Kinship, Incentives and Evolution

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Appendix (supplementary materials)

Proposition 2

Let $\hat{\tau}_i : \Omega \to \left[0, y^H\right]$ be the function that defines, for every state $\omega \in \Omega$, the transfer that individual i would like to make to his or her sibling *if the latter makes no* transfer to i. Then $\hat{\tau}_i(\omega) = 0$ if $u'(y_i) \geq \alpha_i u'(y_j)$ for $j \neq i$, otherwise the optimal transfer $\hat{\tau}_i(\omega)$ is positive and equates i's marginal material utility to that of his sibling's when weighted by his own (i's) degree of altruism:

$$u'(y_i - \hat{\tau}) = \alpha_i u'(y_j + \hat{\tau}). \tag{S1}$$

Thus:

$$\hat{\tau}_i(\omega) = \max\left\{0, \hat{\tau}\right\}, \tag{S2}$$

where $\hat{\tau}$ is uniquely defined by (S1). Since the material utility function is separable in consumption and effort, efforts play no role when determining the transfers, only outputs matter. The claims in Proposition 2 follow from the following lemma:

LEMMA 1: *For each $\omega \in \Omega$, the transfer pair $(\hat{\tau}_A(\omega), \hat{\tau}_B(\omega))$ constitutes a Nash equilibrium of $G(\omega)$. If $\alpha_A \alpha_B < 1$, then this equilibrium is unique. If $\alpha_A = \alpha_B = 1$, then there is a continuum of Nash equilibria, all resulting in equal sharing of the total output.*

Proof: We first prove that if $\alpha_A \alpha_B < 1$, in any equilibrium at most one transfer is strictly positive. Suppose that $\alpha_A \alpha_B < 1$, and that (t_A, t_B) is a Nash equilibrium with $t_A, t_B > 0$. Then the following two first-order conditions must both hold:

$$u'(y_A - t_A + t_B) = \alpha_A u'(y_B + t_A - t_B)$$

$$u'(y_B + t_A - t_B) = \alpha_B u'(y_A - t_A + t_B)$$

and hence

$$u'(y_A - t_A + t_B) = \alpha_A \alpha_B u'(y_A - t_A + t_B)$$

implying $\alpha_A \alpha_B = 1$, contradicting our hypothesis that $\alpha_A \alpha_B < 1$.

Suppose that $\alpha_A \alpha_B < 1$. Then the previous observation, together with (S2), implies that $(t_A, t_B) = \left(\hat{\tau}_A(\omega), \hat{\tau}_B(\omega)\right) = (0, 0)$ is the unique equilibrium in the two states ω in which $y_A = y_B$. The previous observation further implies that, in the states ω where $y_A \neq y_B$, the individual with the low output y^L gives no transfer, since $u'(y^L - \hat{\tau}) > \alpha_i u'(y^H + \hat{\tau})$ for all $\alpha_i \in [0, 1]$ and $\hat{\tau} \geq 0$. Hence, if $y_A > y_B$, the unique Nash equilibrium of $G(\omega)$ is $(t_A, t_B) = \left(\hat{\tau}_A(\omega), 0\right)$, and, if $y_B > y_A$, it is $(t_A, t_B) = \left(0, \hat{\tau}_B(\omega)\right)$.

Suppose now that $\alpha_A \alpha_B = 1$; then it is straightforward to verify the following claims:

- if $y_A > y_B$ any $(t_A, t_B) = \left(\hat{\tau}_A(\omega) + \varepsilon, \varepsilon\right)$ is a Nash equilibrium of $G(\omega)$ for all $\varepsilon \in \left(0, y_A - \hat{\tau}_A(\omega)\right)$
- if $y_B > y_A$ any $(t_A, t_B) = \left(\varepsilon, \hat{\tau}_B(\omega) + \varepsilon\right)$ is a Nash equilibrium of $G(\omega)$ for all $\varepsilon \in \left(0, y_B - \hat{\tau}_B(\omega)\right)$
- if $y_A = y_B$ any $(t_A, t_B) = (\varepsilon, \varepsilon)$ is a Nash equilibrium of $G(\omega)$ for all $\varepsilon \in \left[0, y_A\right]$.

Proposition 4

The first-order condition (8) implicitly defines the transfer t as a differentiable function of λ. An application of the implicit function theorem gives

$$\frac{dt}{d\lambda} = -\frac{\alpha u''\left(\lambda y^H + t\right)}{\alpha u''\left(\lambda y^H + t\right) + u''\left(y^H - t\right)} \cdot y^H,$$

where, by strict concavity of u, the ratio on the right-hand side is a number in the open unit interval. Hence

$$\frac{d\left(\lambda y^H + t\right)}{d\lambda} = \left[1 - \frac{\alpha u''\left(\lambda y^H + t\right)}{\alpha u''\left(\lambda y^H + t\right) + u''\left(y^H - t\right)}\right] \cdot y^H > 0,$$

and

$$\frac{d\left(y^H - t\right)}{d\lambda} = \frac{\alpha u''\left(\lambda y^H + t\right)}{\alpha u''\left(\lambda y^H + t\right) + u''\left(y^H - t\right)} \cdot y^H > 0.$$

Proposition 6

First, assume that $\alpha_A, \alpha_B < \hat{\alpha}$. Then $T(\alpha_A) = T(\alpha_B) = 0$, and inspection of (14) shows that the equation system (13) is independent of α_A and α_B. Hence, its solution set is unaffected by a marginal increase in any one, or both, of these parameters.

Second, assume that condition (15) is met. Then the Jacobian of the equation system (13) is non-null, a condition, which, by the Inversion Theorem (see, e.g., Theorem 41.8 in Robert G. Bartle, 1976), guarantees local uniqueness of the solution to (3). Suppose that $\alpha_i > \hat{\alpha}$, and consider an increase in α_i.

Step 1: First, we prove that, for each success probability of the other individual, p_j, individual i's best response is strictly increasing in α_i. From (13) and noting that $\left(\psi'\right)^{-1}$ is an increasing function, this claim holds if

$$\frac{\partial g\left(p_j, \alpha_i, \alpha_j\right)}{\partial \alpha_i} > 0.$$

Using the first-order condition (8) for the transfer $T(\alpha_i)$, we obtain:

$$\frac{\partial g\left(p_j, \alpha_i, \alpha_j\right)}{\partial \alpha_i} = \left(1 - p_j\right) \cdot \left[u\left(y^L + T(\alpha_i)\right) - u\left(y^L\right)\right] + p_j \cdot \left[u\left(y^H\right) - u\left(y^H - T(\alpha_j)\right)\right].$$

The expression on the right-hand side is positive, since $\alpha_i > \hat{\alpha}$ implies $T(\alpha_i) > 0$.

Step 2: Secondly, we prove that, for each success probability p_i, individual j's best response is strictly decreasing in α_i. For this claim, it is sufficient to show that:

$$\frac{\partial g\left(p_i, \alpha_j, \alpha_i\right)}{\partial \alpha_i} < 0.$$

Using the first-order condition (8) for the transfer $T\left(\alpha_j\right)$, we obtain:

$$\frac{\partial g\left(p_i, \alpha_j, \alpha_i\right)}{\partial \alpha_i} = -p_i \cdot \left(1 - \alpha_i \alpha_j\right) \cdot u'\left(y^L + T\left(\alpha_i\right)\right) T'\left(\alpha_i\right).$$

The expression on the right-hand side is strictly negative, since $\alpha_i > \hat{\alpha}$.

In sum: for $\alpha_i > \hat{\alpha}$ an increase in α_i causes an upward shift in i's success-probability reaction function, and a downward shift in j's success-probability reaction function. In the case of a unique equilibrium, the best-response curves intersect in such a way that either shift is sufficient, *per se*, to yield that p_i^* increases and p_j^* decreases.

Proposition 7

To establish the uniqueness claim, note that, by hypothesis, the left-hand side of (16) is continuous and increasing from zero to plus infinity, while the right-hand side is a decreasing affine function with positive intercept. The latter property becomes transparent after some algebraic manipulation: equation (16) can be written in the form

$$\psi'(p) = a - (1 + \alpha) bp$$

for

$$a = u\left(y^H - T(\alpha)\right) - u\left(y^L\right) + \alpha\left[u\left(y^L + T(\alpha)\right) - u\left(y^L\right)\right]$$

and

$$b = u\left(y^L + T(\alpha)\right) - u\left(y^L\right) - \left[u\left(y^H\right) - u\left(y^H - T(\alpha)\right)\right],$$

where $a, b > 0$. That a is positive follows from our earlier observation that a donor remains richer than the recipient, $u\left(y^H - T(\alpha)\right) > u\left(y^L\right)$. That b is positive follows from the concavity of u, implying that the recipient's material utility increases more from the transfer than the donor's material utility decreases.

Proposition 11

We proceed in two steps. First, we characterize the socially optimal probability p and transfer t, to be given by the rich to the poor, under a Benthamite social welfare function. Secondly, we verify that these coincide with the equilibrium probabilities p_A^* and p_B^*, and transfers $T(\alpha_A)$ and $T(\alpha_B)$ if and only if $\alpha_A = \alpha_B = 1$.

Step 1: Consider a hypothetical social planner who chooses a probability p and transfer t so as to maximize the expected material utility of one individual (since they are *ex ante* identical, this also maximizes the sum of their material utilities):

$$\begin{aligned} W(p, t) &= \left[p^2 u\left(y^H\right) + (1 - p)^2 u\left(y^L\right)\right] \\ &\quad + p(1 - p)\left[u\left(y^H - t\right) + u\left(y^L + t\right)\right] - \psi(p). \end{aligned} \tag{S3}$$

The necessary first-order condition for an interior solution for p is

$$\text{(S4)} \qquad 2pu\left(y^H\right) - 2\left(1-p\right)u\left(y^L\right) + \left(1-2p\right)\left[u\left(y^H - t\right) + u\left(y^L + t\right)\right] = \psi'\left(p\right).$$

Moreover, for any value of p, the value of t that maximizes $W(p,t)$ is such that both individuals end up with the same consumption in all states: $y^H - t = y^L + t$.

Step 2: When positive, the equilibrium transfer satisfies (8). Strict concavity of u implies that $y^H - T(\alpha) = y^L + T(\alpha)$ if and only if $\alpha = 1$. Hence, $\alpha = 1$ is a necessary condition for the equilibrium outcome to coincide with the Benthamite optimum. It is also a sufficient condition, since the first-order condition that defines the equilibrium success probability p^*, equation (16), for $T(\alpha) = (1-\lambda)\,y^H/2$, coincides with (S4), the necessary first-order condition for an interior social optimum, if and only if $\alpha = 1$.

Corollary 12

Given the symmetry of the unique equilibrium outcome, this is Pareto efficient if and only if it maximizes the sum of both individuals' expected welfare, as defined in equation (6). If each individual chooses a success probability p and gives a transfer t when rich and the other is poor, the mentioned sum is $S(p,t) = (1+\alpha)W(p,t)$, where $W(p,t)$ is defined in (S3). For any $\alpha \in [0,1]$, $S(p,t)$ is clearly strictly increasing in $W(p,t)$. But, by Proposition 11, the equilibrium expected material utility V^* coincides with the maximum value of $W(p,t)$ if and only if $\alpha = 1$.