

**Supplementary (Online) Materials for:
Inducing Leaders to Take Risky Decisions: Dismissal,
Tenure, and Term Limits
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Appendix B: The benchmark case with public informa-
tion**

We now discuss the case where the state and signal are publicly observed after each period in the commitment case.

Given this symmetric information, we examine a situation in which the leaders simply follow the signal in each period. This is easily enforced simply by using a mechanism in which a leader who does not follow signals is fired immediately. Thus, we simply examine the principal's optimal choice of how long to keep any given leader. This is a variation on a standard "bandit problem", and the optimal strategy for the principal can be expressed via a simple cut-off belief such that the leader is replaced at the end of period t whenever the posterior belief on her competence is lower than the cutoff.

Here, the "typical" replacement probability is bell-shaped over time, i.e., first increasing and then eventually decreasing. The intuition is follows. Given that replacing the leader is costly, unless signals are extremely accurate it will not be optimal for the principal to replace the leader immediately. In other words, there is an initial "honeymoon" period where the leader is not replaced as there is some initial number and fraction of failures before the leader's competence could begin to be revealed. This is for the short run. As for the very long run, if the leader survives for a long enough time, by the law of large numbers she is very likely to be competent, in which case she is unlikely to be replaced. Hence, in the very long run, the replacement probability also becomes small. It is in the middle range where the substantial replacement probability falls, as enough information to identify competence with some confidence has accumulated. Overall, we thus expect a bell-shaped replacement probability of the incumbent leader over time.

We suppose that $u(\lambda_0) - c > 0$. This guarantees that it is better to get a new leader than to keep a leader who is thought sufficiently incompetent that the principal would rather have them not even try to take action y even with a good signal.

Let $P(t)$ be the probability that a principal replaces a leader at time t (and kept the leader in all periods before t). An optimal strategy for the principal is to retain the leader as long as $\lambda_t \geq \lambda$ for some $0 < \lambda \leq \lambda_0$. We now show that the replacement probability follows a sawtooth pattern for any threshold strategy (optimal or not).

Claim: Sawtooth Patterns Suppose that the principal starts with some prior λ_0 and retains the leader at the end of period t if and only if $\lambda_t \geq \lambda$ for some $0 < \lambda \leq \lambda_0$. There exists $t > 1$ such that $P(t) > 0$ while $P(t + 1) = 0$ and $P(t + k) > 0$ for some $k \geq 1$.

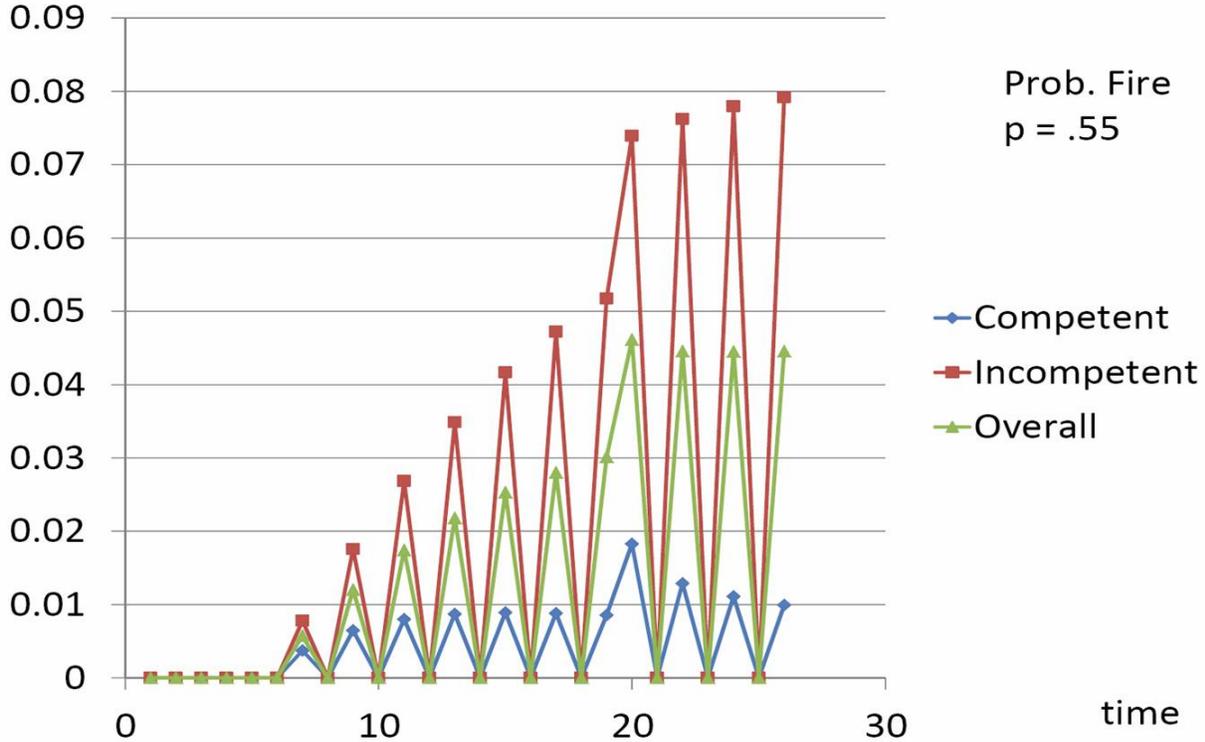


Figure 1: The probability of replacing a leader (for the first time) in various periods for $p = .55$, $\lambda_0 = 1/2$ and $\lambda(c) = 1/3$.

The sawtooth pattern suggested in the proposition is illustrated in Figure 1. That $P(t)$ is strictly positive in period 7 but zero in period 8 can be explained as follows. A leader who is fired in period 7 but not before necessarily had 7 failures in a row over the first seven periods. This follows since a leader is not fired in any previous period regardless of the number of failures, including six straight failures. So consider a leader who survived until period 8. Such a leader could not have had eight failures and no success as in that case she would already been fired at the end of period 7. Nor could she have had failures in all first seven periods followed by one success in period 8 because once again she would have been fired at the end of period 7. Thus the only possibility is that she had at most six failures and at least one success over the first seven periods. But then even if she had another failure in period 8 this is better than having had seven failures in a row over the first seven periods, which was the threshold for replacing the leader, and having seven failures and one success is closer to the posterior of having six failures and no successes than seven failures and no success.⁴⁷ This in turn implies that a leader who survived until period 8 will not be fired

⁴⁷Although our reasoning is particular to our discrete time setting, the sawtooth pattern will not fully disappear if we move to continuous time. In the continuous time case, after any success there will still be periods of time during which the leader is kept for sure. Again, a leader who makes it past some particular

in period 8. In a nutshell: *a success over the first periods buys the current leader some additional "grace period" where she is not fired.*

We can also examine just the positive probability dates as pictured in Figure 2. The fact that successes and failures come in integers leads the curves to be non-monotonic even when we look only at dates with positive probabilities.

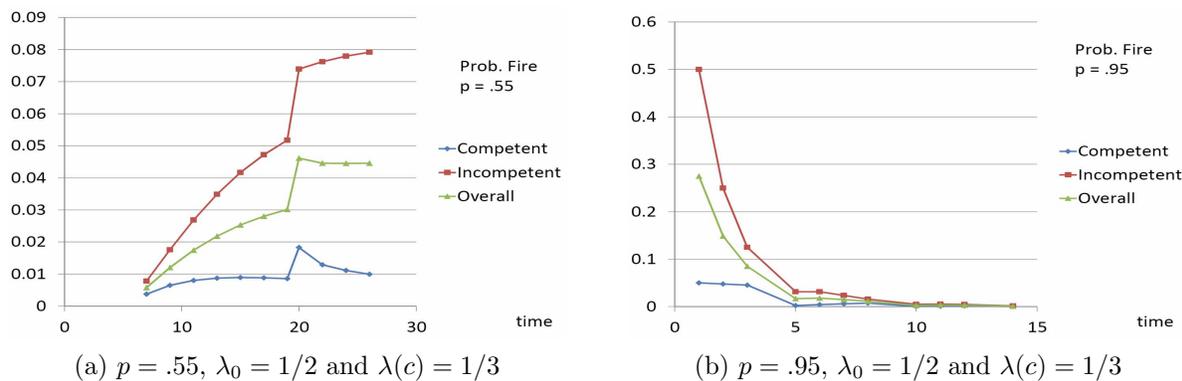


Figure 2: The probability of replacing a leader (for the first time) in various periods, *just the dates positive probabilities.*

We also see some interesting comparisons between situations where competent leaders are “barely competent” so that $p = .55$ and so hard to tell apart from incompetent leaders, compared to situations where competent leaders are “highly competent” so that $p = .95$ and very different from incompetent leaders. In the left-hand panel where competent leaders are barely competent, it takes time to sort out leaders, and so the probability of replacement is growing over time. Also, the probability of making a “false-positive” or type I error (replacing a competent leader) conditional on making a replacement starts out at roughly $1/2$ and then drops over time reaching about $1/8$ by period 25. Notice also that the probability of replacing a leader in any given period is quite small, less than $.1$ in all of the first 25 periods. Also, the first period where any replacement occurs is not even until period 7. In contrast, when competent leaders are “highly competent” then they are much easier to distinguish from incompetent ones, and replacements begin in period 1, and have a much higher probability ($.5$ for incompetent leaders in the first period). Moreover, the probabilities in that case are decreasing over time. The relative probability a type I error conditional on making a replacement starts out at $1/10$ and then actually increases for a few periods.⁴⁸

Cumulative Firing Probabilities in the Complete Information Case

time must have had some success, and so there can be periods in which the leader is fired with positive probability followed by ones in which the leader is not fired at all.

⁴⁸The cumulative probabilities of replacing incompetent leaders, as well as mistakenly replacing competent leaders, also exhibit some interesting patterns as pictured in Figure 3.

In Figure 3 we see that the cumulative probabilities are not ordered with respect to how competent a competent leader is. Although the replacement curves in the case of incompetent leader are ordered by the p 's (see Panel 3a), those for competent leaders are not ordered monotonically (see Panel 3b). The highest probability of replacing a good leader starts out highest for the case of $p = .8$ and lowest for $p = .55$, with $p = .95$ and $p = .65$ in between. The two cases of $.65$ and $.55$ eventually overtake the others, but start out lower because information is slow to accumulate in those cases and so replacement probabilities are low initially.

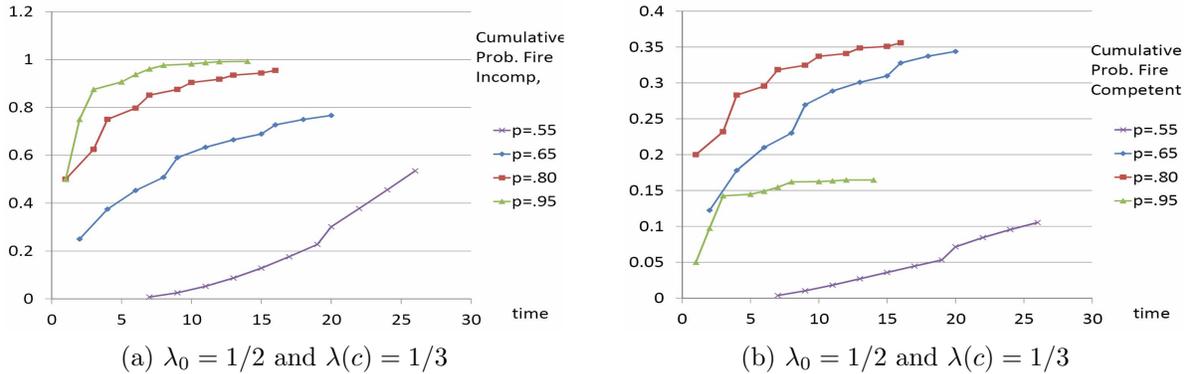


Figure 3: The cumulative probability of replacing a leader (for the first time).

By the law of large numbers, incompetent leaders will be recognized with a probability 1 (so $\lambda_t \rightarrow 0$ with probability 1 in the case of an incompetent leader), and so eventually any incompetent leader will be replaced as long as the cost of replacement is not so high that one would not replace a incompetent leader even if known to be incompetent. There are two main differences for what happens as a function of the cost of replacement: how quickly incompetent leaders are replaced, and with what probability competent leaders are replaced. These present a trade-off.

Proof of the Sawtooth Claim:

In the benchmark case in which states are observed as well as signals, we can let $o_t = 1$ if the leader is correct (a “success”) and $o_t = 0$ if the leader is incorrect (a “failure”). A sufficient statistic for λ_t given λ_0 is the cumulative number of successes through time t :

$$O_t = \sum_{t' \leq t} o_{t'}$$

As noted above, the posterior probability of having a competent leader $\lambda_t = \Lambda(O_t, t, \lambda_0)$ is increasing in the number of successes through a given time, O_t .

Let

$$O_t(\lambda_0, \lambda) = \{O \in \mathbb{N} : \Lambda(O, t, \lambda_0) < \lambda\}$$

denote the set of possible cumulative successes through time t for which the posterior falls below the threshold λ . Given that $\Lambda(O, t, \lambda_0)$ is increasing in O , there exists

$$\underline{O}_t(\lambda_0, \lambda) = \max O_t(\lambda_0, \lambda).$$

Thus, $O \in O_t(\lambda_0, \lambda)$ if and only if $O \leq \underline{O}_t(\lambda_0, \lambda)$ and $O \in \mathbb{N}$.

The proof of the claim makes use of the following two lemmas.

Lemma B1: Suppose that the principal starts with some prior λ_0 and continues to retain the leader as long as $\lambda_t \geq \lambda$ for some $0 < \lambda \leq \lambda_0$. Then $\underline{O}_t(\lambda_0, \lambda)$ is the *unique* value of O_t for which the principal fires the leader at period t and not before t (if $P(t) > 0$).

Proof of Lemma B1:

If $\underline{O}_t(\lambda_0, \lambda) = 0$, then this holds since this is the maximum value out of $O_t(\lambda_0, \lambda)$ and since there are no lower values, it must be unique. So, suppose $\underline{O}_t(\lambda_0, \lambda) > 0$, and also that to the contrary $o = \underline{O}_t(\lambda_0, \lambda) - k$, with $k > 0$, is another value for which the follower stops at period t and not before t . Note that since $k > 0$

$$\Lambda(o, t - 1, \lambda_0) < \Lambda(o + k, t, \lambda_0).$$

Since $\Lambda(o + k, t, \lambda_0) < \lambda$ (noting that $o + k = \underline{O}_t(\lambda_0, \lambda)$), it follows that $\Lambda(o, t - 1, \lambda_0) < \lambda$. This implies that $O_{t-1} \leq o$ would lead to the follower stopping at time $t - 1$ (or possibly before), and so o could not lead to a first stopping at time t since $O_t = o$ implies $O_{t-1} \leq o$ which would lead to stopping by $t - 1$.

Lemma B2: Suppose that the follower starts with some prior λ_0 and continues to retain the leader as long as $\lambda_t \geq \lambda$ for some $0 < \lambda \leq \lambda_0$. Let $P(t') > 0$ and $P(t) > 0$. Then $\underline{O}_{t'}(\lambda_0, \lambda) > \underline{O}_t(\lambda_0, \lambda)$ whenever $t' > t$, and $\underline{O}_{t'}(\lambda_0, \lambda) = 1 + \underline{O}_t(\lambda_0, \lambda)$ if $t' = t + 1$.

Proof of Lemma B2:

First, note that $\underline{O}_{t'}(\lambda_0, \lambda) \geq \underline{O}_t(\lambda_0, \lambda)$. To see this, suppose to the contrary that $o = \underline{O}_{t'}(\lambda_0, \lambda) < \underline{O}_t(\lambda_0, \lambda)$. However, if $O_{t'} = o$ then $O_t \leq o < \underline{O}_t(\lambda_0, \lambda)$ which implies that the follower would have stopped by t .

Next let us show that $\underline{O}_{t'}(\lambda_0, \lambda) \neq \underline{O}_t(\lambda_0, \lambda)$. This follows since $O_t \leq O_{t'}$ and so then the principal would have fired the leader by t if the two were the same. Next, let us show that $\underline{O}_{t+1}(\lambda_0, \lambda) = \underline{O}_t(\lambda_0, \lambda) + 1$. Note that $\underline{O}_t(\lambda_0, \lambda) \geq \underline{O}_{t+1}(\lambda_0, \lambda) - 1$ since

$$\Lambda(o, t, \lambda_0) < \Lambda(o + 1, t + 1, \lambda_0),$$

for any o . So if $\Lambda(o + 1, t + 1, \lambda_0) < \lambda$, it follows that $\Lambda(o, t, \lambda_0) < \lambda$. ■

To show the Sawtooth Claim, let \underline{t} be the first t for which $P(t) > 0$.⁴⁹

We now invoke the two lemmas above. They imply that if $P(t) > 0$ for all $t > \underline{t}$, then it would have to be that

$$\underline{O}_t = \underline{O}_{\underline{t}} + t - \underline{t}.$$

⁴⁹There exists such a t since a long enough string 0's will lead λ_t into any given neighborhood of 0. Any long enough string of 0's has positive probability.

This would imply that $\underline{Q}_t/t \rightarrow 1$, and so $\lambda_t \rightarrow 1$, which contradicts the fact that the leader would be followed. Thus, there is some finite t for which $P(t) = 0$. Note also, that $P(t) > 0$ infinitely often: it is possible that the leader is correct for any arbitrary initial number of periods and then incorrect thereafter for any given number of periods. Thus, for any τ and $\varepsilon > 0$ it is possible to have $\lambda_t > \lambda_0$ for all $t < \tau$, and then $\lambda_t < \varepsilon$ for large enough $t > \tau$. This implies that $P(t) > 0$ for some $t > \tau$. Since this is true for any τ , it is true infinitely often. ■