locality $\ell$. The priority list of $\ell$ is $f_1 \triangleleft_\ell f_2 \triangleleft_\ell f_3 \triangleleft_\ell f_4$. There are two dimensions and the sizes and capacities are displayed below:

\[
\nu = d_1 \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \kappa = d_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]

Locality $\ell$ cannot accommodate $\{f_2, f_3\}$; therefore, when $f_3$ and $f_4$ propose, $f_3$ and $f_4$’s thresholds are 1 so $C^T_\ell(\{f_3, f_4\}) = \{f_3\}$. When $f_2$ proposes alongside $f_3$ and $f_4$, $f_3$ and $f_4$’s thresholds are 0 so $C^T_\ell(\{f_2, f_3, f_4\}) = \{f_2\}$. Consider the alternative choice function $C_\ell$ such that $C_\ell(\{f_3, f_4\}) = \{f_3, f_4\}$, $C_\ell(\{f_2, f_3, f_4\}) = \{f_2, f_4\}$, and $C_\ell(G) = C^T_\ell(G)$ for every other subset of families $G$. First, $C_\ell$ is feasible since $\ell$ can accommodate $\{f_3, f_4\}$ and $\{f_2, f_4\}$. Second, $C_\ell$ is interference-free as $\nu^{f_4}_d = 0 < \nu^{f_3}_d$ and $\ell$ can weakly accommodate $f_4$ alongside $\{f_2, f_3\}$ (hence alongside $\{f_3\}$). Third, $C_\ell$ satisfies the cardinal monotonicity condition since $|C_\ell(\{f_1, f_3, f_4\})| = |\{f_1, f_3\}| = 2$ and $|C_\ell(\{f_1, f_2, f_3, f_4\})| = |\{f_1, f_2\}| = 2$. □

Therefore, Example D.2 shows that Proposition D.3 does not hold when there are multiple dimensions. Unfortunately, as the next example shows, improving the efficiency of the TKDA choice function is computationally difficult.

**Example D.3.** There are three families $f_{12}$, $f_3$, and $f_4$ and one locality $\ell$. The priority list of $\ell$ is $f_{12} \triangleleft_\ell f_3 \triangleleft_\ell f_4$. There are two dimensions and the sizes and capacities are displayed below:

\[
\nu = d_1 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \kappa = d_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]

This example is identical to Example D.2 except that the families $f_1$ and $f_2$ are merged into family $f_{12}$. For any feasible and interference-free choice function $C_\ell$ that satisfies the cardinal monotonicity condition, $C_\ell(\{f_{12}, f_3, f_4\}) \subseteq \{f_{12}\}$ as $\ell$ can weakly accommodate neither $f_3$ alongside $f_{12}$ nor $f_4$ alongside $\{f_{12}, f_3\}$. Therefore, in contrast to Example D.2, if
Proposition D.3. The KDA and TKDA mechanisms match at least one family to a locality in $L \setminus \{\emptyset\}$. If all families can be accommodated on their own at all localities, then the KDA and TKDA mechanisms match at least $\min\{|F|, |L| - 1\}$ families to localities in $L \setminus \{\emptyset\}$. 

---

**D.2. Lower bound on the efficiency of KDA and TKDA mechanisms.** We now show the lower bound on the efficiency of the KDA and TKDA mechanisms.
The intuition for Proposition D.4 is as follows. First, if a family \( f \) that can be accommodated at some locality \( \ell \) is matched to the null locality, then \( f \) has been permanently rejected by \( \ell \). Therefore, there exists a family \( f' \) with a higher priority than \( f \) at \( \ell \) that (i) can be accommodated on its own at \( \ell \) and (ii) proposes to \( \ell \) in some round of the KDA or TKDA algorithm. In turn, family \( f' \) would only be matched to the null locality if there is yet another higher priority family \( f'' \) and, by induction, it is not possible that all families be matched to the null locality. Second, consider the case where all families can be accommodated on their own at all localities. If fewer than \(|L| - 1\) families are matched to non-null localities, then either there are fewer than \(|L| - 1\) families in the market or at least one family \( f \) is matched to the null locality while a non-null locality \( \ell \) is not matched to any family. This yields a contradiction since \( f \) proposes to and is permanently rejected by \( \ell \) in some round of the KDA or TKDA algorithms.

D.3. TKDA with Clinching. In this section, we present a modification of the TKDA algorithm that improves its efficiency without affecting its properties. The TKDA with Clinching (TKDAC) algorithm (Algorithm D.1) starts with a Clinching Round that creates a new priority profile. The TKDA algorithm is then run using the new priority profile.

Proposition D.5. The TKDAC mechanism is strategy-proof and interference-free. Moreover, \( \mu^{TKDAC} \succeq \mu^{TKDA} \).

The idea of the TKDAC algorithm is to identify family-locality pairs that will necessarily be matched together by the TKDA algorithm. If locality \( \ell \) is family \( f \)'s first preference and \( \ell \) can weakly accommodate \( f \) alongside all higher-priority families, then \( \mu^{TKDA}(f) = \ell: \) \( f \) proposes to \( \ell \) and \( \ell \) tentatively accepts \( f \)'s proposal throughout the algorithm because \( f \)'s threshold at \( \ell \) is \( \infty \). In the Clinching Round of the TKDAC algorithm, family \( f \) clinches locality \( \ell \). Then we construct a new priority profile in which \( f \) moves to the bottom of the priority list of each locality \( \ell' \) such that \( \ell >_f \ell' \). The change in the priority profile does not affect \( f \)'s match in the TKDA algorithm, since \( f \) will be matched to \( \ell \), no matter what \( f \)'s priority at less preferred localities. However, the change in the priority profile may positively affect families that propose to \( f \)'s less preferred localities since their thresholds at these localities are no longer affected by \( f \).

To illustrate how clinching can improve families’ welfare, consider a locality \( \ell_1 \) and three families \( f_1, f_2, \) and \( f_3 \) such that \( >_{\ell_1} : f_1, f_2, f_3, \ldots \). There is only one dimension \( d_1 \) and the sizes and capacities are

\[
\begin{pmatrix}
    f_1 & f_2 & f_3 & \ell_1 \\
    d_1 & 2 & 1 & 1 & 2
\end{pmatrix}.
\]

Consider what happens if \( f_2 \) and \( f_3 \) both propose to \( \ell_1 \) in some round of the TKDA algorithm. The thresholds are \( \theta^{f_1}_{\ell_1} = \infty \) (since \( f_1 \) has the highest priority at \( \ell_1 \) and \( \ell_1 \) can accommodate
Algorithm D.1. TKDA with Clinching (TKDAC)

Clinching Round

Step 0
No locality rejects or proposes to any family and no family clinches any locality.
Set \( e^{\triangleright 1} = \triangleright \) and continue to Step 1.

Step \( j \geq 1 \)

(a) Each locality \( \ell \) rejects family \( f \) if \( \ell \) cannot weakly accommodate \( f \) alongside the families that (i) clinched \( \ell \) in Step \( j - 1 \) and (ii) are higher than \( f \) on \( \triangleright^j \).

(b) Each locality \( \ell \) proposes to family \( f \) if \( \ell \) can weakly accommodate \( f \) alongside all families that are higher than \( f \) on \( \triangleright^j \).

(c) Family \( f \) clinches locality \( \ell \) if (i) \( \ell \) proposed to \( f \) in part (b) and (ii) \( \ell \) is \( f \)'s most preferred locality that did not reject \( f \) in part (a).

(d) If at least one clinch occurred in part (c) that did not occur in Step \( j - 1 \), continue to part (e). Otherwise, set \( e^{\triangleright j} = e^{\triangleright j - 1} \) and continue to the TKDA algorithm.

(e) Construct \( e^{\triangleright j + 1} \) as follows, then continue to Step \( j + 1 \). For each \( \ell \in L \) and each \( f, f' \in F \) with \( f \triangleright^j \ell \)

- \( f' \triangleright^{j+1} \ell \) if, in part (c), (i) \( f \) clinched a locality that \( f \) strictly prefers to \( \ell \) and (ii) \( f' \) did not clinch a locality that \( f' \) strictly prefers to \( \ell \);

- \( f \triangleright^{j+1} f' \) otherwise.

TKDA
Run the TKDA algorithm with the priority profile \( e^{\triangleright} \).

---

Suppose, however, that \( f_1 \)'s first preference is another locality \( \ell_2 \) and that \( \ell_2 \) can (weakly) accommodate \( f_1 \) alongside all higher-priority families. Then, the clinching round identifies this pair and \( f_1 \) clinches \( \ell_2 \). The Clinching Round produces a priority profile \( \triangleright \) such that \( f_2 \triangleright_{\ell_1} f_3 \triangleright_{\ell_1} f_1 \). If the TKDA algorithm is run with the new priority profile \( \triangleright \) and \( f_2 \) and \( f_3 \) both propose to \( \ell_1 \), we have that \( \theta_{f_2}^{\ell_1} = \theta_{f_3}^{\ell_1} = \infty \) since \( f_2 \) and \( f_3 \) now have the two highest priorities at \( \ell_1 \) and therefore \( \ell_1 \) can accommodate both \( f_2 \) and \( f_3 \). As a consequence, following the Clinching Round, \( \ell_1 \) no longer permanently rejects \( f_3 \) resulting in a Pareto improvement over the original TKDA mechanism.

The TKDAC algorithm also allows families to clinch localities that are not their first preferences. In Step 1 of the Clinching Round, locality \( \ell \) rejects family \( f \) if \( \ell \) cannot accommodate \( f \) on its own (as no clinches have occurred before the start of Step 1). If \( \ell \) rejects \( f \) and \( f \) receives a proposal from its second-preference locality \( \ell' \), i.e., if \( \ell' \) can weakly accommodate
f alongside all higher-priority families, then it can be established that the TKDA algorithm
would match f to ℓ'. Therefore, f clinches ℓ', i.e., f goes to the bottom of the priority list
of all of f's less preferred localities. In Step 2, these localities may propose to new families
as a result. In addition, ℓ' now rejects every family with a lower priority than f at ℓ' that
ℓ' cannot weakly accommodate alongside f. The Clinching Round continues until there is a
step in which no family clinches any locality. Online Appendix E.3 provides an example of
the TKDAC algorithm.

D.4. Why clinching any proposing locality affects incentives for truth-telling. As
clinching improves efficiency, one might consider allowing families to clinch any locality that
proposes to them (whether or not they have been rejected by all of their more preferred
localities). Suppose we allowed family f to clinch a locality ℓ as long as ℓ can weakly
accommodate f alongside all higher-priority families, even if there are other localities that
f prefers and that have not rejected f.

Formally, consider the following modification to the Clinching Round in Algorithm D.1:
remove condition (ii) of part (c) in every Step j. We call TKDAC* algorithm the TKDAC
algorithm with the modified Clinching Round. Therefore, the TKDAC* algorithm differs
from the TKDAC algorithm by the fact that, in the Clinching Round, a family f clinches
any locality that proposes to f.

Unfortunately, the TKDAC* algorithm is not strategy-proof, as the following example
shows.

Example D.4. There are three families, three localities, and one dimension. The prefer-
ences, priorities, family sizes, and locality capacities are

\[
\succ_{f_1}: \ell_1, \ell_2, \ell_3 \quad \succ_{f_2}: \ell_2, \ell_1, \ell_3 \quad \succ_{f_3}: \ell_1, \ell_2, \ell_3
\]

\[
\prec_{\ell_1}: f_2, f_3, f_1 \quad \prec_{\ell_2}: f_1, f_2, f_3 \quad \prec_{\ell_3}: f_1, f_2, f_3
\]

ν = \( \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \) \quad κ = \( \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix} \).

Suppose that all families report truthfully. In the modified Clinching Round, every family
receives a proposal from its second-preference locality. Since ℓ_3 is every family’s third- and
last-preference locality, the modified Clinching Round does not affect the localities’ priorities.
As a result, the priority profile $\succeq$ that is used in the TKDA algorithm is the same as the original priority profile $\succ$.\(^{23}\)

The TKDA algorithm—summarized in Table D.4—yields the following matching:

\[
\begin{pmatrix}
  f_1 & f_2 & f_3 \\
  \ell_2 & \ell_1 & \ell_1
\end{pmatrix}.
\]

Suppose now that family $f_1$ reports $\succ'_1 : \ell_1, \ell_3, \ell_2$ while the other two families report their true preferences. In the modified Clinching Round, $f_1$ clinches $\ell_3$ and drops to the bottom of $\ell_2$’s priority list; therefore, we have a new priority list $\succeq_{\ell_2} : f_2, f_3, f_1$.\(^{24}\) Families $f_2$ and $f_3$ clinch locality $f_1$, but this proposal is inconsequential since $f_3$ already has the highest priority at $\ell_3$. Therefore, the modified Clinching Round does not modify the priorities of $\ell_1$ and $\ell_3$, i.e., $\succeq_{\ell_1} = \succeq_{\ell_1}$ and $\succeq_{\ell_3} = \succeq_{\ell_3}$.

The TKDA algorithm is then run with the priority profile $\succeq$. In the first round, $f_1$ proposes to $\ell_1$ and is tentatively accepted as $\theta_{\ell_1} = 1$ and no other family proposes to $\ell_1$. Families $f_2$ and $f_3$ both propose to $\ell_2$. Since $f_2$ and $f_3$ both now have a higher priority than $f_1$ and $\ell_2$ can accommodate them together and $\ell_2$ tentatively accepts both families (as $\theta_{\ell_2} = \theta_{\ell_2} = \infty$). The TKDA algorithm ends and yields the matching

\[
\begin{pmatrix}
  f_1 & f_2 & f_3 \\
  \ell_1 & \ell_2 & \ell_2
\end{pmatrix}.
\]

Clearly, $f_1$’s manipulation has been successful since $f_1$ is now matched to its first-preference locality $\ell_1$ instead of its second-preferences locality $\ell_2$. The reason why $f$ can successfully manipulate is that by clinching $\ell_3$, $f_1$ allows $f_3$ to be tentatively accepted by $\ell_2$; as a result, $f_3$ does not compete with $f_1$ for $\ell_1$.

Note that the TKDAC algorithm precludes $f_1$’s manipulation opportunity in Example D.4 because the TKDAC algorithm only allows $f_1$ to clinch a $\ell_3$ when it has been established that the TKDA algorithm will not match $f_1$ to any locality that $f_1$ prefers to $\ell_3$ (i.e., $\ell_1$).

D.5. **Proofs of results in Online Appendix D.** Throughout, for any family $f \in F$, any locality $\ell$, and any subset of families $G \subseteq F$, we denote by $\widehat{G}_f^\ell = G \cap \widehat{F}_f^\ell$ the subset of families in $G$ that have a higher priority for $\ell$ than $f$. We also denote by $\widehat{\theta}_f^\ell(G)$ (resp. $\theta_f^\ell(G)$) the temporary threshold (resp. the threshold) of $f$ for $\ell$ when all families in $G$ propose to $\ell$, as determined by Algorithm 5.

---

\(^{23}\)Family $f_1$ also receives a proposal from $\ell_3$, but this proposal is inconsequential since $\ell_3$ is $f_1$’s least preferred locality.

\(^{24}\)Family $f_1$ also receives a proposal from $\ell_2$, but this proposal is inconsequential since $\ell_2$ is $f_1$’s reported least preferred locality.
Proof of Proposition D.1. We have already shown in the proof of Theorem 6 that $C^T_\ell$ is feasible and satisfies the cardinal monotonicity condition. It remains to show that $C^T_\ell$ is interference-free and, when $|D| = 1$, envy-free.

Interference-freeness. Consider a subset of families $G$ and a family $f \in C^T_\ell(G)$. We need to show that $\ell$ can weakly accommodate $f$ alongside $\widehat{G}^f_\ell$. Suppose towards a contradiction that this is not the case. Then, by definition (see Algorithm 5), $\theta^f_\ell(G) = 0$ and so $f \notin C^T_\ell(G)$, a contradiction.

Envy-freeness when $|D| = 1$. Suppose that $|D| = 1$ and consider a subset of families $G$ as well as two families $f \in C^T_\ell(G)$ and $g \in G \setminus C^T_\ell(G)$. We need to show that $f \triangleright_\ell g$. Suppose towards a contradiction that $g \triangleright_\ell f$.

If $\theta^f_\ell(G) = \infty$, then by definition (see Algorithm 5), $\ell$ can weakly accommodate $f$ alongside $\widehat{G}^f_\ell$. As $g \triangleright_\ell f$ and $|D| = 1$, it follows that $\ell$ can weakly accommodate $g$ alongside $\widehat{G}^g_\ell$ so $g \in C^T_\ell(G)$, a contradiction.

If $\theta^f_\ell(G) \neq \infty$, then by definition (see Algorithm 5) $\theta^f_\ell(G) \leq \theta^g_\ell(G)$ (since $g \triangleright_\ell f$). As $f \in C^T_\ell(G)$ and $g \triangleright_\ell f$, it follows that $\theta^g_\ell(G) \geq \theta^f_\ell(G) \geq |\widehat{G}^f_\ell| + 1 \geq |\widehat{G}^g_\ell| + 1$ so $g \in C^T_\ell(G)$, a contradiction.

Proof of Proposition D.2. We need to show that $C_\ell(G) \subseteq C^T_\ell(G)$. The result holds trivially if $C^T_\ell(G) = G$; otherwise, let $f$ be the highest-priority family in $G \setminus C^T_\ell(G)$. As $C_\ell$ is envy-free, it is sufficient to show that $f \notin C_\ell(G)$. If $\ell$ cannot accommodate $\widehat{G}^f_\ell \cup \{f\}$, then $f \notin C_\ell(G)$ since $C_\ell$ is feasible and envy-free. If $\ell$ can accommodate $\widehat{G}^f_\ell \cup \{f\}$, then $\ell$ can weakly accommodate $f$ alongside $\widehat{G}^f_\ell$; hence, $f$’s temporary threshold $\tilde{\theta}^f_\ell(G)$ (as calculated by Algorithm 5) is at least $|\widehat{G}^f_\ell| + 1$. As $f$ has the $(|\widehat{G}^f_\ell| + 1)$-st highest priority among the families in $G$ and $f \notin C^T_\ell(G)$, $f$’s threshold $\theta^f_\ell$ is at most $|\widehat{G}^f_\ell|$. Therefore, we have that $\theta^f_\ell < \tilde{\theta}^f_\ell(G)$ so there exists a family $g \in \widehat{F}^f_\ell$ such that $\tilde{\theta}^g_\ell(G) \leq |\widehat{G}^f_\ell|$. Then, there exists a subset of families $H \subseteq \widehat{F}^g_\ell$ with $|H| = |\widehat{G}^f_\ell|$ and $\widehat{G}^g_\ell \subseteq H$ such that $\ell$ cannot weakly accommodate $g$ alongside $H$, which implies that $\ell$ cannot accommodate $H \cup \{g\}$. Then, all families in $C_\ell(G \cup H \cup \{g\})$ have a higher priority than $g$ at $\ell$, as otherwise the assumption that $C_\ell$ is envy-free would imply that $(H \cup \{g\}) \subseteq C_\ell(G \cup H \cup \{g\})$, which would violate the assumption that $C_\ell$ is feasible. As $\widehat{G}^g_\ell \subseteq H$, it follows that $C_\ell(G \cup H \cup \{g\}) \subseteq H$, which combined with the fact that $|\widehat{G}^f_\ell| = |H|$ implies that $|C_\ell(G \cup H \cup \{g\})| \leq |\widehat{G}^f_\ell|$. Finally, as $C_\ell$ satisfies the cardinal monotonicity condition, we have that $|C_\ell(G)| \leq |\widehat{G}^f_\ell|$; hence, the assumption that $C_\ell$ is envy-free yields $f \notin C_\ell(G)$.

Proof of Proposition D.3. Recall that, by assumption, there is only one dimension, which we denote throughout by $d$. Therefore, weak accommodation is equivalent to accommodation.

Denote by $C^K_\ell$ the choice function of the KDA algorithm: for any subset of families $G \subseteq F$ and any family $f \in G$, $f \in C^K_\ell(G)$ if and only if $\ell$ can weakly accommodate $f$ alongside $\widehat{G}^f_\ell$. 
Claim D.1. $C^K_\ell$ is envy-free, $C_\ell \subseteq C^K_\ell$, and $C^T_\ell \subseteq C^K_\ell$.

Proof of Claim D.1. We first show that $C^K_\ell$ is envy-free. Towards a contradiction, suppose that there exists a subset of families $G \subseteq F$ and two families $f \in C^K_\ell(G)$ and $g \in G \setminus C^K_\ell(G)$ such that $g \triangleright_\ell f$. Locality $\ell$ can (weakly) accommodate $f$ alongside $\hat{G}^g_\ell$ but cannot (weakly) accommodate $g$ alongside $\hat{G}^g_\ell$. It follows that $\nu'_f + \sum_{h \in \hat{G}^g_\ell} \nu'_h \leq \kappa'_f < \nu'_g + \sum_{h \in \hat{G}^g_\ell} \nu'_h$, a contradiction since $g \triangleright_\ell f$ implies that $(\{g\} \cup \hat{G}^g_\ell) \subseteq \hat{G}^g_\ell$.

We next show that $C_\ell \subseteq C^K_\ell$. Towards a contradiction, suppose that there exists a family $f \in C_\ell(G) \setminus C^K_\ell(G)$. As $f \notin C^K_\ell(G)$, $\ell$ cannot (weakly) accommodate $f$ alongside $\hat{G}^f_\ell$. Then, $f \in C_\ell(G)$ implies that $C_\ell(G)$ is not interference-free, a contradiction.

The proof that $C^T_\ell \subseteq C^K_\ell$ is analogous, one simply needs to replace $C_\ell(G)$ by $C^T_\ell(G)$ in the preceding paragraph. \hfill $\square$

Letting $G \subseteq F$ be any subset of families, we show that $|C_\ell(G)| \leq |C^T_\ell(G)|$. A direct implication of Claim D.1 is that the proof is complete if $C^T_\ell(G) = C^K_\ell(G)$ so we focus on the case in which $C^T_\ell(G) \subset C^K_\ell(G)$. Let $f$ be the highest-priority family in $G \setminus C^T_\ell(G)$. As $C^T_\ell$ is envy-free (Proposition D.1), we have that $C^T_\ell(G) = \hat{G}^f_\ell$. Moreover, as $C^T_\ell(G) \subset C^K_\ell(G)$ by assumption and $C^K_\ell(G)$ is envy-free by Claim D.1, $f \in C^K_\ell(G)$. It follows that $\ell$ can (weakly) accommodate $f$ alongside $\hat{G}^f_\ell$; hence, $\theta'_f(G) \geq |\hat{G}^f_\ell(G)| + 1$ (see Algorithm 5). However, as $f \notin C^T_\ell(G)$, we know that $\theta'_f(G) < |\hat{G}^f_\ell| + 1$; therefore, there exists a family $g \in \hat{F}^f_\ell$ such that $\theta'_g(G) = \theta'_f(G)$. We consider two cases separately.

Case 1: $\theta'_g(G) = \theta'_f(G) = 0$. By definition (see Algorithm 5), $\ell$ cannot (weakly) accommodate $g$ alongside $\hat{G}^g_\ell$. It follows that $g \notin C^K_\ell(G \cup \{g\})$, which implies that $C^K_\ell(G \cup \{g\}) \subseteq \hat{G}^g_\ell$ since $C^K_\ell$ is envy-free. Then, by Claim D.1, we have that $C_\ell(G \cup \{g\}) \subseteq \hat{G}^g_\ell$ so $|C_\ell(G \cup \{g\})| \leq |\hat{G}^g_\ell|$. As $C_\ell$ satisfies the cardinal monotonicity condition, it follows that $|C_\ell(G)| \leq |\hat{G}^g_\ell|$. As $C^T_\ell(G) = \hat{G}^f_\ell$ and $g \triangleright_\ell f$, we have that $\hat{G}^g_\ell \subseteq C^T_\ell(G)$; hence, we conclude that $|C_\ell(G)| \leq |\hat{G}^g_\ell| \leq |C^T_\ell(G)|$.

Case 2: $\theta'_g(G) = \theta'_f(G) > 0$. By definition (see Algorithm 5), there exists a subset of families $G^*$ such that $\hat{G}^g_\ell \subseteq G^* \subseteq \hat{F}^f_\ell$ and $|G^*| = \theta'_f(G)$ alongside which $\ell$ cannot (weakly) accommodate $g$. Let us define the set $H = G^* \cup \{g\}$ and observe that, by construction, $G^* = \hat{H}^g_\ell$. As $\ell$ cannot (weakly) accommodate $g$ alongside $G^* = \hat{H}^g_\ell$, we have that $g \notin C^K_\ell(H)$, which implies that $C^K_\ell(H) \subseteq \hat{H}^g_\ell = G^*$ since $C^K_\ell$ is envy-free (Claim D.1). By Claim D.1, $C_\ell(H) \subseteq G^*$ so $|C_\ell(H)| \leq |G^*| = \theta'_f(G)$, which implies that $|C_\ell(G)| \leq \theta'_f(G)$ since $C_\ell$ satisfies the cardinal monotonicity condition. As $\theta'_f(G) < |\hat{G}^f_\ell| + 1$ (equivalently, $\theta'_f(G) \leq |\hat{G}^f_\ell|$) and $C^T_\ell(G) = \hat{G}^f_\ell$, we conclude that $|C_\ell(G)| \leq \theta'_f(G) \leq |C^T_\ell(G)|$.

Having established that $|C_\ell(G)| \leq |C^T_\ell(G)|$, we now show that every family in $C^T_\ell(G) \setminus C_\ell(G)$ has a higher priority than any family in $C_\ell(G) \setminus C^T_\ell(G)$. Considering any two families $f \in C^T_\ell(G) \setminus C_\ell(G)$ and $g \in C_\ell(G) \setminus C^T_\ell(G)$, it is immediate that $f \triangleright g$ since $f \in C^T_\ell(G)$, $g \in G \setminus C^T_\ell(G)$, and $C^T_\ell(G)$ is envy-free (Proposition D.1). \hfill $\square$
Proof of Proposition D.4. We first show that the TKDA mechanism matches at least one family to a locality. Recall that we assume throughout that for each family \( f \in F \), there exists a locality \( \ell \in L \setminus \{\emptyset\} \) such that \( \ell \) can accommodate \( f \) on its own (see Section 3). Without loss of generality, let \( f \) be the highest-priority family among those that \( \ell \) can accommodate on their own. Recall that we assume throughout that localities prioritize families they can accommodate on their own over families that they cannot accommodate (see Section 3) so \( f \) has the highest priority at \( \ell \) among all families. Therefore, \( \theta^f_\ell = \infty \) throughout the TKDA algorithm, which means that \( \ell \) does not permanently reject \( f \) in any round of the TKDA algorithm. If \( f \) proposes to \( \ell \), \( \mu^{TKDA}(f) = \ell \succ_f \emptyset \); otherwise, \( f \) does not propose to \( \ell \), so \( \mu^{TKDA}(f) \succ_f \ell \succ_f \emptyset \). In both cases, \( f \) is matched to a locality that is not the null.

We next show that if all families can be accommodated on their own at all localities then the TKDA mechanism matches at least \( \min\{|F|, |L| - 1\} \) families to a locality other than the null. Towards a contradiction, suppose that \( |F \setminus \mu^{TKDA}(\emptyset)| < \min\{|F|, |L| - 1\} \). Then, \( \mu^{TKDA}(\emptyset) \neq \emptyset \) and there exists \( \ell \in L \setminus \{\emptyset\} \) such that \( \mu^{TKDA}(\ell) = \emptyset \). Since we assume that \( \emptyset \) is every family’s last preference and \( \mu^{TKDA}(\ell) = \emptyset \), \( \ell \) receives at least one proposal. Let \( f \) be the highest-priority family among those that propose to \( \ell \) at least once in the TKDA algorithm. Since the hypothesis states that \( \ell \) can accommodate \( f \) on its own, \( \theta^f_\ell \geq 1 \) throughout the TKDA algorithm. In every round in which \( f \) proposes to \( \ell \), \( f \) has the highest priority among proposing families so \( \ell \) never permanently rejects \( f \). This means that \( f \in \mu^{TKDA}(\ell) \), which contradicts \( \mu^{TKDA}(\ell) = \emptyset \).

Since the KDA mechanism is family optimal, we have that \( \mu^{KDA} \succeq \mu^{TKDA} \). Moreover, \( \emptyset \) is every family’s last preference. Hence, the result also holds for the KDA mechanism. \( \square \)

Proof of Proposition D.5. First notice that the TKDAC algorithm simply runs the TKDA algorithm with a different priority profile. Since the matching produced by the TKDA algorithm does not violate any matching constraints (see the proof of Theorem 6), the same is true of the TKDAC algorithm, i.e., for every \( \ell \in L \), \( \ell \) can accommodate \( \mu^{TKDAC}(\ell) \).

We next introduce some notation, which we use throughout the proof. Consider the Clinching Round and let \( N \) be its total number of steps. We use throughout the convention that \( \triangleright^0 = \triangleright \). For each Step \( j = 0, 1, \ldots, N \), each family \( f \in F \), and each locality \( \ell \in L \), let

- \( \widehat{F}_\ell^j(\triangleright^0) \) be the set of families that are higher than \( f \) on \( \triangleright^j \);
- \( \Delta^j_f \) be the set of localities that reject \( f \) in Step \( j \);
- \( \Gamma^j_f \) be the set of localities that propose to \( f \) in Step \( j \); and
- \( \Theta^j_f \) be the set of families that clinch \( \ell \) in Step \( j \).

Note that \( \widehat{F}_\ell^0(\triangleright^0) = \widehat{F}_\ell^0(\triangleright^1) = \widehat{F}_\ell^0(\triangleright) = \widehat{F}_\ell^0 \) and \( \widehat{F}_\ell^j(\triangleright^N) = \widehat{F}_\ell^j(\triangleright) \). Since no rejections, proposals, or clinches occur in Step 0, we also have \( \Delta^0_f = \Gamma^0_f = \Theta^0_f = \emptyset \).

The following two lemmata are key to our analysis. Their proofs can be found directly after the current proof.
Lemma D.1. For each Step $j = 1, \ldots, N$, each family $f \in F$, and each locality $\ell \in L$:

(i) If $f \notin \Theta^j_{\ell - 1}$ for all $\ell' \in L$ such that $\ell' \succ_f \ell$, then $\widehat{F}_\ell^j(\succ^j) \subseteq \widehat{F}_\ell^{j-1}(\succ^{j-1})$;

(ii) $\Delta_f^{j-1} \subseteq \Delta_f^j$;

(iii) If $\ell \in \Gamma_f^{j-1} \setminus \Gamma_f^j$, then there exists $\ell' \in L$ such that $\ell' \succ_f \ell$ and $f \in \Theta^j_{\ell'}$; and

(iv) $\Theta^j_{\ell} \subseteq \Theta^j_{\ell'}$.

(The proof of Lemma D.3 follows at the end of the proof of Proposition D.5.) Part (i) of Lemma D.1 says that if family $f$ has not clinched a locality that $f$ prefers to $\ell$, then the set of families that are higher than $f$ at $\ell$ shrinks throughout the Clinching Round. Part (ii) says that the set of localities that are rejecting $f$ grows throughout the Clinching Round. Part (iii) says that if $\ell$ stops proposing to $f$, then $f$ has clinched a more preferred locality. Part (iv) says that the set of families that have clinched $\ell$ grows throughout the Clinching Round.

Lemma D.2. For each Step $j = 1, \ldots, N$, each family $f \in F$, and each locality $\ell \in L$:

(i) If $\ell \in \Delta_f^j$, then $\mu_{TKDAC}(f) \neq \ell$;

(ii) If $\ell \in \Gamma_f^j$, then $\mu_{TKDAC}(f) \succeq_f \ell$; and

(iii) If $f \in \Theta_f^j$, then $\mu_{TKDAC}(f) = \ell$.

(The proof of Lemma D.3 follows at the end of the proof of Proposition D.5.) Part (i) of Lemma D.2 says that if a locality $\ell$ rejects a family $f$ in the Clinching Round, then the TKDAC algorithm will not match $f$ to $\ell$. Part (ii) says that if $\ell$ proposes to $f$ in the Clinching Round, then $f$ will be matched to $\ell$ or a more preferred locality under the TKDAC algorithm. Part (iii) says that if $f$ clinches $\ell$ in the Clinching Round, then the TKDAC algorithm will match $f$ to $\ell$.

We now use Lemmata D.1 and D.2 to show that the TKDAC mechanism is strategy-proof and interference-free and that $\mu_{TKDAC} \succeq \mu_{TKDA}$.

Proof that TKDAC is strategy-proof. We consider a family $f$ and a report $\succ_f$. We need to show that

$$\varphi^{TKDAC}(\succ)(f) \succeq \varphi^{TKDAC}(\succ_f, \succ_f)(f).$$

We use our usual notation—$N$, $\succ^j$, $\Delta_f^j$, $\Gamma_f^j$, and $\Theta_f^j$—for the TKDAC algorithm run with the preference profile $\succ$, i.e., when $f$ reports truthfully. We denote by $\overline{N}$, $\overline{\succ}^j$, $\overline{\Delta}_f^j$, $\overline{\Gamma}_f^j$, and $\overline{\Theta}_f^j$ the counterparts in the TKDAC algorithm run with the preference profile $(\succ_f', \succ_f)$, i.e., when $f$ misreports its preferences.

Consider the Clinching Round when the preference profile is $\succ$. If $f$ clinches a locality, let Step $m$ be the first step in which $f$ clinches the locality; formally, $f \notin \cup_{\ell \in L} \{\Theta^j_{\ell}\}$ for all $j = 1, \ldots, m - 1$ and $f \in \cup_{\ell \in L} \{\Theta^j_{\ell}\}$ for all $j = m, \ldots, N$. (By Lemma D.1(iv), once a family clinches a locality, it continues to clinch the same locality in all remaining steps so
$m$ is well defined.) If $f$ does not clinch any locality, let $m = \infty$; formally, $m = \infty$ whenever $f \notin \bigcup_{\ell \in L} (\Theta_{\ell}^N)$. We define $\overline{m}$ analogously for the preference profile $(\succ_{-}, \succ_{-})$. The following lemma guarantees that the Clinching Round is unaffected by $f$’s report until $f$ clinches a locality.

**Lemma D.3.** Let $q = \min\{m, \overline{m}, N, \overline{N}\}$. For each $j = 1, \ldots, q$, each $g \in F$, and each $\ell \in L$,

$$\Theta_{\ell}^j \setminus \{f\} = \overline{\Theta}_{\ell}^j \setminus \{f\}, \quad \Delta_g^j = \overline{\Delta}_g^j, \quad \Gamma_g^j = \overline{\Gamma}_g^j, \quad \text{and} \quad \overline{\nu}^j = \overline{\nu}^j.$$

Moreover, if $j < \min\{m, \overline{m}\}$, then $\Theta_{\ell}^j = \overline{\Theta}_{\ell}^j$ for all $\ell \in L$.

(The proof of Lemma D.3 follows at the end of the proof of Proposition D.5.) There are four cases to consider.

**Case 1: $m = \infty$ and $\overline{m} = \infty$.** In this case, $f$ does not clinch any locality in the Clinching Round, irrespective of whether $f$ reports $\succ_{-}$ or $\succ_{-}$. Then, $q = \min\{N, \overline{N}\} < \min\{m, \overline{m}\}$. If $N \leq \overline{N}$, then $q = N$ and $\Theta_{\ell}^{N-1} = \Theta_{\ell}^N$ for all $\ell \in L$ since, by construction, the Clinching Round ends when the same clinches occur in two consecutive steps. By Lemma D.3, $\Theta_{\ell}^{N-1} = \overline{\Theta}_{\ell}^{N-1}$ and $\Theta_{\ell}^N = \overline{\Theta}_{\ell}^N$ for all $\ell \in L$; therefore, $\overline{\Theta}_{\ell}^{N-1} = \overline{\Theta}_{\ell}^N$ for all $\ell \in L$, which means that $N = \overline{N}$. Then, since $\overline{\nu}^N = \nu^N$ by Lemma D.3, we conclude that $\overline{\nu} = \nu$. We have established that, whether $f$ reports $\succ_{-}$ or $\succ_{-}$, the Clinching Round ends in the same step and produces the same adjusted priority profile. Then, $\varphi^{TKDAC}(\succ)(f)$ is the matching produced by the TKDA algorithm when the preference and priority profiles are $\succ$ and $\overline{\nu}$ respectively while $\varphi^{TKDAC}(\succ_{-}, \succ_{-})(f)$ is the matching produced by the TKDA algorithm when the preference and priority profiles are $(\succ_{-}, \succ_{-})$ and $\nu = \overline{\nu}$ respectively. Since the TKDA mechanism is strategy-proof (Theorem 6), we conclude that $\varphi^{TKDAC}(\succ)(f) \succeq_f \varphi^{TKDAC}(\succ_{-}, \succ_{-})(f)$, as required. Analogous reasoning yields the same result for the case where $N \geq \overline{N}$.

**Case 2: $m \leq \min\{\overline{m}, N\}$.** In this case, if $f$ reports truthfully, then $f$ clinches a locality in Step $m$ of the Clinching Round and, if $f$ reports $\succ_{-}$, then $f$ either clinches a locality in Step $\overline{m} \geq m$ or does not clinch any locality. Since $m \leq \min\{\overline{m}, N\}$, $q = \min\{m, \overline{N}\}$. Towards a contradiction, suppose that $m > \overline{N}$. Then, $q = \overline{N} < \min\{m, \overline{m}\}$. As the Clinching Round ends whenever the same clinches occur in two consecutive rounds, we have that $\Theta_{\ell}^{\overline{N}-1} = \Theta_{\ell}^{\overline{N}}$ for all $\ell \in L$. Moreover, Lemma D.3 implies that $\Theta_{\ell}^{\overline{N}-1} = \overline{\Theta}_{\ell}^{\overline{N}-1}$ and $\Theta_{\ell}^{\overline{N}} = \overline{\Theta}_{\ell}^{\overline{N}}$ for all $\ell \in L$. It follows that $\overline{\Theta}_{\ell}^{\overline{N}-1} = \overline{\Theta}_{\ell}^{\overline{N}}$, so $N = \overline{N}$, which contradicts our assumption that $\overline{N} < m \leq N$.

We have established that $m \leq \overline{N}$; hence $q = m$. When $f$ reports truthfully, $f$ clinches a locality denoted by $\ell$ in Step $m$ of the Clinching Round, i.e., $f \in \Theta_{\ell}^m$. By Lemma D.2(iii), we have that $\varphi^{TKDAC}(\succ)(f) = \ell$. By construction (Step $m(c)$ of the Clinching Round), $f \in \Theta_{\ell}^m$ implies that, for all $\ell' \in L$ with $\ell' \succ_f \ell$, $\ell' \in \Delta_{\ell'}^m$. By Lemma D.3, $\Delta_{\ell'}^m = \overline{\Delta}_{\ell'}^m$; therefore
\( \ell' \in \bigcup_{f}^{m} \) for all \( \ell' \in L \) such that \( \ell' \succ_f \ell \). By Lemma D.2(i), it follows that \( \varphi^{TKDAC}(\succ_f, \succ_{-f})(f) \neq \ell' \) for all \( \ell' \in L \) such that \( \ell' \succ_f \ell \); therefore \( \ell \succeq_f \varphi^{TKDAC}(\succ_f, \succ_{-f})(f) \), as required.

**Case 3:** \( m \leq \min\{m, \overline{N}\} \). In this case, if \( f \) reports \( \succ_f' \), then \( f \) clinches a locality in Step \( \overline{m} \) of the Clinching Round and, if \( f \) reports truthfully, then \( f \) either clinches a locality in Step \( m \geq \overline{m} \) or does not clinch any locality. Since \( m \leq \{m, \overline{N}\} \), \( q = \min\{m, \overline{N}\} \). Using analogous reasoning to Case 2, we establish that \( \overline{m} \leq N \); hence \( q = \overline{m} \). When \( f \) reports \( \succ_f' \), \( f \) clinches a locality denoted by \( \ell \) in Step \( \overline{m} \) of the Clinching Round, i.e., \( f \in \overline{\Theta}_\ell^{\overline{m}} \). By Lemma D.2(iii), we have that \( \varphi^{TKDAC}(\succ_{-f}, \succ_{-f}) = \ell \). By construction (Step \( \overline{m}(c) \) of the Clinching Round), \( f \in \overline{\Theta}_\ell^{\overline{m}} \) implies that \( \ell \in \Gamma_f^{\overline{m}} \). By Lemma D.3, \( \Gamma_f^{\overline{m}} = \Gamma_f^{\overline{m}} \) so \( \ell \in \Gamma_f^{\overline{m}} \); therefore, by Lemma D.2(ii), \( \varphi^{TKDAC}(\succ_f)(f) \succeq_f \ell \), as required.

**Proof that TKDAC is interference-free.** Arbitrarily fix a family \( f \in F \) and, for ease of notation, let \( \ell = \mu^{TKDAC}(f) \). We need to show that \( f \) does not interfere with \( \mu^{TKDAC} \). Towards a contradiction, suppose that \( f \) interferes with \( \mu^{TKDAC} \); that is, \( \ell \) cannot weakly accommodate \( f \) alongside \( \widehat{F}_\ell^{\mu^{TKDAC}} = \{g \in F : g \succ_\ell f \text{ and } \ell \succeq_g \mu^{TKDAC}(g)\} \). As \( \mu^{TKDAC} \) is the outcome of the TKDA mechanism with the priority profile \( \overline{\nu} \) and the TKDA mechanism is interference-free, \( \ell \) can weakly accommodate \( f \) alongside \( \{g \in F : g \succ_\ell f \text{ and } \ell \succeq_g \mu^{TKDAC}(g)\} \). Therefore, there exists a family

\[
\{g \in F : g \succ_\ell f \text{ and } \ell \succeq_g \mu^{TKDAC}(g)\} \setminus \{g \in F : g \succ_\ell f \text{ and } \ell \succeq_g \mu^{TKDAC}(g)\}.
\]

By assumption, \( h \succ_\ell f \) and \( f \succ_\ell h \); equivalently, \( f \in \widehat{F}_\ell^h(\overline{\nu}) \setminus \widehat{F}_\ell^h \). Therefore, there exists a Step \( j = 1, \ldots, N \) of the Clinching Round such that \( f \in \widehat{F}_\ell^h(\overline{\nu}^j) \setminus \widehat{F}_\ell^h(\overline{\nu}^{j-1}) \). By the contrapositive of Lemma D.1(i), it follows that \( h \in \overline{\Theta}_\ell^{j-1} \) for some \( \ell' \in L \) such that \( \ell' \succ_h \ell \). By Lemma D.2(iii), we have that \( \mu^{TKDAC}(h) = \ell' \). We conclude that \( \mu^{TKDAC}(h) \succ_h \ell \), a contradiction since, by definition, \( \ell \succeq_h \mu^{TKDAC}(h) \).

**Proof that TKDAC is weakly more efficient than TKDA (\( \mu^{TKDAC} \succeq \mu^{TKDA} \)).**

We first introduce some additional notation. Let \( M \) be the number of rounds of the TKDA algorithm and, for each Round \( i = 1, \ldots, M \) and each locality \( \ell \in L \), let \( \Pi_i^\ell \) be the set of families that propose to \( \ell \) in Round \( i \). Similarly, let \( \widetilde{M} \) be the number of rounds of the TKDAC algorithm; that is, the TKDAC algorithm consists of a Clinching Round, which lasts \( N \) steps, and then the TKDA algorithm is run with the constructed priority profile \( \overline{\nu} \) and lasts \( \widetilde{M} \) rounds. For each Round \( i = 1, \ldots, \widetilde{M} \) and each locality \( \ell \in L \), let \( \overline{\Pi}_i^\ell \) be the set of families that propose to \( \ell \) in Round \( i \) of the TKDAC algorithm. By definition (Algorithm 5), the threshold of a family \( f \) at a locality \( \ell \) only depends on higher-priority families; therefore, \( \theta_i^f(\Pi_i^\ell \cap \widehat{F}_\ell^f) \) is the threshold of family \( f \) for locality \( \ell \) in Round \( i \) of the TKDA algorithm and \( \theta_i^f(\overline{\Pi}_i^\ell \cap \widehat{F}_\ell^f) \) is the threshold of family \( f \) for locality \( \ell \) in Round \( i \) of the TKDAC algorithm.
Claim D.2. In every Round $i = 1, \ldots, \widetilde{M}$ of the TKDAC algorithm, every family $f \in F$ proposes to a locality that $f$ weakly prefers to $\mu_{\text{TKDA}}(f)$.

By construction, for every family $f \in F$, $\mu_{\text{TKDAC}}(f)$ is the locality to which $f$ proposes in Round $\widetilde{M}$ of the TKDAC algorithm. Therefore, Claim D.2 implies that $\mu_{\text{TKDAC}}(f) \succeq_f \mu_{\text{TKDA}}(f)$ for all $f \in F$, as required. To complete the proof of Proposition D.5 it only remains to prove Claim D.2 as well as Lemmata D.1, D.2 and D.3.

Proof of Claim D.2. In Round 1 of the TKDAC algorithm, every family proposes to its most preferred locality; therefore Claim D.2 holds for $i = 1$. The remainder of the proof is by induction. We suppose that Claim D.2 holds for some $i = 1, \ldots, \widetilde{M} - 1$ (induction hypothesis) and show that Claim D.2 holds for $i + 1$.

Consider any family $f \in F$ and, for ease of notation, let $\ell = \mu_{\text{TKDA}}(f)$. We need to show that, in Round $i + 1$ of the TKDAC algorithm, $f$ proposes to a locality that $f$ weakly prefers to $\ell$. By our induction hypothesis, $f$ proposes to a locality that $f$ weakly prefers to $\ell$ in Round $i$. If $f$ proposes to a locality that $f$ strictly prefers to $\ell$ in Round $i$, then $\ell$ does not permanently reject $f$ in Round $i$ so $f$ proposes to a locality that $f$ weakly prefers to $\ell$ in Round $i + 1$, as required. We therefore focus on the case where $f$ proposes to $\ell$ in Round $i$ of the TKDAC algorithm and need to show that $\ell$ tentatively accepts $f$’s proposal. That is, by the construction of the TKDA part of the TKDAC algorithm, we need to show that

$$\theta^f_\ell (\Pi^f_\ell \cap \widehat{F}^f_\ell) \geq |\Pi^f_\ell \cap \widehat{F}^f_\ell| + 1. \tag{24}$$

First, as $f$ proposes to $\ell$ in Round $i$, $\ell \succeq_f \mu_{\text{TKDAC}}(f)$ by construction. Then, by Lemma D.2(iii), $f$ does not clinch any locality that $f$ strictly prefers to $\ell$ in the Clinching Round. By Lemma D.1(i), it follows that $\widehat{F}^f_\ell (\triangledown) \subseteq \widehat{F}^f_\ell$.

Second, by construction, family $f$ proposes to and is tentatively accepted by $\ell$ in the last round of the TKDA algorithm; hence

$$\theta^f_\ell (\Pi^M_\ell \cap \widehat{F}^f_\ell) \geq |\Pi^M_\ell \cap \widehat{F}^f_\ell| + 1. \tag{25}$$

There are two cases: $\theta^f_\ell (\Pi^M_\ell \cap \widehat{F}^f_\ell) = \infty$ and $\theta^f_\ell (\Pi^M_\ell \cap \widehat{F}^f_\ell) \neq \infty$.

Case 1: $\theta^f_\ell (\Pi^M_\ell \cap \widehat{F}^f_\ell) = \infty$. In this case, by the definition of thresholds (Algorithm 5), $\ell$ can weakly accommodate $f$ alongside $\widehat{F}^f_\ell$. As $\widehat{F}^f_\ell (\triangledown) \subseteq \widehat{F}^f_\ell$, $\ell$ can weakly accommodate $f$ alongside $\widehat{F}^f_\ell (\triangledown)$. Again, by the definition of thresholds (Algorithm 5), we have that $\theta^f_\ell (\Pi^f_\ell \cap \widehat{F}^f_\ell (\triangledown)) = \infty$; hence inequality (24) is satisfied, as required.

Case 2: $\theta^f_\ell (\Pi^M_\ell \cap \widehat{F}^f_\ell) \neq \infty$. There are two sub-cases: $\ell$ permanently rejects at least one family with a higher priority than $f$ at $\ell$ in the TKDA algorithm and $\ell$ does not permanently reject any family with a higher priority than $f$ at $\ell$ in the TKDA algorithm.
Sub-case 2.1: Locality $\ell$ permanently rejects at least one family with a higher priority than $f$ at $\ell$ in the TKDA algorithm. In this case, let $g \in F$ be the highest-priority family that $\ell$ permanently rejects in the TKDA algorithm. Therefore, there exists a Round $j = 1, \ldots, M$ of the TKDA algorithm, $g \in \Pi^j_\ell$ and

$$\theta^g_\ell(\Pi^j_\ell \cap \hat{F}^g_\ell) < |\Pi^j_\ell \cap \hat{F}^g_\ell| + 1. \quad (26)$$

As $g$ is the highest-priority family that $\ell$ permanently rejects in the TKDA algorithm, any family with a higher priority that proposes to $\ell$ in Round $j$ continues to propose to $\ell$ until the end of the algorithm; therefore, $\Pi^j_\ell \cap \hat{F}^g_\ell \subseteq \Pi^M_\ell \cap \hat{F}^g_\ell$.

We next show that the following inequality holds:

$$\theta^g_\ell(\Pi^M_\ell \cap \hat{F}^g_\ell) < |\Pi^M_\ell \cap \hat{F}^g_\ell| + 1. \quad (27)$$

First, inequality (27) holds trivially if $\theta^g_\ell(\Pi^M_\ell \cap \hat{F}^g_\ell) = 0$. Second, inequality (26) implies that $\theta^g_\ell(\Pi^j_\ell \cap \hat{F}^g_\ell) \neq \infty$; hence Lemma A.1(i) implies that $\theta^g_\ell(\Pi^M_\ell \cap \hat{F}^g_\ell) \neq \infty$. Third, if $\theta^g_\ell(\Pi^M_\ell \cap \hat{F}^g_\ell) \in \mathbb{Z}_{>0}$, then, as $\Pi^j_\ell \cap \hat{F}^g_\ell \subseteq \Pi^M_\ell \cap \hat{F}^g_\ell$, we can apply Lemma A.1(ii) to obtain that

$$\theta^g_\ell(\Pi^M_\ell \cap \hat{F}^g_\ell) \leq \theta^g_\ell(\Pi^j_\ell \cap \hat{F}^g_\ell) + |\Pi^M_\ell \cap \hat{F}^g_\ell| - |\Pi^j_\ell \cap \hat{F}^g_\ell|. \quad (28)$$

Combined with inequality (26), inequality (28) implies inequality (27).

As $g \triangleright_\ell f$, we have that $\Pi^M_\ell \cap \hat{F}^g_\ell \subseteq \Pi^M_\ell \cap \hat{F}^f_\ell$. Moreover, as a family’s threshold only depends on higher-priority families (Algorithm 5), $\theta^g_\ell(\Pi^M_\ell \cap \hat{F}^f_\ell) = \theta^g_\ell(\Pi^M_\ell \cap \hat{F}^g_\ell)$. Combining these observations with inequalities (27) and (25) yields

$$\theta^g_\ell(\Pi^M_\ell \cap \hat{F}^f_\ell) = \theta^g_\ell(\Pi^M_\ell \cap \hat{F}^g_\ell) < |\Pi^M_\ell \cap \hat{F}^g_\ell| + 1 \leq |\Pi^M_\ell \cap \hat{F}^g_\ell| + 1 \leq \theta^f_\ell(\Pi^M_\ell \cap \hat{F}^f_\ell). \quad (29)$$

By inequality (26), $\theta^g_\ell(\Pi^M_\ell \cap \hat{F}^f_\ell) \neq \infty$; hence $\theta^g_\ell(\Pi^M_\ell \cap \hat{F}^g_\ell) \neq \infty$. Hence, we can apply Lemma A.1(iii) to obtain that

$$\theta^g_\ell(\Pi^M_\ell \cap \hat{F}^f_\ell) \geq \theta^f_\ell(\Pi^M_\ell \cap \hat{F}^f_\ell),$$

which contradicts inequality (29). We therefore conclude that Sub-case 2.1 cannot occur.

Sub-case 2.2: Locality $\ell$ does not permanently reject any family with a higher priority than $f$ at $\ell$ in the TKDA algorithm. In this case, we first show that $\Pi^j_\ell \cap \hat{F}^j_\ell \subseteq \Pi^M_\ell \cap \hat{F}^j_\ell$. Towards a contradiction, suppose that there exists a family $g \in (\Pi^j_\ell \cap \hat{F}^j_\ell) \setminus (\Pi^M_\ell \cap \hat{F}^j_\ell)$. As $(\Pi^j_\ell \cap \hat{F}^j_\ell) \setminus (\Pi^M_\ell \cap \hat{F}^j_\ell) = (\Pi^j_\ell \setminus \Pi^M_\ell) \cap \hat{F}^j_\ell$, we have that $g \in \Pi^j_\ell$, $g \notin \Pi^M_\ell$, and $g \in \hat{F}^j_\ell$.

First, by our induction hypothesis, $g$ proposes in Round $i$ of the TKDAC algorithm to a locality that $g$ weakly prefers to $\mu^{\text{TKDA}}(g)$; therefore, the fact that $g \in \Pi^j_\ell$ implies that $\ell \succeq_g \mu^{\text{TKDA}}(g)$. Second, as the TKDA algorithm matches $g$ to the last locality to which $g$ proposes, the fact that $g \notin \Pi^M_\ell$ implies that $\ell \neq \mu^{\text{TKDA}}(g)$. Third, by the assumption of Sub-case 2.2, the fact that $g \in \hat{F}^j_\ell$ implies that $\ell$ does not permanently reject $g$ in the TKDA
algorithm, hence $\mu_{\text{TKDA}}(g) \geq g$. Combining these observations, we have that $\ell \geq g \mu_{\text{TKDA}}(g)$, $\ell \neq \mu_{\text{TKDA}}(g)$, and $\mu_{\text{TKDA}}(g) \geq g$, a contradiction.

Having established that $\tilde{\Pi}_i \cap \tilde{F}_f^j \subseteq \Pi_i^M \cap \tilde{F}_f^j$ and that $\tilde{F}_f^j(\triangledown) \subseteq \tilde{F}_f^j$, we conclude that $\tilde{\Pi}_i \cap \tilde{F}_f^j(\triangledown) \subseteq \Pi_i^M \cap \tilde{F}_f^j$. It follows that $|\tilde{\Pi}_i \cap \tilde{F}_f^j(\triangledown)| \leq |\Pi_i^M \cap \tilde{F}_f^j|$. Moreover, as $\theta_f(\Pi_i^M \cap \tilde{F}_f^j) \neq \infty$ by the assumption of Case 2, and as $\theta_f(\Pi_i^M \cap \tilde{F}_f^j) \neq 0$ by inequality (25), we have that $\theta_f(\Pi_i^M \cap \tilde{F}_f^j) \in \mathbb{Z}_{>0}$ so we can apply Lemma A.1(ii) to obtain that

$$
\theta_f(\Pi_i^M \cap \tilde{F}_f^j) \leq \theta_f(\tilde{\Pi}_i \cap \tilde{F}_f^j(\triangledown)) + |\Pi_i^M \cap \tilde{F}_f^j| - |\tilde{\Pi}_i \cap \tilde{F}_f^j(\triangledown)|.
$$

Combining inequality (30) with inequality (25) yields inequality (24), as required. □

**Proof of Lemma D.1.** We prove the lemma by a single induction argument. To show that the lemma holds for $j = 1$, note that (i) $\tilde{e}^0 = \tilde{e}^1 = \triangledown$, (ii) $\triangledown = \Delta_f^j \subseteq \Delta_f^j$ for every $f \in F$, (iii) $\triangledown = \Gamma_f^j \subseteq \Gamma_f^j$ for every $f \in F$, and (iv) $\triangledown = \Theta_f^0 \subseteq \Theta_f^j$ for every $\ell \in L$.

For the induction step, let us assume that part (iv) of the lemma holds for some $j = 1, \ldots, N$, i.e., $\Theta_f^{j-1} \subseteq \Theta_f^j$ for every $\ell \in L$ and every $j = 1, \ldots, N$. We will show that the assumption implies parts (i)-(iv) of the lemma in Step $j+1$. That is, we consider an arbitrary family $f \in F$ and an arbitrary locality $\ell \in L$ and show the following:

(i) If $f \notin \Theta_f^j$ for all $\ell' \in L$ such that $\ell' \succ_f \ell$, then $\tilde{F}_f^j(\tilde{e}^{j+1}) \subseteq \tilde{F}_f^j(\tilde{e}^j)$;

(ii) $\Delta_f^j \subseteq \Delta_f^{j+1}$;

(iii) If $\ell \in \Gamma_f^j \setminus \Gamma_f^{j+1}$, then there exists $\ell' \in L$ such that $\ell' \succ_f \ell$ and $f \in \Theta_f^{j+1}$; and

(iv) $\Theta_f^j \subseteq \Theta_f^{j+1}$.

**Proof of (i).** We prove the contrapositive. Suppose there exists a family $g \in \tilde{F}_f^j(\tilde{e}^{j+1}) \setminus \tilde{F}_f^j(\tilde{e}^j)$, then $f \triangledown_{\ell} g$ and $g \triangledown_{\ell}^{j+1} f$. We need to show that $f$ clinches a locality that $f$ strictly prefers to $\ell$ in Step $j$. There are two cases: $f \triangledown \ell g$ and $g \triangledown \ell f$.

**Case 1:** $f \triangledown \ell g$. Since $g \triangledown_{\ell}^{j+1} f$, then by construction (Step $j(e)$ of the Clinching Round) we have that $f$ clinches a locality that $f$ strictly prefers to $\ell$ in Step $j$, of the Clinching Round.

**Case 2:** $g \triangledown \ell f$. Since $f \triangledown_{\ell}^j g$, then by construction (Step $j(e)$ of the Clinching Round), $g$ clinches a locality that $g$ strictly prefers to $\ell$ in Step $j-1(c)$ of the Clinching Round. Since we have assumed that $\Theta_f^{j-1} \subseteq \Theta_f^j$, $g$ continues to clinch that locality in Step $j(c)$ of the Clinching Round. Since $g \triangledown_{\ell}^{j+1} f$, we have that $f$ clinches a locality that $f$ strictly prefers to $\ell$ in Step $j(c)$ of the Clinching Round.

**Proof of (ii).** Suppose that $\ell \in \Delta_f^j$, we need to show that $\ell \in \Delta_f^{j+1}$. As $\ell \in \Delta_f^j$, $\ell$ cannot weakly accommodate $f$ alongside $\Theta_f^{j-1} \cap \tilde{F}_f^j(\tilde{e}^j)$. Similarly, $\ell \in \Delta_f^{j+1}$ if $\ell$ cannot weakly accommodate $f$ alongside $\Theta_f^j \cap \tilde{F}_f^j(\tilde{e}^{j+1})$. Therefore, it is sufficient to show that

$$(\Theta_f^{j-1} \cap \tilde{F}_f^j(\tilde{e}^j)) \subseteq (\Theta_f^j \cap \tilde{F}_f^j(\tilde{e}^{j+1})).$$
Towards a contradiction, suppose that there exists a family
\[ g \in (\Theta^{j-1}_i \cap \hat{F}^j(\mathcal{G}^j)) \setminus (\Theta^{j}_i \cap \hat{F}^j(\mathcal{G}^{j+1})). \]
Since we have assumed that \( \Theta^{j-1}_i \subseteq \Theta^{j}_i \), we have that
\[ g \in (\Theta^{j-1}_i \cap \hat{F}^j(\mathcal{G}^j)) \setminus \hat{F}^j(\mathcal{G}^{j+1}) = \Theta^{j-1}_i \setminus (\hat{F}^j(\mathcal{G}^j) \setminus \hat{F}^j(\mathcal{G}^{j+1})). \]
Since \( g \in \hat{F}^j(\mathcal{G}^j) \setminus \hat{F}^j(\mathcal{G}^{j+1}) \), we have that \( g \triangleright f \) and \( f \triangleright g \). Therefore, \( f \in \hat{F}^j(\mathcal{G}^{j+1}) \setminus \hat{F}^j(\mathcal{G}^j) \) and, as a result, \( \hat{F}^j(\mathcal{G}^j) \setminus \hat{F}^j(\mathcal{G}^{j+1}) \). Then, the contrapositive of (i) implies that \( g \in \Theta^j_{\ell'} \) for some \( \ell' \in L \) such that \( \ell' > \mu \). Since \( g \in \Theta^{j-1}_i \) and we have assumed that \( \Theta^{j-1}_i \subseteq \Theta^{j}_i \), it follows that \( g \) clinches both \( \ell \) and \( \ell' \) in Step \( j \). This is a contradiction since, by construction (Step \( j(c) \) of the Clinching Round), a family can clinch at most one locality in any given step of the Clinching Round.

**Proof of (iii).** Suppose that \( \ell \in \Gamma^j_j \setminus \Gamma^{j+1}_j \). Then, \( \ell \) can weakly accommodate \( f \) alongside \( \hat{F}^j(\mathcal{G}^j) \) but not alongside \( \hat{F}^j(\mathcal{G}^{j+1}) \), which implies that \( \hat{F}^j(\mathcal{G}^{j+1}) \notin \hat{F}^j(\mathcal{G}^j) \). By the contrapositive of (i), we have that \( f \in \Theta^j_{\ell'} \) for some \( \ell' \in L \) such that \( \ell' > \mu \).

**Proof of (iv).** Suppose that \( f \in \Theta^j_{\ell} \), we need to show that \( f \in \Theta^{j+1}_\ell \). By construction (Step \( j(c) \) of Clinching Round), \( f \) proposes to \( f \) in Step \( j(b) \) and all localities that \( f \) prefers to \( f \) reject \( f \) in Step \( j(a) \); that is, \( \ell \in \Gamma^j_j \) and, for all \( \ell' \in L \) such that \( \ell' > \mu \), \( \ell' \in \Delta^j_j \). Since a family can clinch at most one locality in each step, \( f \notin \Theta^j_{\ell'} \) for all \( \ell' \in L \) such that \( \ell' > \mu \).

By the contrapositive of (iii), \( \ell \notin \Gamma^j_j \setminus \Gamma^{j+1}_j \); therefore \( \ell \in \Gamma^{j+1}_j \). Moreover, by (ii), \( \ell' \in \Delta^{j+1}_f \) for all \( \ell' \in L \) such that \( \ell' > \mu \). Therefore, \( f \) proposes to \( f \) in Step \( j+1 \) and all localities that \( f \) prefers to \( f \) reject \( f \) in Step \( j+1 \), which means that \( f \) clinches \( \ell \) in Step \( j+1 \), i.e., \( f \in \Theta^{j+1}_\ell \).

**Proof of Lemma D.2.** For ease of notation, let \( \mu = \mu^{\text{TKDAC}} \). We consider a Step \( j = 1, \ldots, N \) of the Clinching Round, a family \( f \in F \), and a locality \( \ell \in L \). We prove each part of the lemma in turn.

**Proof of (i).** We first show that if a family \( g \) is rejected by \( \mu(g) \) in Step \( i = 2, \ldots, N \) of the Clinching Round, then a family \( h \) is rejected by \( \mu(h) \) in Step \( i - 1 \).

Suppose that a family \( g \in F \) is rejected by \( \mu(g) \) in Step \( i = 2, \ldots, N \) of the Clinching Round. Since \( \mu(g) \) rejects \( g \) in Step \( i(a) \) of the Clinching Round, \( \mu(g) \) cannot weakly accommodate \( g \) alongside \( \Theta^{i-1}_{\mu(g)} \cap \hat{F}^g_{\mu(g)}(\mathcal{G}^i) \). If all families in \( \Theta^{i-1}_{\mu(g)} \cap \hat{F}^g_{\mu(g)}(\mathcal{G}^i) \) are matched to \( \mu(g) \) at the end of the TKDAC algorithm (i.e., if \( \Theta^{i-1}_{\mu(g)} \cap \hat{F}^g_{\mu(g)}(\mathcal{G}^i) \subseteq \mu(\mu(g)) \)), then \( \mu(g) \) can accommodate \( \Theta^{i-1}_{\mu(g)} \cap \hat{F}^g_{\mu(g)}(\mathcal{G}^i) \cup \{g\} \). Therefore, \( \mu(g) \) can weakly accommodate \( g \) alongside \( \Theta^{i-1}_{\mu(g)} \cap \hat{F}^g_{\mu(g)}(\mathcal{G}^i) \), a contradiction. We conclude that there exists a family \( h \in \Theta^{i-1}_{\mu(g)} \cap \hat{F}^g_{\mu(g)}(\mathcal{G}^i) \) such that \( \mu(h) \neq \mu(g) \). Since \( h \in \Theta^{i-1}_{\mu(g)} \), we have that \( h \in \Theta^N_{\mu(g)} \) by Lemma D.1(iv). It follows that \( \mu(g) \) proposes to \( h \) in Step \( N \) of the Clinching Round, which in turn implies
that \( \mu(g) \) can weakly accommodate \( h \) along one \( F_{\ell}^h(\overrightarrow{\gamma}^N) = \overrightarrow{F}_{\ell}^h(\overrightarrow{\gamma}) \). Then, in every round of the TKDAC algorithm, \( h \)'s threshold at \( \mu(g) \) is \( \infty \) (see Algorithm 5). It follows that \( \mu(g) \) tentatively accepts any proposal from \( h \) so the fact that \( \mu(h) \neq \mu(g) \) implies that \( h \) does not propose to \( \mu(g) \) in the TKDAC algorithm, hence \( \mu(h) \succ_h \mu(g) \). Finally, since \( h \in \Theta_{\mu(g)}^{-1} \) and \( \mu(h) \succ_f \mu(g) \), \( \mu(h) \) rejects \( h \) in Step \( i - 1(c) \) of the Clinching Round, as required.

Now, suppose towards a contradiction that \( \mu(f) \in \Delta_f^j \). The preceding argument implies, by induction, that there exists a family \( f' \) such that \( \mu(f') \in \Delta_f^1 \), i.e., \( \mu(f') \) rejects \( f' \) in Step 1 of the Clinching Round. Then, \( \mu(f') \) cannot weakly accommodate \( f' \) alongside \( \Theta_{\mu(f')}^0 \cap \overrightarrow{F}_{\mu(f')}^\ell(\overrightarrow{\gamma}^1) \). Since \( \Theta_{\mu(f')}^0 = \emptyset \), it follows that \( \mu(f') \) cannot weakly accommodate \( f' \) on its own. Therefore, the threshold of \( f' \) at \( \mu(f') \) is \( 0 \) in every round of the TKDAC algorithm. We conclude that \( \mu(f') \) permanently rejects any proposal from \( f' \), which contradicts the assumption that \( f' \) is matched to \( \mu(f') \) at the end of the TKDAC algorithm.

**Proof of (ii).** Suppose that \( \ell \in \Gamma_f^j \). We need to show that \( \mu(f) \succeq_f \ell \). Then, by Lemma D.1(iii), \( \ell \in \Gamma_f^j \) implies that one of the following two cases holds: either \( \ell \in \Gamma_f^j \) or \( f \in \Theta_{\ell'}^{-1} \) for some \( \ell' \in L \) such that \( \ell' \succ_f \ell \).

**Case 1:** \( \ell \in \Gamma_f^j \). In this case, by construction (Step \( N(c) \) of the Clinching Round), \( \ell \) can weakly accommodate \( f \) alongside \( \overrightarrow{F}_{\ell}^f(\overrightarrow{\gamma}^N) = \overrightarrow{F}_{\ell}^f(\overrightarrow{\gamma}) \). Therefore, in every round of the TKDAC algorithm, \( f \)'s threshold at \( \ell \) is \( \infty \) (see Algorithm 5). It follows that \( \ell \) tentatively accepts any proposal from \( f \) in the TKDAC algorithm, hence \( \mu(f) \succeq_f \ell \), as required.

**Case 2:** \( f \in \Theta_{\ell'}^{-1} \) for some \( \ell' \in L \) such that \( \ell' \succ_f \ell \). In this case, Lemma D.1(iv), we have that \( f \in \Theta_{\ell'}^N \) for some \( \ell' \in L \) such that \( \ell' \succ_f \ell \). Then, by construction (Step \( N(c) \) of the Clinching Round), \( \ell' \) proposes to \( f \) in Step \( N \) of the Clinching Round, i.e., \( \ell' \in \Gamma_f^N \). Then, \( f \)'s threshold at \( \ell' \) is \( \infty \) in every round of the TKDAC algorithm, hence \( \mu(f) \succeq_f \ell' \succ_f \ell \), as required.

**Proof of (iii).** Suppose that \( f \in \Theta_{\ell}^j \). Then, by construction (Step \( j(c) \) of the Clinching Round) we have that \( \ell \in \Gamma_j^f \) and, for all \( \ell' \in L \) such that \( \ell' \succ_f \ell \), it is the case that \( \ell' \in \Delta_f^j \). From parts (i) and (ii) of the lemma, we obtain that \( \mu(f) \succeq_f \ell \) and that \( \mu(f) \neq \ell' \) for all \( \ell' \in L \) such that \( \ell' \succ_f \ell \). We conclude that \( \mu(f) = \ell \), as required. \( \square \)

**Proof of Lemma D.3.** We prove the first part of the lemma by a single induction argument. For the initial step, we have that \( \Theta_{\ell}^0 = \overrightarrow{\Theta}_{\ell}^0 = \emptyset \) for all \( \ell \in L \).

For the induction step, let us assume that \( \Theta_{\ell}^{j-1} \setminus \{f\} = \overrightarrow{\Theta}_{\ell}^{j-1} \setminus \{f\} \) holds for some \( j = 1, \ldots, q \) and all \( \ell \in L \). We will show that \( \overrightarrow{\gamma}^j = \overrightarrow{\gamma}^j, \overrightarrow{\Delta}_g = \overrightarrow{\Delta}_g, \overrightarrow{\Gamma}_g = \overrightarrow{\Gamma}_g \), and, finally, that \( \Theta_{\ell}^j \setminus \{f\} = \overrightarrow{\Theta}_{\ell}^j \setminus \{f\} \) for all \( g \in F \) and \( \ell \in L \).

Since \( j \leq q \), we have that \( j - 1 < \min\{m, \overline{m}\} \). Therefore, \( f \) does not clinch any locality in Step \( j - 1 \) of the Clinching Round with either report; therefore, \( \Theta_{\ell}^{j-1} \setminus \{f\} = \overrightarrow{\Theta}_{\ell}^{j-1} \setminus \{f\} \) for all \( \ell \in L \) implies that \( \Theta_{\ell}^{j-1} = \overrightarrow{\Theta}_{\ell}^{j-1} \) for all \( \ell \in L \). In Step \( j - 1(c) \) of the Clinching Round
the construction of \( \overline{\theta} \) only depends on which families have clinched which localities, hence \( \Theta^{-1} = \overline{\Theta}^{-1} \) for all \( \ell \in L \) implies that \( \overline{\theta} = \theta \). In Step \( j(a) \), the fact that \( \Theta^{-1} = \overline{\Theta}^{-1} \) for all \( \ell \in L \) and \( \overline{\theta} = \theta \) implies that every locality rejects the same families under both reports so \( \Delta^j = \overline{\Delta}^j \) for all \( g \in F \). Similarly, in Step \( j(b) \), \( \overline{\theta} = \theta \) implies that \( \Gamma^j = \overline{\Gamma}^j \) for all \( g \in F \). Finally, consider any family \( h \neq f \) in Step \( j(c) \). As \( \Delta_h^j = \overline{\Delta}_h^j \), \( \Gamma_h^j = \overline{\Gamma}_h^j \), and \( h \) does not misreport its preferences (only \( f \)'s report changes from \( \succ_f \) to \( \succ'_f \)), \( h \) clinches the same locality (if any), whether \( f \) reports \( \succ_f \) or \( \succ'_f \). We therefore conclude that \( \Theta^j \setminus \{ f \} = \overline{\Theta}^j \setminus \{ f \} \) for all \( \ell \in L \), as required.

We now turn to the second part of the lemma. Consider any \( j = 1, \ldots, q \) with \( j < \{ m, m' \} \). We have established that \( \Theta^j \setminus \{ f \} = \overline{\Theta}^j \setminus \{ f \} \) for all \( \ell \in L \). As \( j < \{ m, m' \} \), whether \( f \) reports \( \succ_f \) or \( \succ'_f \), \( f \) does not clinch any locality in Step \( j \) of the Clinching Round. Therefore \( \Theta^j = \overline{\Theta}^j \) for all \( \ell \in L \), as required. \( \square \)
Appendix E. Additional Examples

We present an illustrative example of the KTTCE algorithm in Online Appendix E.1 and of the TKDA algorithm in Online Appendix E.2. In Online Appendix E.3, we revisit the latter example to show how clinching can improve the efficiency of the TKDA algorithm.

E.1. Knapsack Top Trading Cycles with Endowment. We illustrate the KTTCE algorithm using the example from the proof of Theorem 1. We add the following lexicographic priorities:

\[ \triangleright_{\ell_1} : f_1, f_2, \ldots \triangleright_{\ell_2} : f_1, f_2, \ldots \triangleright_{\ell_3} : f_3, \ldots \triangleright_{\ell_4} : f_4, \ldots \]

The KTTCE algorithm also depends on the order in which families are picked, should the algorithm enter the Rejection Stage. With four families, there are 4! = 24 such orderings; however, we will see that the only part of that ordering that matters for the outcome is whether \( f_3 \) or \( f_4 \) is picked first. Therefore, we describe the KTTCE algorithm under two different orderings: KTTCE3 picks \( f_3 \) before \( f_4 \) and KTTCE4 picks \( f_4 \) before \( f_3 \).

KTTCE3. The workings of KTTCE3 are displayed in Figure E.1. In Round 1, the unique trading cycle is \( f_1 \rightarrow \ell_3 \rightarrow f_3 \rightarrow \ell_2 \rightarrow f_1 \). That trading cycle is not feasible since \( \ell_2 \) cannot accommodate \( f_3 \) alongside \( f_2 \). Therefore, the algorithm enters the Rejection Stage. As \( f_1 \) and \( f_2 \) require only one unit of capacity, they can replace any family at any locality; hence no permanent rejection occurs if one of these families is picked. In contrast, neither \( f_3 \) nor \( f_4 \) can replace \( f_1 \) at \( \ell_2 \). By assumption, \( f_3 \) is picked before \( f_4 \) and permanently rejected by \( \ell_2 \).

In Round 2, \( f_3 \) points at its second preference \( \ell_3 \) and the feasible (and trivial) cycle \( f_3 \rightarrow \ell_3 \rightarrow f_3 \) appears so \( f_3 \) is permanently matched to \( \ell_3 \). As a result, \( \ell_3 \) is “full” and permanently rejects all other families, including \( f_1 \). In Round 3, \( f_1 \) points at its second preference \( \ell_1 \) and is permanently matched to it, effectively taking advantage of \( \ell_1 \)’s unassigned unit of capacity. In Round 4, since \( f_1 \) has been permanently matched, \( \ell_2 \) points at its second-priority family \( f_2 \). In addition, \( \mu^4(\ell_2) = \emptyset \) since \( f_1 \) has left \( \ell_2 \) to be permanently matched to \( \ell_1 \). As a result, the cycle \( f_2 \rightarrow \ell_4 \rightarrow f_4 \rightarrow \ell_2 \rightarrow f_2 \) is feasible so both families are permanently matched to the locality at which they are pointing. Since all families are permanently matched, the algorithm ends and produces the following matching:

\[
\begin{pmatrix}
 f_1 & f_2 & f_3 & f_4 \\
 \ell_1 & \ell_4 & \ell_3 & \ell_2 
\end{pmatrix}
\]

KTTCE4. The workings of KTTCE4 are displayed in Figure E.2. As for KTTCE3, the unique cycle in Round 1 is \( f_1 \rightarrow \ell_3 \rightarrow f_3 \rightarrow \ell_2 \rightarrow f_1 \), which is not feasible. The difference in KTTCE4 compared to KTTCE3 is that in the Rejection Stage \( f_4 \) is picked and permanently rejected by \( \ell_2 \). In Round 2, \( f_4 \) points at its second preference \( \ell_4 \) (which points back at \( f_4 \)) so \( f_4 \) is permanently matched to \( \ell_4 \). In Round 3, the unique cycle is again \( f_1 \rightarrow \ell_3 \rightarrow f_3 \rightarrow \ell_2 \rightarrow f_1 \),
which is still infeasible. This time, $\ell_2$ permanently rejects $f_3$ (because $f_3$ must be picked in the Rejection Stage). In Round 4, $f_3$ points at and is permanently matched to $\ell_3$. In Round 5, $\ell_3$ is “full” and permanently rejects $f_1$, which points at and is permanently matched to $\ell_1$. In Round 6 (not displayed), $f_2$ and $\ell_2$ point at one another and are permanently matched. The algorithm ends and produces the following matching:

$$\begin{pmatrix}
  f_1 & f_2 & f_3 & f_4 \\
  \ell_1 & \ell_2 & \ell_3 & \ell_4
\end{pmatrix}.$$

**Discussion.** Observe first that both matchings Pareto dominate the endowment. This is not surprising since there is only one dimension and the priorities are lexicographic, hence Theorem 4 applies. Perhaps more surprising is the fact that KTTCE3 produces a chain-efficient matching, which may appear at odds with Theorem 1. What is more, we showed in the proof of Theorem 1 that no individually rational, chain-efficient, and strategy-proof mechanism exists in this specific market. However, the fact that KTTCE3 produces a chain-efficient matching in this instance does not mean it is a chain-efficient mechanism. In fact, one can show that, if $f_1$ reports its preferences to be $\ell_3, \ell_2, \ldots$, KTTCE3 produces

$$\begin{pmatrix}
  f_1 & f_2 & f_3 & f_4 \\
  \ell_2 & \ell_1 & \ell_3 & \ell_4
\end{pmatrix},$$

which is not chain-efficient.

Second, the matching produced by KTTCE3 Pareto dominates the one produced by KTTCE4. This is due to the fact that picking $f_3$ allows the algorithm to match $f_1$ to $\ell_1$ in Round 3, which makes the cycle $f_2 \rightarrow \ell_4 \rightarrow f_4 \rightarrow \ell_2 \rightarrow f_2$ feasible in Round 4. In contrast, picking $f_4$ in KTTCE4 does not allow matching $f_2$ to $\ell_1$ in Round 3 (since $\ell_1$ points at $f_1$), and therefore the cycle $f_1 \rightarrow \ell_3 \rightarrow f_3 \rightarrow \ell_2 \rightarrow f_1$ remains infeasible. One might therefore wonder whether the picking order can be designed in a way that maximizes the efficiency of the mechanism. This would require picking the “best family” (from an efficiency point of view) every time the algorithm enters the Rejection Stage. Unfortunately, what constitutes the best family depends on preferences; therefore such a mechanism would violate strategy-proofness. In order for the mechanism to be strategy-proof, the picking order must be entirely independent of preferences, which has an efficiency cost. Third, the KTTCE mechanism may produce different outcomes with different priorities. If the priorities are $\triangleright_{\ell_1} : f_1, f_2, \ldots$ and $\triangleright_{\ell_2} : f_2, f_1, \ldots$ (without changing the priorities of $\ell_3$ and $\ell_4$ so that priorities remain lexicographic), KTTCE3 and KTTCE4 produce the same matchings as they do in our example above. If the priorities are either $\triangleright_{\ell_1} : f_2, f_1, \ldots$ and $\triangleright_{\ell_2} : f_2, f_1, \ldots$ or $\triangleright_{\ell_1} : f_2, f_1, \ldots$ and $\triangleright_{\ell_2} : f_1, f_2, \ldots$, KTTCE3 and KTTCE4 respectively produce the following matchings:

$$\begin{pmatrix}
  f_1 & f_2 & f_3 & f_4 \\
  \ell_2 & \ell_1 & \ell_3 & \ell_4
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
  f_1 & f_2 & f_3 & f_4 \\
  \ell_3 & \ell_1 & \ell_2 & \ell_4
\end{pmatrix}.$$
Figure E.1. Workings of KTTC3.
Figure E.2. Workings of KTTCE4.
E.2. Threshold Knapsack Deferred Acceptance. We illustrate the TKDA algorithm with the following example. There are seven families, four localities, and two dimensions.

Preferences:
\[
\succ_{f_1}: \ell_2, \ell_3, \ldots \succ_{f_2}: \ell_4, \ell_1, \ldots \succ_{f_3}: \ell_2, \ell_1, \ldots \succ_{f_4}: \ell_1, \ldots
\]

\[
\succ_{f_5}: \ell_1, \ell_2, \ldots \succ_{f_6}: \ell_1, \ell_2, \ell_3, \ldots \succ_{f_7}: \ell_4, \ldots
\]

Priorities:
\[
\triangleright_{\ell_3}: f_1, f_2, f_3, f_4, f_5, f_6, f_7 \quad \triangleright_{\ell_2}: f_5, f_1, f_2, f_7, f_6, f_3, f_4 \\
\triangleright_{\ell_3}: f_4, f_6, f_1, \ldots \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \triangleright_{\ell_4}: f_7, f_2, \ldots
\]

Sizes and capacities:
\[
\nu = \begin{pmatrix}
d_1 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\
d_2 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} \quad \kappa = \begin{pmatrix}
d_1 & \ell_1 & \ell_2 & \ell_3 & \ell_4 \\
d_2 & 4 & 4 & 3 & 1
\end{pmatrix}
\]

We show that, in this example, the TKDA algorithm produces the following matching:
\[
\mu_{\text{TKDA}} = \begin{pmatrix}
f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\
\ell_3 & \ell_1 & \ell_1 & \ell_2 & \ell_3 & \ell_4
\end{pmatrix}
\]

The TKDA algorithm lasts four rounds. The first three rounds are displayed in Table E.1. In Round 1, every family proposes to its first-preference locality. Families \( f_4, f_5, \) and \( f_6 \) propose to \( \ell_1 \). The threshold at \( \ell_1 \) of \( f_4, f_5, \) and \( f_6 \) is \( \infty \) since \( \ell_1 \) can accommodate all three families together. Family \( f_4 \)’s threshold at \( \ell_1 \) is 2 because \( \ell_1 \) can weakly accommodate \( f_4 \) alongside any one of \( f_1, f_2, \) or \( f_3 \), but not alongside \( \{ f_1, f_2 \} \) or \( \{ f_1, f_3 \} \). Family \( f_5 \)’s threshold at \( \ell_1 \) is also 2 since \( \ell_1 \) can weakly accommodate \( f_5 \) alongside \( f_4 \) but not alongside \( \{ f_1, f_4 \} \). Family \( f_6 \)’s temporary threshold at \( \ell_1 \) is 3 since \( \ell_1 \) can weakly accommodate \( f_6 \) alongside \( \{ f_4, f_5 \} \) but not alongside \( \{ f_1, f_4, f_5 \} \). However, \( f_6 \)’s threshold at \( \ell_1 \) is 2 since \( \ell_1 \)’s threshold at \( \ell_1 \) cannot exceed \( f_5 \)’s because \( f_5 \) has a higher priority and \( f_6 \)’s temporary threshold is finite. Finally, \( f_7 \)’s threshold at \( \ell_1 \) is 0 since \( \ell_1 \) cannot weakly accommodate \( f_7 \) alongside \( \{ f_4, f_5, f_6 \} \). All three proposing families—\( f_4, f_5, \) and \( f_6 \)—have a threshold of 2 at \( \ell_1 \); therefore, \( \ell_1 \) tentatively accepts the two higher-priority proposing families—\( f_4 \) and \( f_5 \)—and permanently rejects the lowest-priority proposing family—\( f_6 \). As \( \ell_1 \) is able to accommodate \( \{ f_4, f_5, f_6 \} \), one might be tempted to allow \( \ell_1 \) to tentatively accept \( f_6 \)’s proposal (as it would in the KDA algorithm). However, an interference-free choice function that allowed this would violate the cardinal monotonicity condition. Suppose that \( f_1, f_2, f_4, f_5, \) and \( f_6 \) propose to \( \ell_1 \). Locality \( \ell_1 \) cannot weakly accommodate \( f_4 \) alongside \( \{ f_1, f_2 \} \), \( f_5 \) alongside \( \{ f_1, f_2, f_4 \} \), and \( f_6 \) alongside \( \{ f_1, f_2, f_4, f_5 \} \); therefore, interference-freeness dictates that \( \ell_1 \) must only tentatively accept two families: \( f_1 \) and \( f_2 \). However, the cardinal monotonicity condition dictates that at most two families can be tentatively accepted when \( f_1, f_5, \) and \( f_6 \) propose.
<table>
<thead>
<tr>
<th>TKDA – Round 1</th>
<th>TKDA – Round 2</th>
<th>TKDA – Round 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ell_1 ) ((4, 2) ) ( \theta )</td>
<td>( \ell_2 ) ((4, 1) ) ( \theta )</td>
<td>( \ell_3 ) ((3, 1) ) ( \theta )</td>
</tr>
<tr>
<td>( f_1 ) ( (2, 1) ) ( \infty )</td>
<td>( f_5 ) ( (1, 1) ) ( \infty )</td>
<td>( f_4 ) ( (2, 0) ) ( \infty )</td>
</tr>
<tr>
<td>( f_2 ) ( (1, 0) ) ( \infty )</td>
<td>( \cmark ) ( (2, 1) ) ( 1 )</td>
<td>( f_6 ) ( (1, 0) ) ( \infty )</td>
</tr>
<tr>
<td>( f_3 ) ( (1, 0) ) ( \infty )</td>
<td>( f_2 ) ( (1, 0) ) ( \infty )</td>
<td>( f_1 ) ( (2, 1) ) ( 1 )</td>
</tr>
<tr>
<td>( \cancel{f_4} ) ( (2, 0) ) ( 2 )</td>
<td>( f_7 ) ( (1, 0) ) ( 1 )</td>
<td>( \cdot )</td>
</tr>
<tr>
<td>( \cmark ) ( (1, 0) ) ( 2 )</td>
<td>( \cmark ) ( (1, 0) ) ( 1 )</td>
<td>( \cdot )</td>
</tr>
<tr>
<td>( f_7 ) ( (1, 0) ) ( 0 )</td>
<td>( f_4 ) ( (2, 0) ) ( 0 )</td>
<td>( \cdot )</td>
</tr>
</tbody>
</table>

**Table E.1.** Rounds 1-3 of the TKDA algorithm. Sizes and capacities in parentheses. \( \cmark \) : \( f_i \) proposes and is tentatively accepted. \( \cancel{f_i} \) : \( f_i \) proposes and is permanently rejected.
Families $f_1$ and $f_3$ propose to $\ell_2$. Family $f_5$'s threshold at $\ell_2$ is $\infty$ since $f_5$ has the highest priority and $\ell_2$ can accommodate $f_5$ on its own. However, $\ell_2$ cannot weakly accommodate $f_1$ alongside $f_5$ as this would require two units of $d_2$ and $\ell_2$ has only one unit available; therefore, $f_1$’s threshold at $\ell_2$ is 1. In contrast, $f_2$’s threshold at $\ell_2$ is $\infty$ because $\ell_2$ can weakly accommodate $f_2$ alongside $\{f_1,f_5\}$. (Recall from Algorithm 5 that if a family’s threshold is $\infty$, then it is allowed to exceed the thresholds of higher-priority families.) This situation illustrates how interference-freeness can improve efficiency over envy-freeness. Locality $\ell_2$ cannot accommodate $f_2$ alongside $\{f_1,f_5\}$ as this would violate $\ell_2$’s capacity for $d_2$; however, as $f_2$ does not require any unit of $d_2$, $\ell_2$ can weakly accommodate $f_2$ alongside $\{f_1,f_5\}$. The temporary threshold of $f_7$ at $\ell_2$ is 3 since $\ell_2$ can weakly accommodate $f_7$ alongside either one of $\{f_1,f_5\}$ or $\{f_1,f_2\}$, but not alongside $\{f_1,f_2,f_5\}$. However, since $f_1$’s threshold is 1, we set $f_2$’s threshold to 1 as well. The same reasoning applies to $f_6$ and $f_3$ while $f_4$’s threshold at $\ell_2$ is 0 because $\ell_2$ cannot weakly accommodate $f_4$ alongside $\{f_1,f_3\}$. Locality $\ell_2$ tentatively accepts $f_1$ but permanently rejects $f_3$.

No family proposes to $\ell_3$. The threshold of both $f_4$ and $f_6$ at $\ell_3$ is $\infty$ since $\ell_3$ can accommodate $\{f_4,f_6\}$. Family $f_1$’s threshold at $\ell_3$ is 1 since $\ell_3$ cannot weakly accommodate $f_1$ alongside $f_4$. Finally, $f_2$ and $f_7$ propose to $\ell_4$. As $f_7$ has the highest priority at $\ell_4$ and $\ell_4$ can accommodate $f_7$ on its own, $f_7$’s threshold at $\ell_4$ is $\infty$, which means that $\ell_4$ tentatively accepts $f_7$. In contrast, $\ell_4$ cannot weakly accommodate $f_2$ alongside $f_7$; therefore $f_2$’s threshold at $\ell_4$ is 0 and $\ell_4$ permanently rejects $f_2$.

In Round 2, $f_2$ and $f_3$ both propose to $\ell_1$ after having been permanently rejected by their respective first preferences ($\ell_4$ and $\ell_2$) in Round 1. As a result, $f_4$’s threshold at $\ell_1$ rises to 3. This situation illustrates how a family’s threshold can increase from one round to the next. In Round 1, $f_4$’s threshold at $\ell_1$ is 2 because $\ell_1$ cannot weakly accommodate $f_4$ alongside either one of $\{f_1,f_2\}$ or $\{f_1,f_3\}$. However, in Round 2, $f_2$ and $f_3$ propose to $\ell_1$ but $f_1$ does not. As $\ell_1$ can weakly accommodate $f_4$ alongside $\{f_2,f_3\}$, $f_4$’s threshold at $\ell_1$ is 3. It follows that $\ell_1$ continues to tentatively accept $f_4$. In contrast, $\ell_4$ cannot weakly accommodate $f_5$ alongside $\{f_2,f_3,f_4\}$; therefore, $f_5$’s threshold at $\ell_1$ is 0 and $\ell_1$ permanently rejects $f_5$. The third family that was permanently rejected in Round 1, $f_6$, proposes to $\ell_2$ in Round 2. As $f_6$’s threshold at $\ell_2$ remains 1 and $f_6$ has the second-highest priority (after $f_1$) among proposing families, $\ell_2$ permanently rejects $f_6$.

In Round 3, $f_5$ proposes to $\ell_2$ and is tentatively accepted since $f_5$’s threshold at $\ell_2$ is $\infty$. As a result, however, $\ell_2$ permanently rejects $f_1$ which now has a threshold of 0. Family $f_6$ proposes to $\ell_3$ and is also tentatively accepted since $f_6$’s threshold at $\ell_3$ is $\infty$. A consequence of $f_6$’s proposal to $\ell_3$ is that $f_1$’s threshold at $\ell_3$ rises to 2 because $\ell_3$ can weakly accommodate $f_1$ alongside $f_6$. Therefore, in Round 4, $\ell_3$ tentatively accepts $f_1$’s proposal and the algorithm ends.
Threshold Knapsack Deferred Acceptance with Clinching. We use the example introduced in Online Appendix E.2 to illustrate the TKDAC algorithm. We show that

$$\mu_{\text{TKDAC}} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\ \ell_3 & \ell_1 & \ell_1 & \ell_1 & \ell_2 & \ell_2 & \ell_4 \end{pmatrix}.$$  

The only difference between $\mu_{\text{TKDA}}$ and $\mu_{\text{TKDAC}}$ is that $\mu_{\text{TKDA}}(f_6) = \ell_3$ while $\mu_{\text{TKDAC}}(f_6) = \ell_2$. Since $\ell_2 > f_6 \ell_3$, $\mu_{\text{TKDAC}} > \mu_{\text{TKDA}}$.

The three steps of the Clinching Round are displayed in Table E.2. In Step 1(a), localities $\ell_1$, $\ell_2$ and $\ell_3$ do not reject any family since they can accommodate every family on its own. Locality $\ell_4$ rejects every family that requires either two units of $d_1$ or one unit of $d_2$, i.e., $\ell_4$ rejects $f_1$, $f_4$, and $f_5$. In Step 1(b), $\ell_1$ is able to accommodate its three highest-priority families, which all receive a proposal. Locality $\ell_1$ cannot weakly accommodate any other family alongside $\{f_1, f_2, f_3\}$ since every family takes up at least one unit of capacity in the first dimension; therefore none of the other families receive a proposal from $\ell_1$. Locality $\ell_2$ can accommodate $f_5$ on its own but cannot weakly accommodate $f_1$ alongside $f_5$ since both families require a unit of $d_2$ and only one unit is available. In contrast, $\ell_2$ can weakly accommodate $f_2$ alongside $\{f_1, f_5\}$ because $f_2$ does not require any units of $d_2$. Therefore, $\ell_2$ proposes to both $f_5$ and $f_2$. None of the other families receive a proposal as this would violate the capacity of $d_1$. Locality $\ell_3$ is able to accommodate its two highest-priority families while $\ell_4$ is only able to accommodate its highest-priority family. No other family receives a proposal from either $\ell_3$ or $\ell_4$ as this would violate their respective capacities in both dimensions. Family $f_7$ is the only family to receive a proposal from its first-preference locality, $\ell_4$. As a result, $f_7$ clinches $\ell_4$ in Step 1(c). In Step 1(e), $\succsim^2$ is constructed by giving $f_7$ the lowest priority at $\ell_1$, $\ell_2$, and $\ell_3$. In particular, $\ell_2$’s priority list is updated to

$$\succsim^2_{\ell_2} : f_5, f_1, f_2, f_6, f_3, f_4, f_7.$$  

In Step 2(a), $\ell_4$ rejects all families except $f_7$ because $f_7$ has clinched $\ell_4$ and $\ell_4$ cannot weakly accommodate any family alongside $f_7$. In particular, $\ell_4$ rejects its second-priority family $f_2$. In Step 2(b), the same proposals occur as in Step 1; however, the fact that $\ell_4$ has rejected $f_2$ means that $\ell_1$ is now $f_2$’s most preferred locality that has not rejected $f_2$. As $\ell_1$, proposes to $f_2$, $f_2$ clinches $\ell_1$ in Step 2(c). In Step 2(e), $\succsim^3$ is constructed by moving $f_2$ to the bottom of $\ell_2$ and $\ell_3$’s priorities. $f_2$’s priority list is updated to

$$\succsim^3_{\ell_2} : f_5, f_1, f_6, f_3, f_4, f_2, f_7.$$  

In Step 3(a), there are no new rejections as $\ell_1$ can weakly accommodate any family alongside $f_2$. In Step 3(b), there is one new proposal: $\ell_2$ proposes to $f_6$ as $\ell_2$ can weakly accommodate $f_6$ alongside $\{f_1, f_5\}$. This proposal was made possible by the fact that $f_2$ and $f_7$ have both moved below $f_6$ in $\ell_2$’s priority list. However, $\ell_2$ is $f_6$’s second preference and $f_6$’s...
### Clinching Round – Step 1

<table>
<thead>
<tr>
<th>$\ell_1$ (4,2)</th>
<th>$\ell_2$ (4,1)</th>
<th>$\ell_3$ (3,1)</th>
<th>$\ell_4$ (1,0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$ (2,1)</td>
<td>$f_5$ (1,1)</td>
<td>$f_4$ (2,0)</td>
<td>$\boxed{f_7}$ (1,0)</td>
</tr>
<tr>
<td>$f_2$ (1,0)</td>
<td>$f_1$ (2,1)</td>
<td>$f_6$ (1,0)</td>
<td>$f_2$ (1,0)</td>
</tr>
<tr>
<td>$f_3$ (1,0)</td>
<td>$f_2$ (1,0)</td>
<td>$f_1$ (2,1)</td>
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</tr>
<tr>
<td>$f_4$ (2,0)</td>
<td>$f_7$ (1,0)</td>
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<td></td>
</tr>
<tr>
<td>$f_5$ (1,1)</td>
<td>$f_6$ (1,0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_6$ (1,0)</td>
<td>$f_3$ (1,0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_7$ (1,0)</td>
<td>$f_4$ (2,0)</td>
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<td></td>
</tr>
</tbody>
</table>

### Clinching Round – Step 2

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<th>$\ell_4$ (1,0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$ (2,1)</td>
<td>$f_5$ (1,1)</td>
<td>$f_4$ (2,0)</td>
<td>$\boxed{f_7}$ (1,0)</td>
</tr>
<tr>
<td>$\boxed{f_2}$ (1,0)</td>
<td>$f_1$ (2,1)</td>
<td>$f_6$ (1,0)</td>
<td>$f_2$ (1,0)  X</td>
</tr>
<tr>
<td>$f_3$ (1,0)</td>
<td>$f_2$ (1,0)</td>
<td>$f_1$ (2,1)</td>
<td>:</td>
</tr>
<tr>
<td>$f_4$ (2,0)</td>
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</tr>
<tr>
<td>$f_5$ (1,1)</td>
<td>$f_3$ (1,0)</td>
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<td></td>
</tr>
<tr>
<td>$f_6$ (1,0)</td>
<td>$f_4$ (2,0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_7$ (1,0)</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

### Clinching Round – Step 3

<table>
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<th>$\ell_1$ (4,2)</th>
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<th>$\ell_3$ (3,1)</th>
<th>$\ell_4$ (1,0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$ (2,1)</td>
<td>$f_5$ (1,1)</td>
<td>$f_4$ (2,0)</td>
<td>$\boxed{f_7}$ (1,0)</td>
</tr>
<tr>
<td>$\boxed{f_2}$ (1,0)</td>
<td>$f_1$ (2,1)</td>
<td>$f_6$ (1,0)</td>
<td>$f_2$ (1,0)  X</td>
</tr>
<tr>
<td>$f_3$ (1,0)</td>
<td>$f_6$ (1,0)</td>
<td>$f_1$ (2,1)</td>
<td>:</td>
</tr>
<tr>
<td>$f_4$ (2,0)</td>
<td>$f_3$ (1,0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_5$ (1,1)</td>
<td>$f_4$ (2,0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_6$ (1,0)</td>
<td>$f_2$ (1,0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_7$ (1,0)</td>
<td>$f_7$ (1,0)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table E.2.** Clinching Round of the TKDAC algorithm. Sizes and capacities in parentheses. ✓: a locality proposes to a family. X: a locality rejects a family. $\boxed{f_i}$: family $f_i$ clinches a locality.
first preference, $\ell_1$, has not rejected $f_6$. It follows that no new clinch occurs in Step 3(c) so the Clinching Round ends in Step 3(d) and outputs $\triangleright = \triangleright^3$ such that

$$\triangleright_{\ell_1} : f_1, f_2, f_3, f_4, f_5, f_6, f_7$$
$$\triangleright_{\ell_2} : f_5, f_1, f_6, f_3, f_4, f_2, f_7$$
$$\triangleright_{\ell_3} : f_4, f_6, f_1, \ldots$$
$$\triangleright_{\ell_4} : f_7, f_2, \ldots$$

With the priority profile $\triangleright$ constructed in the Clinching Round, the TKDA algorithm lasts four rounds. The first three rounds are displayed in Table E.3. Changing the priority profile from $\triangleright$ to $\triangleright$ affects the TKDA algorithm in one important way. With the constructed priority profile $\triangleright$, $f_6$ is the third highest-priority family at $\ell_2$. Since $\ell_2$ can accommodate $f_6$ alongside $\{f_1, f_5\}$, $f_6$’s threshold at $\ell_2$ is $\infty$. It follows that $\ell_2$ tentatively accepts $\ell_6$’s proposal in Rounds 2-4, which is why $\mu^{\text{TKDAC}}(f_6) = \ell_2$.

When the TKDA algorithm is run with the true priority profile $\triangleright$, $f_6$ does not get a threshold of $\infty$ at $\ell_2$ because $f_2$ and $f_7$ remain above $f_6$ on $\ell_2$’s priority list. The Clinching Round’s contribution in this example is to identify that $f_2$ and $f_7$ will necessarily be matched to a more preferred locality, i.e., $f_2$ will be matched to $\ell_1$ (with $\ell_1 \succ f_2, \ell_2$) and $f_7$ will be matched to $\ell_4$ (with $\ell_4 \succ f_2, \ell_2$). Thus, the Clinching Round identifies that $f_2$ and $f_7$ cannot cause a violation of the interference-freeness or cardinal monotonicity conditions when $f_6$ is matched to $\ell_2$.

While the preceding example has two categories, clinching can also yield efficiency gains when $|D| = 1$. Consider a simple example with three families $f_1$ (size 2), $f_2$ (size 1), and $f_3$ (size 1) and two localities $\ell_1$ (capacity 2) and $\ell_2$ (capacity 2). All families prefer $\ell_1$ to $\ell_2$ and $\ell_2$ to the null. Both localities prioritize $f_1$ over $f_2$ and $f_2$ over $f_3$.

It is easy to verify that the TKDA algorithm matches $f_1$ to $\ell_1$, $f_2$ to $\ell_2$, and $f_3$ to the null. The reason $f_3$ is matched to the null is that the cardinal monotonicity condition forces $\ell_2$ to permanently reject $f_3$ when $f_2$ and $f_3$ propose, as both families would be rejected if $f_1$ also proposed. However, in the TKDAC algorithm, $f_1$ clinches $\ell_1$ and goes to the bottom of $\ell_2$’s priority; as a result, $f_2$ and $f_3$ clinch $\ell_2$. Therefore, clinching allows identifying that $f_1$ will not propose to $\ell_2$; hence, $\ell_2$ can accept both $f_2$ and $f_3$ as the violation of the cardinal monotonicity condition created will not materialize.
<table>
<thead>
<tr>
<th>Round 1</th>
<th>Round 2</th>
<th>Round 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$ (2, 1) $\infty$</td>
<td>$f_1$ (2, 1) $\infty$</td>
<td>$f_1$ (2, 1) $\infty$</td>
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<td>$f_2$ (1, 0) $\infty$</td>
<td>$f_2$ (1, 0) $\infty$</td>
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</tr>
<tr>
<td>$f_4$ (2, 0) 2</td>
<td>$f_4$ (2, 0) 3</td>
<td>$f_4$ (2, 0) 3</td>
</tr>
<tr>
<td>$f_5$ (1, 1) 2</td>
<td>$f_5$ (1, 1) 0</td>
<td>$f_5$ (1, 1) 0</td>
</tr>
<tr>
<td>$f_6$ (1, 0) 1</td>
<td>$f_6$ (1, 0) 1</td>
<td>$f_6$ (1, 0) 1</td>
</tr>
<tr>
<td>$f_7$ (1, 0) 0</td>
<td>$f_7$ (1, 0) 0</td>
<td>$f_7$ (1, 0) 0</td>
</tr>
</tbody>
</table>

Table E.3. Rounds 1-3 of the TKDAC algorithm. Sizes and capacities in parentheses. $f_i$: $f_i$ proposes and is tentatively accepted. $\Box$: $f_i$ proposes and is permanently rejected.
Table F.1. Number (percentage) of families made better off for different orderings in the Rejection Stage of the KTTCE (one-dimensional constraints). Averages over 100 simulation rounds. Numbers are rounded to 1 d.p.

<table>
<thead>
<tr>
<th>Preference type</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Largest-to-smallest family</td>
<td>5.9 (1.8)</td>
<td>50.0 (15.2)</td>
<td>49.5 (15.0)</td>
<td>23.7 (7.2)</td>
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<tr>
<td>Smallest-to-largest family</td>
<td>6.6 (2.0)</td>
<td>44.6 (13.5)</td>
<td>46.0 (14.0)</td>
<td>23.3 (7.1)</td>
</tr>
</tbody>
</table>

Table F.2. Number (fraction) of families made better off by the KTTCE mechanism (three-dimensional constraints). Averages over 100 simulation rounds.

<table>
<thead>
<tr>
<th>Preference type</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Families made better off</td>
<td>6.8 (2.1%)</td>
<td>10.7 (3.3%)</td>
<td>11.7 (3.6%)</td>
<td>9.7 (3.0%)</td>
</tr>
</tbody>
</table>

Appendix F. Further Simulation Results

F.1. Effect of the order of families in the Rejection Stage in the KTTCE algorithm. A part of the design of the KTTCE mechanism that we leave open is the order in which families are picked in the Rejection Stage. In our main simulations, that order is determined randomly (with an equal probability for each possible order). We also test two alternative rules: ordering families from largest to smallest and from smallest to largest (same-size families continue to be ordered randomly). The results of the two alternative family orderings in the Rejection Stage are displayed in Table F.1. We see that the largest-to-smallest ordering is at least as efficient as the smallest-to-largest ordering: the differences are pronounced for Type 2 and Type 3 preferences which generate the most improvements and are not significant for Type 1 and Type 4 preferences. Intuitively, larger families are more likely to cause a cycle not to be feasible (see Algorithm 2) as the locality at which they are pointing may not be able to accommodate them. Therefore, by rejecting larger families first, the algorithm needs to reject fewer families before finding a feasible cycle. Compared to a random order, one would then naturally expect rejecting larger families first to have a positive impact on efficiency and rejecting smaller families first to have a negative one. Why is the efficiency gain of the largest-to-smallest ordering small relative to a random ordering? The reason is that families that can be rejected in the Rejection Stage tend to be large, meaning that, when families are ordered randomly, it is likely that larger families will be picked first and the algorithm will find a feasible cycle before picking one of the few small families. Consequently, a random ordering does not result in large efficiency losses compared to the largest-to-smallest ordering.
<table>
<thead>
<tr>
<th>Preference type</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Interference violations</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KTTCE</td>
<td>18888 (17.4%)</td>
<td>18225 (16.8%)</td>
<td>18378 (17.0%)</td>
<td>18617 (17.2%)</td>
</tr>
<tr>
<td>KTTC</td>
<td>603 (9.6%)</td>
<td>1302 (1.2%)</td>
<td>1306 (1.2%)</td>
<td>4390 (4.1%)</td>
</tr>
<tr>
<td>KDA</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>TKDA</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>Average priority rank</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KTTCE</td>
<td>108</td>
<td>108</td>
<td>108</td>
<td>108</td>
</tr>
<tr>
<td>KTTC</td>
<td>36</td>
<td>97</td>
<td>96</td>
<td>78</td>
</tr>
<tr>
<td>KDA</td>
<td>25</td>
<td>27</td>
<td>27</td>
<td>26</td>
</tr>
<tr>
<td>TKDA</td>
<td>14</td>
<td>13</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td><strong>Number of matched families</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KTTCE</td>
<td>315 (95.8%)</td>
<td>315 (95.8%)</td>
<td>315 (95.8%)</td>
<td>315 (95.8%)</td>
</tr>
<tr>
<td>KTTC</td>
<td>303 (92.1%)</td>
<td>307 (93.4%)</td>
<td>308 (93.4%)</td>
<td>302 (91.9%)</td>
</tr>
<tr>
<td>KDA</td>
<td>286 (86.7%)</td>
<td>287 (87.1%)</td>
<td>289 (86.1%)</td>
<td>286 (86.9%)</td>
</tr>
<tr>
<td>TKDA</td>
<td>234 (71.1%)</td>
<td>230 (70.0%)</td>
<td>231 (70.2%)</td>
<td>232 (70.7%)</td>
</tr>
<tr>
<td><strong>Fraction of unfilled capacity</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KTTCE</td>
<td>9.5%</td>
<td>9.4%</td>
<td>9.4%</td>
<td>9.4%</td>
</tr>
<tr>
<td>KTTC</td>
<td>11.6%</td>
<td>12.3%</td>
<td>12.3%</td>
<td>13.1%</td>
</tr>
<tr>
<td>KDA</td>
<td>17.8%</td>
<td>15.8%</td>
<td>15.9%</td>
<td>16.8%</td>
</tr>
<tr>
<td>TKDA</td>
<td>26.6%</td>
<td>27.3%</td>
<td>27.5%</td>
<td>27.1%</td>
</tr>
</tbody>
</table>

Table F.3. Outcomes of KTTCE, KTTC, KDA, and TKDA algorithms (one-dimensional constraints). Averages over 100 simulation rounds (and, for average priority rank, over all localities). Numbers (percentages) are rounded to the nearest integer (to 1 d.p.).

F.2. Performance of KTTC, KTTCE, KDA, and TKDA algorithms in the three-dimensional setting. Table F.2 summarizes the performance of the KTTCE mechanism in the three-dimensional environment. Compared to the one-dimensional case, fewer families are made better off on average. This finding is not surprising since the families’ needs can differ across three dimensions, which makes feasible trades harder to find. However, despite our result that no mechanism is guaranteed to improve upon an endowment when there are multiple services (Theorem 2), we see that the KTTCE mechanism continues to find a significant number of improvements.

Table F.3 and Figure F.1 display the results for the KTTC, KDA, and TKDA algorithms in the three-dimensional environment. The results are consistent with the one-dimensional environment (Table 6) across preference types and mechanisms. The second panel of Table F.3 shows that the average rank of matched families across different algorithms in the

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25Sizes are not monotonic because a family can have more adults but fewer children than another family.
three-dimensional setting is similar to that of the one-dimensional setting. Figure F.1 shows that the ranking of the mechanisms by efficiency in the one-dimensional setting (Figure 2) is preserved in the three-dimensional setting. Comparing Table 6 and Table F.3, we see that our mechanisms tend not to perform as well in the three-dimensional environment as in the one-dimensional environment: the KTTC mechanism generates more interference violations and all three mechanisms are less efficient. These findings are intuitive in the sense that having three dimensions in effect increases the heterogeneity among families’ sizes. Therefore, the KTTC mechanism has more difficulty finding Pareto-improving trades and those trades are more likely to create interference violations; moreover, in the KDA and TKDA mechanism, localities may have to reject more families in order to prevent interference violations.

Finally, we assess the efficiency gain associated with using our solution concept of interference-freeness as opposed to envy-freeness. We modify the KDA algorithm so that it produces the family-optimal envy-free matching and the TKDA algorithm so that it produces an envy-free matching (that is, by only replacing “weak accommodation” with “accommodation” in Algorithms 3 and 5). With one dimension, there is no impact on the outcome of either algorithm (see Proposition C.3 and Corollary C.1). However, in the presence of multidimensional constraints, Figure F.2 shows that the impact on efficiency of using interference-freeness over envy-freeness is substantial both for the KDA and the TKDA algorithms.
MATCHING MECHANISMS FOR REFUGEE RESETTLEMENT

Figure F.1. Preference distributions for outcomes under different matching mechanisms (three-dimensional constraints). Labeled numbers: fractions of matched families.
Figure F.2. Preference distributions for KDA and TKDA algorithms when using accommodation vs. weak accommodation (three-dimensional constraints).
Appendix G. Relationships to Prior Models

Our model generalizes a number of existing matching models, including the following:

- **School choice** (Abdulkadiroğlu and Sönmez, 2003): Each student takes up a single seat at any school. Let us relabel a student as a family and a school as a locality. In our model, this corresponds to having only one dimension \( d \) (\(|D| = 1\)) and any family \( f \in F \) having size \( \nu_d^f = 1 \).

- **Controlled school choice or college admissions with affirmative action and \( m \) type-specific quotas** (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, 2005; Westkamp, 2013): Each student is one of \( m \) types and each school has a quota for each of the \( m \) types. Let us again relabel a student as a family, a school as a locality and a type as a dimension. In our model, this corresponds to having \( m \) dimensions (\(|D| = m\)). Each family takes up one unit of capacity in exactly one of the dimensions (\( \nu_f^d = 1 \) is a \( m \)-dimensional unit vectors for every \( f \in F \)).

- **School choice with majority quotas** (Kojima, 2012; Hafalir et al., 2013): Each student is either a majority or a minority student. Each school has an overall cap on the number of students, which includes a cap for majority students. Let us again relabel a majority/minority student as a majority/minority family and a school as a locality. Let us also relabel “any student seats” as dimension \( d_1 \) and “majority student seats” as dimension \( d_2 \) (\(|D| = 2\)). In our model, the capacity of any locality for \( d_1 \) is greater than the capacity for \( d_2 \) (\( \kappa_{d_1}^\ell > \kappa_{d_2}^\ell \) for all \( \ell \in L \)). A majority family \( f \) takes up one unit of capacity in each dimension (\( \nu_f^d = (1,1) \)) whereas a minority family \( f' \) only takes up a unit of capacity in \( d_1 \) (\( \nu_{f'}^d = (1,0) \)).

- **Hungarian college admissions** (Biró et al., 2010): Students take up a college seat as well as a faculty seat. Both colleges and faculties have their own capacities. Let us relabel a student as a family and a college as a locality. Let us also relabel “college capacity” as the capacity of the locality in dimension \( d_1 \) (\( \kappa_{d_1}^\ell \)). Let us relabel the faculties as the remaining dimensions \( D \setminus \{d_1\} \). Therefore, each family \( f \)'s size is \( \nu_f^d = (1,0,0,\ldots,1,\ldots,0,0) \) where the second “1” is the unit of capacity taken up in \( d \in D \setminus \{d_1\} \).

- **Allocation of trainee teachers to schools in Slovakia and Czechia** (Cechlárová et al., 2015): Teachers are required to teach two out of three subjects and each school has a capacity for all three subjects. Let us relabel a teacher as a family, a school as a locality, and a subject as a dimension. In our model, this corresponds to having three dimensions (\(|D| = 3\)) and the size of every family \( f \) being either \( \nu_f^d = (0,1,1) \), \( \nu_f^d = (1,0,1) \), or \( \nu_f^d = (1,1,0) \).

- **College admission with multidimensional privileges in Brazil** (Aygün and Bó, 2020): Students can claim any combination of three privileges. Colleges have quotas for
each privilege, but a single student can claim more than one privilege. Let us relabel a student as a family, a college as a locality, and a privilege as a dimension. In our model, this corresponds to having three dimensions ($|D| = 3$) and the size of every family being an element of $\{0, 1\}^3$.

- Resident-hospital matching with sizes (McDermid and Manlove, 2010): Doctors apply to hospitals, but the doctors can take up more than one seat at a hospital, e.g., because they arrive as couples. Let us relabel doctors as families and hospitals as localities. In our model, this corresponds to having one dimension ($|D| = 1$) and families having an arbitrary size.26

Most of the models described above use further assumptions and develop solution approaches that suit their particular contexts but differ substantially from ours. Nevertheless, as we note throughout the paper, several impossibility and complexity results established in these papers will apply immediately to our framework.

References


26This model in turn generalizes the resident-hospital matching with inseparable couples (i.e., when couples have the same preference list and prefer to be unmatched to being in different hospitals) as well as resident-hospital matching with couples which have “consistent” preferences (McDermid and Manlove, 2010, Lemma 2.1). In both cases, we set $|D| = 1$ and $\nu_d \in \{1, 2\}$ in our model.


