Appendix 3: Cognitive Hierarchy

As a robustness check, we conduct our analysis with the cognitive hierarchy model of Camerer, Ho and Chong (2004). There, the distribution of types is Poisson distributed, i.e., the proportion of T_k is given by

$$p_k = \frac{e^{-\tau}\tau^k}{k!}.$$

T_k best responds given the belief that the others players are T_0 up to T_{k-1}. T_k's belief about the proportion of $T_{l<k}$ is

$$g_k(l) = \frac{p_l}{\sum_{h=0}^{k-1} p_h}.$$

The cognitive hierarchy model is developed for normal form games only. In order to adapt the model to games with pre-play communication we must specify how beliefs are updated after messages have been received. We assume Bayesian updating. For the games preceded by one round of communication, let $q_{ki}(m_i)$ denote the probability that a T_k player i sends the message m_i (and is allowed to send a message). T_k's belief that the sender i is a $T_{l<k}$ player conditional upon receiving the message m_i is

$$g_{ki}(l|m_i) = \frac{g_k(l)\,q_{li}(m_i)}{\sum_{h=0}^{k-1} g_k(h)\,q_{hi}(m_i)} = \frac{p_l q_{li}(m_i)}{\sum_{h=0}^{k-1} p_h q_{hi}(m_i)},$$

where the latter equality follows from the definition of $g_k(l)$.

We retain the assumption that players randomize uniformly when indifferent, but that they prefer to be honest if it does not affect expected payoffs. This implies that the behavior of T_0 and T_1 is the same in the cognitive hierarchy model and in the level-k model.

A feature of the cognitive hierarchy model is that if T_k plays a strategy that is a best response to T_k opponent in a two-player game with one round of pre-play communication, then $T_{m>k}$ will play the same strategy. Since this result will be used repeatedly we state it separately in Lemma 1.

LEMMA 1: *Let G be a symmetric two-player normal form game. If T_k plays a strategy profile that is a best response to a T_k opponent in G, $\Gamma_I(G)$ or $\Gamma_{II}(G)$, then $T_{m>k}$ play this strategy too.*

PROOF:

Consider the case of one-way communication (the proof for two-way communication and without communication is analogous). Let the strategy played by T_k be denoted $s^* = \langle m^*, a^*, f^*(m)\rangle$. Consider a T_k player that received the message m. We know that $f^*(m)$ is the action that maximizes expected payoff conditional on receiving m given the belief that the opponent is $T_{l<k}$ with probability

$$g_k(l|m) = \frac{p_l q_l(m)}{\sum_{h=0}^{k-1} p_h q_h(m)}.$$

Similarly, a T_{k+1} player that receives the same message m best responds given the belief that the opponent is a $T_{l<k+1}$ player with probability

$$g_{(k+1)}(l|m) = \frac{p_l q_l(m)}{\sum_{h=0}^{k} p_h q_h(m)}.$$

Since $f^*(m)$ maximizes the expected payoff of T_k and is a best response to a T_k sender, by linearity of expected payoffs it must be a best response also to the mixture of types T_{k+1} believes to be facing (note that this argument does not extend to more than two players).

Now consider the communication stage of the game. The message m^* followed by the action a^* is a best response given the belief that the opponent is $T_{l<k}$ with probability

$$g_k(l) = p_l / \sum_{h=0}^{k-1} p_h.$$

Similarly a T_{k+1} player believes that the opponent is $T_{l<k+1}$ with probability

$$g_{k+1}(l) = p_l / \sum_{h=0}^{k} p_h.$$

Since m^* and a^* maximizes the payoff of T_k and is a best response against another T_k player, it must be a best response also to the mixture of types T_{k+1} believes to be facing.

By induction this reasoning holds for all $T_{m>k}$ players.

In the cognitive hierarchy model, predicted behavior depends both on the payoff configuration and the average of the type distribution, τ. A complete characterization of behavior is therefore intractable, and the remainder of this appendix focuses on T_2 and T_3 in the class of symmetric and generic 2×2 games. However, a general characterization of the behavior of T_3 in mixed motive games with two-way communication is also intractable, so in this case we focus on T_2 only. For simplicity, we finally disregard cases in which the combination of τ and the payoff structure of the game implies that T_{2+} is indifferent between strategies as well as games in which neither action is risk dominant.

Two general findings emerge from the analysis. First, when τ is close to zero, T_2 and T_3 players are practically certain that the opponent is T_0 and consequently play the same action as T_1. However, T_2 and T_3 may send another message since they take into account the possibility that the opponent is (a responsive) T_1. Second, for sufficiently large τ, T_{2+} play as in the level-k model in all games except possibly T_{3+} in mixed motive games with two-way communication. In this part of the parameter space, the level-k model is robust to the assumption about lexicographic beliefs. For intermediate levels of τ, T_2 and T_3 best-respond to the mixture of lower-level types they believe they are facing.

An interesting new finding is that the behavior in the Stag Hunt hypothesized by Aumann (1990) emerges endogenously in the model. With the payoffs in Aumann's original example, depicted in Figure 2, a T_{3+} player sends the message h and plays L as sender, and plays L irrespective of the received message as receiver, whenever τ is between 0.547 and 1.646. As a sender, T_{3+} does so in order to induce T_1 and T_2 to play H, but believes that there such a high probability of meeting a randomizing T_0 that it is better to play L. T_{3+} ignores the received message because of the likelihood of meeting a T_2 opponent, who sends h messages that are not self-signalling.

In dominance solvable games, T_1 sends and plays the dominant strategy, so by Lemma 1, T_{1+} does so too (irrespective of whether communication is possible). We now proceed to characterize the behavior in the two remaining classes of games.

Coordination games

As before, we assume that H(igh) is the payoff dominant equilibrium, i.e., $u_{HH} > u_{LL}$.

OBSERVATION 8: *(No communication) T_{1+} plays the risk dominant action.*

PROOF:

T_1 plays the risk dominant action. Consequently, by Lemma 1 all higher level types do the same.

Absent communication, behavior is the same in the level-k and cognitive hierarchy models.

OBSERVATION 9: *(One-way communication) If H is the risk dominant action, T_{1+} sends h and plays H as sender and responds to messages as receiver. If L is the risk dominant action, T_1 sends l and plays L as sender and respond to messages as receiver, but the behavior of T_{2+} depends on the payoff structure of the game:*
Case 1 ($u_{LH} > u_{LL}$): *Let $\alpha \equiv (u_{LL} - u_{HL}) / (u_{HH} - u_{LH})$. If $\tau < (\alpha - 1)/2$, then T_2 plays $\langle h, L, H, L\rangle$ and T_3 plays as follows:*

T_3 plays $\langle h, L, H, L\rangle$ if $\tau < \sqrt{\alpha} - 1$ and $\tau < \left(\sqrt{\alpha+1}+1\right)/\alpha$,
T_{3+} plays $\langle h, L, L, L\rangle$ if $\left(\sqrt{\alpha+1}+1\right)/\alpha < \tau < \sqrt{\alpha} - 1$,
T_{3+} plays $\langle h, H, H, L\rangle$ if $\sqrt{\alpha} - 1 < \tau < \left(\sqrt{\alpha+1}+1\right)/\alpha$,
T_3 plays $\langle h, H, L, L\rangle$ if $\tau > \sqrt{\alpha} - 1$ and $\tau > \left(\sqrt{\alpha+1}+1\right)/\alpha$.

If $\tau > (\alpha - 1)/2$, then T_{2+} play $\langle h, H, H, L\rangle$.
Case 2 ($u_{LH} < u_{LL}$): *Let $\beta \equiv (u_{LH} - u_{HL}) / (u_{HH} - u_{LL})$. If $\tau < (\beta - 1)/2$, then T_2 plays $\langle l, L, H, L\rangle$ and T_3 plays $\langle l, L, H, L\rangle$ if $\tau < \sqrt{\beta} - 1$ and $\langle h, H, H, L\rangle$ if $\tau > \sqrt{\beta} - 1$. If $\tau > (\beta - 1)/2$, then T_{2+} plays $\langle h, H, H, L\rangle$.*

PROOF:

First consider the case when H is risk dominant. As in the level-k model, T_1 plays $\langle h, H, H, L\rangle$ (facing randomizing T_0 receivers and truthful T_0 senders). Since this strategy is a best-response to itself, T_{2+} plays the same strategy.

Now consider the case when L is risk dominant so that T_1 plays $\langle l, L, H, L\rangle$. For T_2 senders, the strategy $\langle l, H\rangle$ is dominated by $\langle h, H\rangle$, so we need not consider that strategy. The expected payoff for the remaining three sender strategies are

$$\pi(\langle l, L\rangle) = g_2(0)\frac{1}{2}(u_{LL} + u_{LH}) + g_2(1)\,u_{LL},$$
$$\pi(\langle h, L\rangle) = g_2(0)\frac{1}{2}(u_{LL} + u_{LH}) + g_2(1)\,u_{LH},$$
$$\pi(\langle h, H\rangle) = g_2(0)\frac{1}{2}(u_{HL} + u_{HH}) + g_2(1)\,u_{HH}.$$

If $u_{LH} > u_{LL}$, then it is clear that T_2 senders play either $\langle h, L\rangle$ or $\langle h, H\rangle$. The payoff from playing $\langle h, L\rangle$ is higher whenever τ is sufficiently low,

$$\tau < \frac{(u_{LL} + u_{LH}) - (u_{HL} + u_{HH})}{2(u_{HH} - u_{LH})} = (\alpha - 1)/2.$$

Similarly, if $u_{LH} < u_{LL}$, then T_2 senders prefer $\langle l, L\rangle$ over $\langle h, H\rangle$ whenever

$$\tau < \frac{(u_{LL} + u_{LH}) - (u_{HL} + u_{HH})}{2(u_{HH} - u_{LL})} = (\beta - 1)/2.$$

T_2 receivers face truthful T_1 and T_2 senders, so they respond to messages. It is clear that for sufficiently high τ, T_{2+} plays $\langle h, H, H, L\rangle$.

We now go on to consider the behavior of T_3 when τ is below the thresholds above. First suppose that $u_{LH} > u_{LL}$ and $\tau < (\alpha - 1)/2$. Then T_3 senders prefer $\langle h, L\rangle$ over $\langle h, H\rangle$ whenever

$$g_3(0)\frac{1}{2}(u_{LL}+u_{LH}) + (g_3(1)+g_3(2))\,u_{LH} > g_3(0)\frac{1}{2}(u_{HL}+u_{HH}) + (g_3(1)+g_3(2))\,u_{HH},$$

which simplifies to $(1+\tau/2)\,\tau < (\alpha-1)/2$. Since the left hand side is larger than τ, this condition may or may not hold. Both sides of the inequality are positive, so the condition is equivalent to $\tau < \sqrt{\alpha} - 1$. Suppose now that $u_{LH} < u_{LL}$. Then T_3 senders prefer $\langle l, L\rangle$ over $\langle h, H\rangle$ whenever

$$(1+\tau/2)\,\tau < \frac{(u_{LL}+u_{LH})-(u_{HL}+u_{HH})}{2(u_{HH}-u_{LL})} = (\beta-1)/2$$

Both sides of the inequality are positive, so this condition is equivalent to $\tau < \sqrt{\beta} - 1$.

Finally, T_3's behavior as receivers depend on the T_2 senders. It is only when T_2 senders send h, but play L that T_3 may not respond to messages. If T_3 receives a l message, it comes from a T_0 player and T_3 best responds by playing L. The payoff from each action upon receiving h is

$$\pi(H|h) = g_3(0|h)\,u_{HH} + g_3(1|h)\,u_{HH} + g_3(2|h)\,u_{HL},$$
$$\pi(L|h) = g_3(0|h)\,u_{LH} + g_3(1|h)\,u_{LH} + g_3(2|h)\,u_{LL}.$$

Playing L is preferable whenever

$$\frac{\tau^2}{1+2\tau} > \frac{u_{HH}-u_{LH}}{u_{LL}-u_{HL}} = 1/\alpha.$$

To illustrate the first case when L is risk dominant, Figure A3 displays the behavior of T_3 as a function of τ and the payoffs of the game. First note that α has to be larger than 1 because L is risk dominant. Above the thick line in Figure A3, T_{2+} plays $\langle h, H, H, L\rangle$ and below it T_2 plays $\langle h, L, H, L\rangle$. Figure A3 shows the four different cases for the behavior of T_3 in the latter case. For example, for the Stag Hunt depicted in Figure 2, $\alpha = 7$, implying that T_{3+} plays $\langle h, L, L, L\rangle$ whenever $0.547 < \tau < 1.646$.

OBSERVATION 10: *(Two-way communication) T_1 randomizes messages and responds to received messages. Let $\lambda \equiv (u_{HH} - u_{LH})/(2u_{LL} - u_{LH} - u_{HH})$. If L is the risk dominant action and $0 < \lambda < (\beta-1)/2$, then T_{2+} plays $\langle h, H, L\rangle$ if $\tau < \lambda$, $\langle l, L, L\rangle$ if $\lambda < \tau < (\beta-1)/2$, and $\langle h, H, H\rangle$ if $\tau > (\beta-1)/2$. If either $\lambda < 0$ or $0 < (\beta-1)/2 < \lambda$, T_{2+} plays $\langle h, H, L\rangle$ if $\tau < \alpha$ and $\langle h, H, H\rangle$ if $\tau > \alpha$. If H is the risk dominant action, the behavior of T_2 depends on the payoff structure of the game:*
Case 1 ($u_{LH} + u_{HH} > u_{LL} + u_{HL}$): *$T_{2+}$ plays $\langle h, H, L\rangle$ if $\tau < \alpha$ and $\langle h, H, H\rangle$ if $\tau > \alpha$.*
Case 2 ($u_{LH} + u_{HH} < u_{LL} + u_{HL}$): *Let $\gamma \equiv (u_{LL} - u_{HL})/(2u_{HH} - u_{LL} - u_{HL})$ and $\delta \equiv (u_{HH} - u_{LL})/(u_{LL} - u_{HL})$. If $\tau < \gamma$, then T_2 plays $\langle l, H, L\rangle$; T_3 plays $\langle l, H, L\rangle$ if in addition $\tau < \left(\sqrt{4\delta\gamma^2+1}-1\right)/2\gamma\delta$, but plays $\langle h, H, H\rangle$ if $\tau > \left(\sqrt{4\delta\gamma^2+1}-1\right)/2\gamma\delta$. If $\tau > \gamma$, then T_{2+} plays $\langle h, H, H\rangle$.*

PROOF:

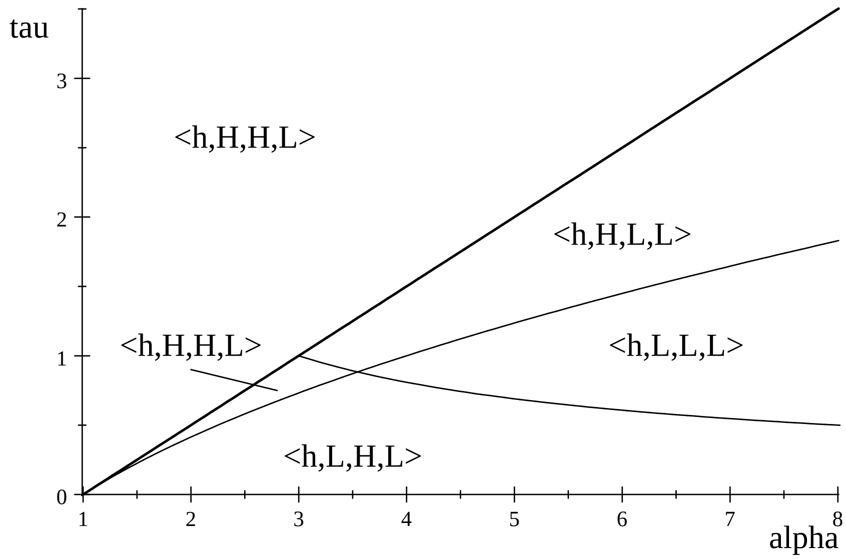

FIGURE A3. LEVEL-3 IN COORDINATION GAMES

T_1 believes that the opponent is truthful and therefore best responds to the received message, while sending random messages (not knowing what action will be taken).

T_2 faces truthful T_0 and responding T_1. Since zero-step and one-step thinkers send both messages with equal probabilities, $g_2(l|m) = g_2(l)$. The strategy $\langle l, H, H\rangle$ is clearly dominated by $\langle h, H, H\rangle$ and $\langle h, L, L\rangle$ is dominated by $\langle h, H, L\rangle$. The remaining strategies gives the following expected payoff:

$$\pi(\langle h, H, L\rangle) = g_2(0)\frac{1}{2}(u_{LL} + u_{HH}) + g_2(1)\frac{1}{2}(u_{LH} + u_{HH}),$$
$$\pi(\langle h, H, H\rangle) = g_2(0)\frac{1}{2}(u_{HL} + u_{HH}) + g_2(1)\,u_{HH},$$
$$\pi(\langle l, L, L\rangle) = g_2(0)\frac{1}{2}(u_{LL} + u_{LH}) + g_2(1)\,u_{LL},$$
$$\pi(\langle l, H, L\rangle) = g_2(0)\frac{1}{2}(u_{LL} + u_{HH}) + g_2(1)\frac{1}{2}(u_{LL} + u_{HL}).$$

First suppose that L is risk dominant. This implies that $u_{LL} + u_{LH} > u_{HL} + u_{HH}$ and consequently $u_{LH} > u_{HL}$ so that $\langle h, H, L\rangle$ dominates $\langle l, H, L\rangle$. T_2 prefers $\langle h, H, H\rangle$ over $\langle h, H, L\rangle$ whenever

$$\tau > (u_{LL} - u_{HL}) / (u_{HH} - u_{LH}) = \alpha,$$

and $\langle l, L, L\rangle$ over $\langle h, H, H\rangle$ whenever $\tau < (\beta - 1)/2$. Finally, T_2 prefers $\langle l, L, L\rangle$ over $\langle h, H, L\rangle$ whenever

$$\tau > \frac{u_{HH} - u_{LH}}{2u_{LL} - u_{LH} - u_{HH}},$$

given that the right hand side is positive. Hence, $\langle l, L, L\rangle$ is only optimal whenever $0 < \lambda < \tau < (\beta - 1)/2$ and the payoffs satisfy $0 < \lambda < (\beta - 1)/2$.

Second, suppose that H is risk dominant and $u_{LH} + u_{HH} > u_{LL} + u_{HL}$ so that $\langle h, H, L\rangle$ dominates $\langle l, H, L\rangle$ and $\langle h, H, H\rangle$ dominates $\langle l, L, L\rangle$. T_2 plays $\langle h, H, H\rangle$ rather than $\langle h, H, L\rangle$ if $\tau > \alpha$.

Finally, suppose that H is risk dominant and $u_{LH} + u_{HH} < u_{LL} + u_{HL}$. Now $\langle l, H, L\rangle$ dominates $\langle h, H, L\rangle$ and $\langle h, H, H\rangle$ dominates $\langle l, L, L\rangle$. Therefore, T_2 plays $\langle h, H, H\rangle$ if $\tau > (u_{LL} - u_{HL}) / (2u_{HH} - u_{LL} - u_{HL})$.

Since $\langle l, L, L\rangle$, $\langle h, H, L\rangle$ and $\langle h, H, H\rangle$ are best responses if the opponent plays the same strategies, by Lemma 1, T_{3+} play like T_2 in these cases. Finally, consider T_3 when T_2 plays $\langle l, H, L\rangle$. In this case, whenever T_3 receives an h message, he believes that it comes from a T_0 or T_1 player. Suppose first T_3 receives the message h. If T_3 sent h, then it is optimal to play H. If T_3 sent the message l, the payoffs from playing L and H are

$$\pi\left(\langle l, L, \cdot\rangle | h\right) = g_3(0)\, u_{LH} + g_3(1)\, u_{LL},$$
$$\pi\left(\langle l, H, \cdot\rangle | h\right) = g_3(0)\, u_{HH} + g_3(1)\, u_{HL}.$$

Playing H is preferred whenever $\tau < (u_{HH} - u_{LH}) / (u_{LL} - u_{HL}) = 1/\alpha$. Since H is risk dominant, $\alpha < 1$ and since $\gamma < 1$, this condition always hold. Now consider the case when T_3 receives the message l. If T_3 sent l, then it is optimal to play L (since T_0 is truthful and T_1 and T_2 best-responds). Suppose that T_3 sent h. Then expected payoffs are:

$$\pi\left(\langle h, \cdot, L\rangle | l\right) = g_3(0|l)\, u_{LL} + g_3(1|l)\, u_{LH} + g_3(2|l)\, u_{LH}$$
$$\pi\left(\langle h, \cdot, H\rangle | l\right) = g_3(0|l)\, u_{HL} + g_3(1|l)\, u_{HH} + g_3(2|l)\, u_{HH}$$

Playing L is preferred whenever $\tau(1 + \tau) < \alpha$.

Which message will T_3 send? Suppose first that $\tau(1+\tau) < \alpha$ so that T_3 plays $\langle h, H, L\rangle$ or $\langle l, H, L\rangle$. These strategies give the following ex ante payoffs

$$\pi\left(\langle h, H, L\rangle\right) = g_3(0)\frac{1}{2}(u_{LL} + u_{HH}) + g_3(1)\frac{1}{2}(u_{LH} + u_{HH}) + g_3(2)\, u_{LH},$$
$$\pi\left(\langle l, H, L\rangle\right) = g_3(0)\frac{1}{2}(u_{LL} + u_{HH}) + g_3(1)\frac{1}{2}(u_{LL} + u_{HL}) + g_3(2)\, u_{LL}.$$

It follows from the condition $u_{LH} + u_{HH} < u_{LL} + u_{HL}$ that $\langle l, H, L\rangle$ dominates $\langle h, H, H\rangle$. Now consider the case when $\tau(1+\tau) > \alpha$ so that T_3 play either $\langle h, H, H\rangle$ or $\langle l, H, L\rangle$. The payoff from each strategy is

$$\pi\left(\langle h, H, H\rangle\right) = g_3(0)\frac{1}{2}(u_{HL} + u_{HH}) + g_3(1)\, u_{HH} + g_3(2)\, u_{HH},$$
$$\pi\left(\langle l, H, L\rangle\right) = g_3(0)\frac{1}{2}(u_{LL} + u_{HH}) + g_3(1)\frac{1}{2}(u_{LL} + u_{HL}) + g_3(2)\, u_{LL}.$$

Sending h is preferred whenever $\tau + \tau^2\delta\gamma > \gamma$, i.e. when $\tau > \left(\sqrt{4\gamma^2\delta + 1} - 1\right)/2\gamma\delta$ (since $\gamma > 0$ and $\delta > 1$).

Note that two-way communication always entails play of the payoff dominant equilibrium in the Stag Hunt game depicted in Figure 2. For that particular game, $\alpha = 7$ so that T_{2+} plays $\langle h, H, L\rangle$ if $\tau < 7$ and $\langle h, H, H\rangle$ otherwise.

As before, we assume without loss of generality that $u_{HL} > u_{LH}$ so that each player prefers the equilibrium where he is the one to play H(igh).

OBSERVATION 11: *(No communication) Let* $\theta = (u_{LH} - u_{HH}) / (u_{HL} - u_{LL})$. *If* H *is the risk dominant action,* T_1 *plays* H. *If* $\tau < (1/\theta - 1)/2$, T_2 *plays* H; T_3 *plays* H *if in addition* $\tau + \tau^2/2 < (1/\theta - 1)/2$, *but plays* L *if* $\tau + \tau^2/2 > (1/\theta - 1)/2$. *If* $\tau > (1/\theta - 1)/2$, T_2 *plays* L; T_3 *plays* H *if in addition* $(2 - \tau/\theta)\tau < 1/\theta - 1$, *but plays* L *if* $(2 - \tau/\theta)\tau > 1/\theta - 1$. *If* L *is the risk dominant action,* T_1 *plays* L. *If* $\tau < (\theta - 1)/2$, T_2 *plays* L; T_3 *plays* L *if in addition* $\tau + \tau^2/2 < (\theta - 1)/2$, *but plays* H *if* $\tau + \tau^2/2 > (\theta - 1)/2$. *If* $\tau > (\theta - 1)/2$, T_2 *plays* H; T_3 *plays* L *if in addition* $(2 - \tau\theta)\tau < \theta - 1$, *but plays* H *if* $(2 - \tau\theta)\tau > \theta - 1$.

PROOF:

First suppose H is risk dominant (which implies that $\theta < 1$). T_2 plays H rather than L if

$$g_2(0)\frac{1}{2}(u_{HL} + u_{HH}) + g_2(1)u_{HH} > g_2(0)\frac{1}{2}(u_{LL} + u_{LH}) + g_2(1)u_{LH},$$

which is equivalent to $1 + 2\tau < 1/\theta$. Suppose this holds so that T_2 plays H. Then T_3 prefers H over L whenever

$$g_3(0)\frac{1}{2}(u_{HL} + u_{HH}) + g_3(1)u_{HH} + g_3(2)u_{HH} > g_3(0)\frac{1}{2}(u_{LL} + u_{LH}) + g_3(1)u_{LH} + g_3(2)u_{LH},$$

which simplifies to $1 + 2\tau + \tau^2 < 1/\theta$. Suppose instead T_2 plays L. Then T_3 prefers H over L whenever

$$g_3(0)\frac{1}{2}(u_{HL} + u_{HH}) + g_3(1)u_{HH} + g_3(2)u_{HL} > g_3(0)\frac{1}{2}(u_{LL} + u_{LH}) + g_3(1)u_{LH} + g_3(2)u_{LL},$$

which is equivalent to $(2 - \tau/\theta)\tau < 1/\theta - 1$.

Now suppose L is risk dominant. Then T_2 plays H rather than L if

$$g_2(0)\frac{1}{2}(u_{HL} + u_{HH}) + g_2(1)u_{HL} > g_2(0)\frac{1}{2}(u_{LL} + u_{LH}) + g_2(1)u_{LL},$$

which is equivalent to $1 + 2\tau > \theta$. Suppose that this holds so that T_2 plays H. Then T_3 prefers H over L whenever

$$g_3(0)\frac{1}{2}(u_{HL} + u_{HH}) + g_3(1)u_{HL} + g_3(2)u_{HH} > g_3(0)\frac{1}{2}(u_{LL} + u_{LH}) + g_3(1)u_{LL} + g_3(2)u_{LH},$$

which simplifies to $(2 - \tau\theta)\tau > \theta - 1$. If T_2 instead plays L, then T_3 prefers H over L whenever $\tau + \tau^2/2 > (\theta - 1)/2$.

Note that some of the conditions above are quadratic, implying that they may be satisfied both for low and high values of τ.

OBSERVATION 12: *(One-way communication) If H is the risk dominant action, then T_{1+} sends h and plays H as sender and responds to messages as receiver. If L is the risk dominant action, then T_1 sends l and plays L as sender and responds to messages as receiver. The behavior of T_{2+} depends on the payoff structure of the game:*

Case 1 ($u_{LH} > u_{LL}$): *Let $\eta \equiv (u_{LL} + u_{LH} - u_{HL} - u_{HH})/2(u_{HL} - u_{LH})$. If $\tau < \eta$, then T_2 plays $\langle l, L, L, H\rangle$ and T_3 plays $\langle l, L, L, H\rangle$ if $(1 + \tau/2)\tau < \eta$ whereas T_{3+} plays $\langle h, H, L, H\rangle$ if $(1 + \tau/2)\tau > \eta$. If $\tau > \eta$, then T_{2+} plays $\langle h, H, L, H\rangle$.*

Case 2 ($u_{LH} < u_{LL}$): *If $\tau < (\theta - 1)/2$, then T_2 plays $\langle h, L, L, H\rangle$ and T_3 plays as follows:*

T_3 plays $\langle h, L, L, H\rangle$ if $(1 + \tau/2)\tau < (\theta - 1)/2$ and $\tau < \sqrt{\theta}$,
T_{3+} plays $\langle h, L, L, L\rangle$ if $(1 + \tau/2)\tau < (\theta - 1)/2$ and $\tau > \sqrt{\theta}$,
T_{3+} plays $\langle h, H, L, H\rangle$ if $(1 + \tau/2)\tau > (\theta - 1)/2$ and $\tau < \sqrt{\theta}$,
T_{3+} plays $\langle h, H, L, L\rangle$ if $(1 + \tau/2)\tau > (\theta - 1)/2$ and $\tau > \sqrt{\theta}$.

If $\tau > (\theta - 1)/2$, then T_{2+} plays $\langle h, H, L, H\rangle$.

PROOF:

First let H be the risk dominant action. A T_1 sender faces a randomizing receiver and therefore plays H and sends h. A T_1 receiver, on the other hand, responds to the sent message, believing it comes from a truthful T_0 opponent. By Lemma 1, T_{2+} plays the same strategy as T_1.

If instead L is the risk dominant action, a T_1 sender instead sends and plays L, but responds to messages as receiver. A T_2 sender faces a tradeoff between playing L (the best response against T_0) and sending h and playing H (the best response against T_1). The expected payoffs from the three relevant sender strategies are:

$$\pi(\langle l, L\rangle) = g_2(0)\frac{1}{2}(u_{LL} + u_{LH}) + g_2(1)u_{LH},$$
$$\pi(\langle h, H\rangle) = g_2(0)\frac{1}{2}(u_{HL} + u_{HH}) + g_2(1)u_{HL},$$
$$\pi(\langle h, L\rangle) = g_2(0)\frac{1}{2}(u_{LL} + u_{LH}) + g_2(1)u_{LL}.$$

Suppose $u_{LH} > u_{LL}$ so that $\langle l, L\rangle$ dominates $\langle h, L\rangle$. Then a T_2 sender plays $\langle l, L\rangle$ if

$$\tau < \frac{(u_{LL} + u_{LH}) - (u_{HL} + u_{HH})}{2(u_{HL} - u_{LH})} = \eta,$$

but plays $\langle h, H\rangle$ otherwise. T_2 receivers face truthful T_0 and T_1 senders, so they respond to messages. If τ is above the threshold above, T_{2+} play $\langle h, H, L, H\rangle$. However, if τ is below the threshold, T_3 senders trade off truthfully playing L or H. They play L if $(1 + \tau/2)\tau < \eta$ and otherwise play H.

Suppose now that $u_{LL} > u_{LH}$ so that T_2 senders prefer sending h when they intend to play L. They prefer doing so over $\langle h, H\rangle$ whenever

$$\tau < \frac{(u_{LL} + u_{LH}) - (u_{HL} + u_{HH})}{2(u_{HL} - u_{LL})} = (\theta - 1)/2.$$

T_2 receivers face truthful senders, so they respond to messages. If $\tau > (\theta - 1)/2$, T_{2+} plays $\langle h, H, L, H\rangle$. A T_3 sender plays $\langle h, L\rangle$ rather than $\langle h, H\rangle$ if $(1 + \tau/2)\tau < (\theta - 1)/2$.

A T_3 receiver believes that an l message is truthful, so they play H in that case. An h message comes either from a T_0 or T_3 When receiving a h message, the payoff from each action is:

$$\pi\left(H|h\right) = g_3\left(0|h\right)u_{HH} + g_3\left(2|h\right)u_{HL},$$
$$\pi\left(L|h\right) = g_3\left(0|h\right)u_{LH} + g_3\left(2|h\right)u_{LL}.$$

So, T_3 play $\langle L, H\rangle$ if

$$\tau < \sqrt{\frac{u_{LH} - u_{HH}}{u_{HL} - u_{LL}}} = \sqrt{\theta},$$

and play $\langle H, H\rangle$ otherwise.

Two-way communication in mixed motive games is particularly cumbersome to characterize generally. The following observation therefore focuses on the behavior of T_2. (For a particular payoff configuration, however, it is straightforward to derive the behavior of T_{3+} players.)

OBSERVATION 13: *(Two-way communication) T_1 sends h and l with equal probabilities and responds to messages. Let $\nu \equiv \left(u_{HL} - u_{LL}\right)/\left(2u_{LH} - u_{LL} - u_{HL}\right)$. If L is risk dominant and $0 < \nu < \eta$, then T_2 plays $\langle h, L, H\rangle$ if $\tau < \nu$, $\langle l, L, L\rangle$ if $\nu < \tau < \eta$, and $\langle h, H, H\rangle$ if $\tau > \eta$. If either $\nu < 0$ or $0 < \eta < \nu$, then T_2 plays $\langle h, L, H\rangle$ if $\tau < \theta$ and $\langle h, H, H\rangle$ if $\tau > \theta$. If H is risk dominant and $u_{LH} + u_{HH} > u_{LL} + u_{HL}$, then T_2 plays $\langle l, L, H\rangle$ if $\tau < \left(u_{LH} - u_{HH}\right)/\left(2u_{HL} - u_{LH} - u_{HH}\right)$ and $\langle h, H, H\rangle$ otherwise. If instead $u_{LH} + u_{HH} < u_{LL} + u_{HL}$, T_2 plays $\langle h, L, H\rangle$ if $\tau < \theta$ and $\langle h, H, H\rangle$ if $\tau > \theta$.*

PROOF:

T_1 believes that the opponent is truthful and therefore sends random messages, but responds to the received message. The strategy $\langle l, H, H\rangle$ is dominated by $\langle h, H, H\rangle$ and the expected payoff for T_2's other strategies are:

$$\pi\left(\langle h, L, L\rangle\right) = g_2\left(0\right)\frac{1}{2}\left(u_{LL} + u_{LH}\right) + g_2\left(1\right)u_{LL},$$
$$\pi\left(\langle h, L, H\rangle\right) = g_2\left(0\right)\frac{1}{2}\left(u_{HL} + u_{LH}\right) + g_2\left(1\right)\frac{1}{2}\left(u_{LL} + u_{HL}\right),$$
$$\pi\left(\langle h, H, H\rangle\right) = g_2\left(0\right)\frac{1}{2}\left(u_{HL} + u_{HH}\right) + g_2\left(1\right)u_{HL},$$
$$\pi\left(\langle l, L, H\rangle\right) = g_2\left(0\right)\frac{1}{2}\left(u_{HL} + u_{LH}\right) + g_2\left(1\right)\frac{1}{2}\left(u_{LH} + u_{HH}\right),$$
$$\pi\left(\langle l, L, L\rangle\right) = g_2\left(0\right)\frac{1}{2}\left(u_{LL} + u_{LH}\right) + g_2\left(1\right)u_{LH}.$$

Suppose H is risk dominant. Then $\langle h, H, H\rangle$ dominates $\langle l, L, L\rangle$ and $\langle h, L, L\rangle$. First suppose that $u_{LH} + u_{HH} > u_{LL} + u_{HL}$ so that $\langle l, L, H\rangle$ dominates $\langle h, L, H\rangle$. T_2 plays $\langle h, H, H\rangle$ rather than $\langle l, L, H\rangle$ if

$$\tau > \frac{u_{LH} - u_{HH}}{2u_{HL} - u_{LH} - u_{HH}}.$$

If instead $u_{LL} + u_{HL} > u_{LH} + u_{HH}$, then T_2 plays $\langle h, H, H\rangle$ rather than $\langle h, L, H\rangle$ if

$$\tau > \frac{u_{LH} - u_{HH}}{u_{HL} - u_{LL}} = \theta.$$

Now consider the case when L is risk dominant. This implies that $u_{LL} > u_{HH}$, so $\langle h, L, H\rangle$ dominates $\langle l, L, H\rangle$ and $\langle h, L, H\rangle$ dominates $\langle h, L, L\rangle$. There are three remaining strategies to consider. T_2 plays $\langle h, H, H\rangle$ rather than $\langle h, L, H\rangle$ if $\tau > \theta$. T_2 may prefer to play $\langle l, L, L\rangle$. $\langle l, L, L\rangle$ preferred over $\langle h, L, H\rangle$ whenever

$$\tau > \frac{u_{HL} - u_{LL}}{2u_{LH} - u_{LL} - u_{HL}} = \nu,$$

given that the right hand side is positive (otherwise the condition cannot hold). $\langle l, L, L\rangle$ is preferred over $\langle h, H, H\rangle$ whenever

$$\tau < \frac{(u_{LL} + u_{LH}) - (u_{HL} + u_{HH})}{2\,(u_{HL} - u_{LH})} = \eta.$$

Hence, in order for $\langle l, L, L\rangle$ to be optimal, τ must be between ν and η and the payoffs must satisfy $\eta > \nu > 0$.