

# Simple Models of Pareto Income and Wealth Inequality

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## **Abstract**

This note is a companion to “Pareto and Piketty: The Macroeconomics of Top Income and Wealth Inequality.” It contains the details of the models discussed there.

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## 1. Introduction

This note should be viewed as an extended appendix to “Pareto and Piketty: The Macroeconomics of Top Income and Wealth Inequality” in the *Journal of Economic Perspectives*. That paper summarizes the main results from some simple models of income and wealth inequality. This note provides the details backing up the claims there. It is probably useful to read that paper first.

This note seeks to illustrate the simplest models of Pareto inequality. The model for income is about as simple as it can get and is quite useful for intuition and for understanding where Pareto distributions come from. The model for wealth builds on the key insight of the income model. However, it is more complicated, partly by nature and partly so that it can speak to the roles of “ $r - g$ ” and population growth that Piketty (2014) highlights in his book.

## 2. Income Inequality

The simplest models of Pareto inequality are surprisingly easy to understand. Pareto inequality emerges from *exponential growth that occurs for an exponentially-distributed amount of time*. Excellent introductions to Pareto models can be found in Mitzenmacher (2004), Gabaix (2009), Benhabib (2014), and Moll (2012b). Benhabib traces the history of Pareto-generating mechanisms and attributes the earliest instance of a simple model like that outlined here to Cantelli (1921).

To see how this works, we first require some heterogeneity. Suppose people are exponentially distributed across some variable  $x$ , which could denote age or experience or talent. For example,  $\Pr[\text{Age} > x] = e^{-\delta x}$ , where  $\delta$  denotes the death rate in the population.

Next, we need to explain how income varies with age in the population. A natural assumption is exponential growth: suppose income  $y$  rises exponentially with age (or experience or talent) at rate  $\mu$ :  $y = e^{\mu x}$ . Inverting this assumption gives us the age at which an individual earns income  $y$ :  $x(y) = 1/\mu \cdot \log y$ .

That's it, and the Pareto distribution then emerges easily:

$$\begin{aligned} \Pr [\text{Income} > y] &= \Pr [\text{Age} > x(y)] \\ &= e^{-\delta x(y)} \\ &= y^{-\frac{\delta}{\mu}} \end{aligned} \tag{1}$$

Recall that the Pareto inequality index is just the inverse of the exponent in this equation, which gives

$$\eta_{\text{income}} = \frac{\mu}{\delta}. \tag{2}$$

The Pareto exponent is increasing with  $\mu$ , the rate at which incomes grow with age (or experience or talent) and decreasing in the death rate  $\delta$ . Intuitively, the lower is the death rate, the longer some lucky people in the economy can benefit from exponential growth, which widens Pareto inequality. Similarly, faster exponential income growth across age (a higher return to experience?) also widens inequality. Jones and Kim (2014) build a richer model of labor income inequality along these lines that endogenizes  $\mu$  and  $\delta$ .

### 3. Wealth Inequality

A Pareto distribution of wealth can be obtained using a similar logic. Richer models of wealth inequality that motivate the simple model below can be found in Wold and Whittle (1957), Stiglitz (1969), Huggett (1996), Quadrini (2000), Castaneda, Diaz-Gimenez and Rios-Rull (2003), Benhabib and Bisin (2006), Cagetti and Nardi (2006), Nirei (2009), Benhabib, Bisin and Zhu (2011), Moll (2012a), Piketty and Saez (2012), Aoki and Nirei (2013), Moll (2014), and Piketty and Zucman (2014).

#### 3.1. Individual wealth

Let  $a$  denote an individual's wealth, which accumulates over time according to

$$\dot{a} = ra - \tau a - c \tag{3}$$

where  $r$  is the interest rate,  $\tau$  is a wealth tax, and  $c$  is the individual's consumption. Assume consumption is a constant fraction  $\alpha$  of wealth (e.g. as it will be with log utility), which yields

$$\dot{a} = (r - \tau - \alpha)a. \quad (4)$$

With this law of motion, the wealth of an individual of age  $x$  at date  $t$  is

$$a_t(x) = a_{t-x}(0) e^{(r-\tau-\alpha)x} \quad (5)$$

where  $a_{t-x}(0)$  is the initial wealth of a newborn at date  $t-x$ , described further below.

### 3.2. Heterogeneity through a birth-death process

The simple birth-death process here is a canonical model of the demography literature; for example, see Tuljapurkar (2008) or do a google search for “stable population theory”. The number of people born at date  $t$  is

$$B_t = B_0 e^{\bar{n}t}. \quad (6)$$

Death is a Poisson process with arrival rate  $\bar{d}$ . As shown at the end of this note, the stationary distribution for this birth-death process is exponential:

$$\Pr[\text{Age} > x] = e^{-(\bar{n}+\bar{d})x}. \quad (7)$$

To see the intuition behind this equation, notice that the (long-run) birth rate for this process is  $\bar{b} \equiv \bar{n} + \bar{d}$ .<sup>1</sup> That is, a fraction  $\bar{b}$  of the population is newly born at each instant, some to compensate for deaths and some representing net population growth. The age distribution then declines exponentially at rate  $\bar{b}$ .

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<sup>1</sup>The law of motion for the population is

$$\frac{\dot{N}_t}{N_t} = \frac{B_t}{N_t} - \bar{d}.$$

So population growth is constant if and only if  $N$  grows at the same rate as  $B$ , i.e. at rate  $\bar{n}$ . In this case,  $B/N = \bar{n} + \bar{d}$ .

### 3.3. The wealth distribution in partial equilibrium

Newborns equally inherit the wealth of the people who die in this economy:

$$\begin{aligned} a_t(0) &= \frac{\bar{d}K_t}{(\bar{n} + \bar{d})N_t} \\ &= \bar{a}k_t \end{aligned} \quad (8)$$

where  $\bar{a} \equiv \bar{d}/(\bar{n} + \bar{d})$  and  $k_t \equiv K_t/N_t$  is capital (wealth) per person in the economy. To understand this equation, consider the first line. The numerator in the first part of this equation,  $\bar{d}K_t$ , equals aggregate wealth of the people who die, and the denominator is the number of newborns. In the second line, notice that because of population growth, newborns inherit less than the average amount of capital per person in the economy, and this fraction is given by  $\bar{a}$ .

Assume that the macroeconomy is in steady state, so that capital per person grows at a constant and exogenous rate,  $g$ , over time:  $k_t = k_0 e^{gt}$ . Equation (8) can be used to help characterize the cross-section distribution of wealth at date  $t$ . In particular, the amount of wealth that a person of age  $x$  at date  $t$  inherited when they were born (at date  $t - x$ ) is

$$a_{t-x}(0) = \bar{a}k_{t-x} = \bar{a}k_t e^{-gx}. \quad (9)$$

And substituting this expression into (5), we obtain the cross-section of wealth at date  $t$  by age:

$$a_t(x) = \bar{a}k_t e^{(r-g-\tau-\alpha)x} \quad (10)$$

This is the exponential growth process that is one of the two key ingredients that delivers a Pareto distribution for normalized wealth, and one can already see that  $r - g$  plays a role. The other key ingredient is the exponential age distribution in equation (7), providing the heterogeneity. Together, these two building blocks give us our requirement: exponential growth occurs over an exponentially-distributed amount of time.

Inverting equation (10) gives the age at which a person in the cross-section achieves

wealth  $a$ :

$$x(a) = \frac{1}{r - g - \tau - \alpha} \log \left( \frac{a}{\bar{a}k_t} \right). \quad (11)$$

Then the wealth distribution is

$$\begin{aligned} \Pr [\text{Wealth} > a] &= \Pr [\text{Age} > x(a)] \\ &= e^{-(\bar{n} + \bar{d})x(a)} \\ &= \left( \frac{a}{\bar{a}k_t} \right)^{-\frac{\bar{n} + \bar{d}}{r - g - \tau - \alpha}}. \end{aligned} \quad (12)$$

Recall that Pareto inequality is measured by the inverse of the exponent in the expression above, which gives our first main result for wealth inequality:

$$\eta_{\text{wealth}} = \frac{r - g - \tau - \alpha}{\bar{n} + \bar{d}}. \quad (13)$$

### 3.4. The consumption share of wealth, $\alpha$

If expected lifetime utility is

$$\int_0^\infty e^{-(\rho + \bar{d})t} \log c_t dt \quad (14)$$

then it is straightforward to show that  $c_t = (\rho + \bar{d})a_t$ . That is, consumption is a constant fraction of wealth, and we have  $\alpha = (\rho + \bar{d})$ . The linearity of consumption in wealth applies more generally, delivering a richer formula for  $\alpha$ ; see Moll (2014), for example.

It is worth pausing here to address a natural question: why is there no  $N_t$  or  $B_t$  in the utility function? The answer is that leaving  $N_t$  out is the simplest approach. This case corresponds to the assumption that individuals do not care about their offspring, and this is consistent with the structure of the rest of the model — namely, that newborns equally inherit the wealth of the people who die. It would be useful to consider altruism, where newborns inherit wealth from parents who care about their well-being, and such structures have been considered in the literature cited earlier.

### 3.5. The wealth distribution in general equilibrium

We close the model in two different ways, which turn out to yield the same result for Pareto inequality in general equilibrium. Consider an “AK” production function:

$$Y_t = A_t K_t. \quad (15)$$

Our two cases are

1. **Capital model:** Here,  $A_t = \bar{A}$  is constant over time, and capital accumulates endogenously:  $\dot{K}_t = Y_t - C_t - T_t - \delta K_t$ , where  $C$  denotes aggregate consumption and  $T_t = \tau K_t$  denotes aggregate tax revenue. The fact that tax revenue enters the budget constraint (rather than being rebated lump sum) leads to substitution and income effects canceling. This case corresponds to the tax revenue being thrown away or alternatively being spent on a public good that enters utility in an additively separable fashion.
2. **Land model:** Alternatively, suppose  $A_t = A_0 e^{\bar{g}t}$  and let  $K_t = \bar{K}$  denote a fixed supply of land.

Both interpretations generate economic growth. The fact that they lead to identical Pareto wealth inequality highlights the fact that whether wealth is capital that accumulates or just land that does not does not matter from the standpoint of wealth inequality.

Because the details are somewhat involved, we’ll just report the main result first. In both cases, we assume that taxes are taken out of the economy and thrown away. In each, the interest rate in general equilibrium satisfies:

$$r - g - \tau - \alpha = \bar{n}, \quad (16)$$

so wealth inequality in general equilibrium is

$$\eta_{\text{wealth}} = \frac{\bar{n}}{\bar{n} + \bar{d}}. \quad (17)$$

What is going on here? The first intuition comes from the standard Euler equation for the standard neoclassical growth model with log utility, e.g.  $r - g = \rho$ . In particular, the interest rate moves one-for-one with the growth rate, and  $r - g$  is just a constant. Another feature of log utility is that substitution and income effects offset. This, together with the fact that we are throwing away the tax revenue in this setup, delivers the result that the tax rate does not matter for long-run inequality. If taxes are rebated lump sum, the tax parameter will matter once again for inequality in general; I suspect that the progressivity of the tax on wealth could also matter more generally.<sup>2</sup>

A second intuition is even more appropriate here. Recall that  $r - g - \tau - \alpha$  is the growth rate of an individual's normalized wealth. It is this growth rate that turns out to equal the rate of population growth,  $\bar{n}$ . To see why, look back at equation (8) and recall that each newborn inherits less than the average amount of capital per person in the economy; in fact, they get the fraction  $\frac{\bar{d}}{\bar{n} + \bar{d}}$ . Apart from this cohort effect, each person in this economy is essentially the same. In particular, in this setup, the size of each cohort grows at rate  $\bar{n}$ , so that the per capita wealth of each generation falls at rate  $\bar{n}$  as we look at younger and younger cohorts. But this is just another way of saying that normalized wealth — i.e. taking out macroeconomic growth at rate  $g$  — grows over time at rate  $\bar{n}$ . This is why the general equilibrium requires  $r - g - \tau - \alpha = \bar{n}$ .

An important implication of this reasoning can now be seen: if there were no population growth in the model, newborns would each inherit the per capita amount of wealth in the economy. The accumulation of wealth by individuals over time would correspond precisely to the growth in the per capita wealth that newborns inherit, and there would be no inequality in the model!

This section illustrates very nicely an important point about models of Pareto inequality: the general equilibrium of the model must be considered, and it can

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<sup>2</sup>The lump-sum rebate case makes the model more complicated, in that a lump sum rebate adds a form of income that is not directly proportional to wealth, so we lose the simple exponential growth that makes this model so easy, though results should still go through asymptotically. Heathcote, Storesletten and Violante (2014) highlight a related point and note that similar issues arise with progressive taxation.



change the comparative statics. For example, we already noted that in partial equilibrium, an increase in the population growth rate  $\bar{n}$  lowers Pareto inequality, as the concentration of wealth gets diluted by more offspring. In general equilibrium, the effect works in the opposite direction for the reasons discussed above. Similarly,  $r - g$  and  $\tau$  no longer matter for inequality in general equilibrium.

### 3.6. Details of the Capital Model

Since individual consumption is proportional to wealth, aggregate consumption is as well:  $C = \alpha K$ . For the baseline case, we assume that tax revenue is used to pay for government services that enter utility in an additively separable way, so the aggregate resource constraint for this economy is  $Y = C + I + T$ , where  $I$  is gross investment. The capital accumulation equation then implies that aggregate growth is  $g_Y = A - \delta - \tau - \alpha$ , and therefore per capita growth is  $g = A - \delta - \tau - \alpha - \bar{n}$ .

The equilibrium interest rate in this model is just the net marginal product of capital:  $r = A - \delta$ . Combining these last two equations gives the key result needed above:  $r - g - \tau - \alpha = \bar{n}$ .

*Lump-sum rebate of tax revenue:* The case in which the wealth tax is rebated lump sum is different, however. In this case, the exponential growth of normalized wealth across ages breaks down, except at the very top for the wealthiest people: the lump sum rebate is a vanishing fraction of the wealth for the richest households. So the partial equilibrium equation for  $\eta$  continues to apply, but only at the very top of the wealth distribution. Now, however, the aggregate resource constraint is  $Y = C + I$ , so that all of the tax revenue comes back into the economy as consumption or investment. In this case, aggregate growth is  $g_Y = A - \delta - \alpha$ , which is invariant to the tax rate in the log utility case. Now  $r - g - \tau - \alpha = \bar{n} - \tau$ , and top wealth inequality is given by

$$\eta_{\text{wealth}}^{\text{lumpsum}} = \frac{\bar{n} - \tau}{\bar{n} + \bar{d}}. \quad (18)$$

So what happens to the tax revenue matters crucially for the effect of wealth taxes on top wealth inequality.<sup>3</sup>

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<sup>3</sup>This analysis requires  $\bar{n} \geq \tau$ .

### 3.7. Details of the Land Model

For the land model, let  $P_t$  denote the price (measured in units of output) of one unit of land. Aggregate wealth is then  $W_t = P_t \bar{K}$ . The price of land satisfies a standard arbitrage equation:

$$r = \frac{A_t}{P_t} + \frac{\dot{P}_t}{P_t}. \quad (19)$$

That is, one can invest  $P$  units of output in the bank and earn interest on it, or one can buy a unit of land, earn the dividend  $A_t$ , and then sell it, pocketing the capital gain. Along a balanced growth path (no bubbles), this equation implies the capital gain term equals the growth rate of  $A$ ,  $\bar{g}$ , so the price of land is pinned down by

$$P_t = \frac{A_t}{r - \bar{g}}. \quad (20)$$

Aggregate consumption in this economy can be computed in two ways, and this allows us to solve for the interest rate. First,

$$C = \alpha W = \alpha P_t \bar{K} = \frac{\alpha}{r - \bar{g}} \cdot A_t \bar{K} = \frac{\alpha}{r - \bar{g}} \cdot Y_t. \quad (21)$$

Alternatively,

$$\begin{aligned} C_t &= Y_t - T_t = Y_t - \tau W_t = Y_t - \tau P_t \bar{K} \\ &= Y_t - \frac{\tau}{r - \bar{g}} \cdot A_t \bar{K} = \left(1 - \frac{\tau}{r - \bar{g}}\right) Y_t \end{aligned} \quad (22)$$

Equating these two expressions for consumption and noting that  $g_Y = \bar{g}$  so that  $g = \bar{g} - \bar{n}$  gives the required solution for the interest rate:  $r - g - \tau - \alpha = \bar{n}$ . Wealth inequality is therefore given by equation (13).

*Lump-sum rebate:* If tax revenues are rebated lump sum, then  $C = Y$ . Then from (21), we must have  $r - \bar{g} = \alpha$ , so that  $r - g - \alpha = \bar{n}$  and therefore  $r - g - \tau - \alpha = \bar{n} - \tau$ , and inequality with lump sum rebates is also given by equation (18) in the land version of the model.

### 3.8. The stationary distribution of the simple birth-death process

Let  $G(x, t) = \Pr[\text{Age} > x]$  denote the complementary form of the age distribution at time  $t$ . With population growth rate  $\bar{n}$  and death rate  $\bar{d}$ , the distribution evolves over a small time interval  $\Delta t$  as

$$G(x, t + \Delta t) = \frac{1 - \bar{d}\Delta t}{1 + \bar{n}\Delta t} \cdot G(x, t) + G(x - \Delta x, t) - G(x, t). \quad (23)$$

The first term captures the change from deaths and population growth (to keep the distribution proper), while the last two terms capture the inflow of younger people into the higher ages.

Using a Taylor expansion for  $\frac{1}{1 + \bar{n}\Delta t} \approx 1 - \bar{n}\Delta t$  and ignoring the higher order terms leads to

$$\frac{G(x, t + \Delta t) - G(x, t)}{\Delta t} = -(\bar{n} + \bar{d})G(x, t) - \frac{G(x, t) - G(x - \Delta x, t)}{\Delta x}, \quad (24)$$

where we've also used the fact that  $\Delta x = \Delta t$ .

Taking the limits as  $\Delta t \rightarrow 0$  gives

$$\frac{\partial G(x, t)}{\partial t} = -(\bar{n} + \bar{d})G(x, t) - \frac{\partial G(x, t)}{\partial x} \quad (25)$$

Setting the time derivative equal to zero and solving for the stationary distribution yields the desired result:

$$G(x) = e^{-(\bar{n} + \bar{d})x}. \quad (26)$$

## References

- Aoki, Shuhei and Makoto Nirei, "Pareto Distributions and the Evolution of Top Incomes in the U.S," MPRA Paper 47967, University Library of Munich, Germany July 2013.
- Benhabib, Jess, "Wealth Distribution Overview," 2014. NYU teaching slides <http://www.econ.nyu.edu/user/benhabib/wealth%20distribution%20theories%20overview3.pdf>.
- , Alberto Bisin, and Shenghao Zhu, "The Distribution of Wealth and Fiscal Policy in Economies With Finitely Lived Agents," *Econometrica*, 01 2011, 79 (1), 123–157.
- and —, "The distribution of wealth and redistributive policies," 2006 Meeting Papers 368, Society for Economic Dynamics December 2006.
- Cagetti, Marco and Mariacristina De Nardi, "Entrepreneurship, Frictions, and Wealth," *Journal of Political Economy*, October 2006, 114 (5), 835–870.
- Cantelli, F.P., "Sulle applicazioni del calcolo delle probabilita alla fisica molecolare," *Metron*, 1921, 1 (3), 83–91.
- Castaneda, Ana, Javier Diaz-Gimenez, and Jose-Victor Rios-Rull, "Accounting for the U.S. Earnings and Wealth Inequality," *Journal of Political Economy*, August 2003, 111 (4), 818–857.
- Gabaix, Xavier, "Power Laws in Economics and Finance," *Annual Review of Economics*, 2009, 1 (1), 255–294.
- Heathcote, Jonathan, Kjetil Storesletten, and Giovanni L. Violante, "Optimal Tax Progressivity: An Analytical Framework," NBER Working Papers 19899, National Bureau of Economic Research, Inc February 2014.
- Huggett, Mark, "Wealth Distribution in Life-Cycle Economies," *Journal of Monetary Economics*, December 1996, 38 (3), 469–494.
- Jones, Charles I. and Jihee Kim, "A Schumpeterian Model of Top Income Inequality," October 2014. Stanford University manuscript.
- Mitzenmacher, Michael, "A Brief History of Generative Models for Power Law and Lognormal Distributions," *Internet Mathematics*, 2004, 1 (2).

- Moll, Benjamin, "Inequality and Financial Development: A Power-Law Kuznets Curve," 2012. Princeton University working paper.
- , "Lecture 6: Income and Wealth Distribution," 2012. Princeton teaching slides [http://www.princeton.edu/~moll/ECO521Web/Lecture6.ECO521\\_web.pdf](http://www.princeton.edu/~moll/ECO521Web/Lecture6.ECO521_web.pdf).
- , "Why Piketty Says  $r - g$  Matters for Inequality," 2014. Princeton teaching slides [http://www.princeton.edu/~moll/piketty\\_notes.pdf](http://www.princeton.edu/~moll/piketty_notes.pdf).
- Nirei, Makoto, "Pareto Distributions in Economic Growth Models," IIR Working Paper 09-05, Institute of Innovation Research, Hitotsubashi University July 2009.
- Piketty, Thomas, *Capital in the Twenty-first Century*, Harvard University Press, 2014.
- and Emmanuel Saez, "A Theory of Optimal Capital Taxation," NBER Working Papers 17989, National Bureau of Economic Research, Inc April 2012.
- and Gabriel Zucman, "Capital is Back: Wealth-Income Ratios in Rich Countries, 1700-2010," *Quarterly Journal of Economics*, forthcoming 2014, pp. xxx-xxx.
- Quadrini, Vincenzo, "Entrepreneurship, Saving and Social Mobility," *Review of Economic Dynamics*, January 2000, 3 (1), 1-40.
- Stiglitz, Joseph E, "Distribution of Income and Wealth among Individuals," *Econometrica*, July 1969, 37 (3), 382-97.
- Tuljapurkar, Shripad, "Stable Population Theory," in Steven N. Durlauf and Lawrence E. Blume, eds., *The New Palgrave Dictionary of Economics*, second edition ed., Palgrave MacMillan, 2008.
- Wold, Herman OA and Peter Whittle, "A Model Explaining the Pareto Distribution of Wealth," *Econometrica*, 1957, pp. 591-595.