Web-appendix to “The Law of the Few”

1 Proofs from section IV of paper

PROOF OF PROPOSITION 6:

Let $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ be a non-empty network equilibrium in which $x_i > 0$ for all i. Let $n(l)$ be the number of players who sponsor l links, $l = 0, ..., n-1$; also let $x(l)$ be the information acquired by a player who sponsors l links. Note that in a minimally connected network there are $n-1$ links and n nodes and therefore $n(0) > 0$. Since every player acquires information, we have that $x(0) = x(l) + kl$. Thus, aggregate information is $\sum_{l=0}^{n-1} n(l)x(l) = \sum_{l=0}^{n-1} n(l)(x(0) - kl) = nx(0) - k(n-1)$. Since $x(0)$ is part of equilibrium, it is the solution to $f'(nx(0) - k(n-1)) = C'(x(0))$.

For given k, let $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ be a non-empty network equilibrium in which $x_i > 0$ for all $i \in N$; similarly, for given $k' > k$, let $\mathbf{s}' = (\mathbf{x}', \mathbf{g}')$ be a non-empty network equilibrium in which $x'_i > 0$ for all $i \in N$. Let y and y' be the aggregate information under $\mathbf{s}$ and $\mathbf{s}'$, respectively. We claim that $y > y'$. Suppose $y \leq y'$, then $f'(y) = C'(x(0)) \geq f'(y') = C'(x'(0))$ (where recall that $x(0)$ is the information of a player who does not sponsor any links.); so, $x(0) \geq x'(0)$. But then, since $k' > k$, $y = nx(0) - (n-1)k \geq nx'(0) - (n-1)k > nx'(0) - (n-1)k' = y'$, a contradiction.

Next, fix k; for a given n, let $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ be a non-empty network equilibrium with $x_i > 0$ for all $i \in N$; similarly, for given $n' > n$, let $\mathbf{s}' = (\mathbf{x}', \mathbf{g}')$ be a non-empty network equilibrium with $x'_i > 0$ for all $i \in N$. We claim that $y' > y$. Suppose that $y' \leq y$, then $f'(y) = C'(x(0)) \leq f'(y') = C'(x'(0))$; so, $x(0) \leq x'(0)$. But then, $y = nx(0) - (n-1)k \leq nx'(0) - (n-1)k < n'x'(0) - (n'-1)k = y'$, where the last inequality follows because $nx'(0) - (n-1)k$ is increasing in n whenever $x'(0) > k$, which holds because $x'(l) = x'(0) - kl > 0$ for all $l \geq 1$. This concludes the proof.

PROOF OF PROPOSITION 7:

First, suppose that $\mathbf{s}^*$ is pairwise equilibrium and that $x^*_i = 0$ for some $i \in N$. Since $\mathbf{s}$ is Nash, it follows that $\bar{g}^*_{ij} = 0$ for all $j \in N$, which, in view of Lemma 1, it implies that $x^*_i = \hat{y}$. This contradicts our initial hypothesis that $x^*_i = 0$.

Second, let $\mathbf{s}^*$ be such that $\bar{\mathbf{g}}^*$ is a regular network of degree $v = \{1, ..., n-2\}$ and each player exerts effort $x^* = \hat{y}/[v+1]$. Note that each player accesses $\hat{y}$ and that the payoffs to a player i are $\Pi(\mathbf{s}^*) = f(\hat{y}) - cx^* - vk/2$. If player i deletes some of $t \leq v$ of his links, his best new effort would be $x_i = (t+1)x^*$. Hence, player i's payoffs are $f(\hat{y}) - (t+1)x^*c - (v-t)k/2$. Since $\mathbf{s}$ is pairwise equilibrium, condition 2 implies that $f(\hat{y}) - (t+1)x^*c - (v-t)k/2 \leq \Pi(\mathbf{s}^*)$, which is equivalent to $k \leq 2cx^*$. Next, note that since $v < n-1$ there are two players, say i and j, with $\bar{g}^*_{ij} = 0$. Suppose i and j deviates by forming a new link between them. Then the new effort of i and the new effort of j must be such that $x'_i + x'_j = x^*$. Suppose, without loss of generality, that $x'_i \leq x'_j$; it is clear that if j wishes to form the new link with i then also i wishes to form the new link with j. So, we can focus on player j. His payoffs, after adding the new link and revising his effort, are $f(\hat{y}) - cx'_j - (v+1)k/2$. From above we know that $x'_j = x^* - x'_i$ and therefore pairwise equilibrium requires that

$f(\hat{y}) - cx^* - vk/2 \geq f(\hat{y}) - cx'_j - (v+1)k/2 = f(\hat{y}) - c(x^* - x'_i) - (v+1)k/2$ or $k/2 > cx'_i$. Since $x'_i \in [0, x^*/2]$, as long as $k > cx^*$ the condition is always satisfied. Hence the profile s is pairwise stable whenever $k \in [cx^*, 2cx^*]$. The proof for the case in which the network is complete follows similar arguments and therefore it is omitted. This concludes the proof. □

Heterogeneity in indirect transmission model: Suppose that player 1 has the lowest marginal costs of information acquisition, i.e., $c_1 < c_i = c$ for all $i \in N \setminus \{1\}$. Let k_i be the costs that any player has to pay to link with player i. Assume that there is a player j, with $k_j < k = k_i$, for all $i \neq j$. In words, player j is the socially most available player, an individual attribute which may reflect higher communication skills, higher sociability and innate interest in providing public goods. Recall that $\hat{y} = \arg\max_y f(y) - cy$, while $\hat{y}_1 = \arg\max_y f(y) - c_1 y$.

PROPOSITION A: *Suppose payoffs are given by (4) and suppose that $c_1 < c = c_i$, for all $i \neq 1$ and there is some player j such that $k_j < k = k_i$ for all $i \neq j$.*

1. *Suppose $j = 1$. If $k_1 < f(\hat{y}_1) - f(\hat{y}) + c\hat{y}$ then there is a unique equilibrium, $\mathbf{s}^* = (\mathbf{x}^*, \mathbf{g}^*)$; with $x_1^* = \hat{y}_1$, $x_i^* = 0$ for all $i \neq 1$, $\mathbf{g}^*$ is a periphery-sponsored star and player 1 is the hub.*

2. *Suppose $j \neq 1$. If $k < f(\hat{y}_1) - f(\hat{y}) + c\hat{y}$ then in every equilibrium $\mathbf{s}^* = (\mathbf{x}^*, \mathbf{g}^*)$, $x_1^* = \hat{y}_1$, $x_i^* = 0$ for all $i \neq 1$, $\mathbf{g}^*$ is such that there are $l \in \{0, n-2\}$ players who form a link with j, and all other players lie in the path between player j and player 1. Moreover, there is a unique strict equilibrium with $l = n - 2$.*

PROOF OF PROPOSITION A:

The assumption on frictionless information transmission implies that in every equilibrium the network is minimal. Let $z_{C(\mathbf{g})}$ be the aggregate information in component $C(\mathbf{g})$

CLAIM 5: *Suppose $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ is a non-empty equilibrium network and let $C(\mathbf{g})$ be a component. If $1 \in C(\mathbf{g})$ then $z_{C(\mathbf{g})} = \hat{y}_1$. If $1 \notin C(\mathbf{g})$ then $z_{C(\mathbf{g})} = \hat{y}$.*

PROOF OF CLAIM 5: First, note that in every component $C(\mathbf{g})$, $x_i > 0$ for some i. Second, let $1 \in C(\mathbf{g})$. Suppose $x_1 = 0$, then from Lemma 1 we know that $x_1 + y_1 \geq \hat{y}_1$; but this contradicts Lemma 1 because $\hat{y}_1 > \hat{y}$ and there exists a player $i \in C(\mathbf{g})$ with $x_i > 0$. So, $x_1 > 0$, and again from Lemma 1 it follows that $x_1 + y_1 = \hat{y}_1 = z_{C(\mathbf{g})}$. The second part of the claim can be proved using analogous arguments, the details of which are omitted. This completes the proof of claim 5.

Proof of Part 1. Suppose $j = 1$, $k_1 < f(\hat{y}_1) - f(\hat{y}) + c\hat{y}$ and $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ is equilibrium. We first show that $\mathbf{g}$ is connected. Suppose $\mathbf{g}$ is not connected. First, assume that $1 \in C(\mathbf{g})$ and $C(\mathbf{g})$ is singleton. Claim 5 implies that $z_{C(\mathbf{g})} = \hat{y}_1$. Select a component $\tilde{C}(\mathbf{g})$, and let $i \in \tilde{C}(\mathbf{g})$, $i \neq 1$; claim 5 implies that $z_{\tilde{C}(\mathbf{g})} = \hat{y}$. If $\tilde{C}(\mathbf{g})$ is singleton, $x_i = \hat{y}$ and $\Pi_i(\mathbf{s}) = f(\hat{y}) - c\hat{y}$. If player i forms a link to 1 and chooses zero information, he gets $f(\hat{y}_1) - k_1 > \Pi_i(\mathbf{s})$, where the inequality follows from $k_1 < f(\hat{y}_1) - f(\hat{y}) + c\hat{y}$. This contradicts the hypothesis that $\mathbf{s}$ is equilibrium. Thus, $i, j \in \tilde{C}(\mathbf{g})$ and assume, without loss of generality, that $g_{ij} = 1$. The payoffs to i are $\Pi_i(\mathbf{s}) = f(\hat{y}) - cx_i - k\eta_i(\mathbf{g})$, where $x_i \geq 0$ and $\eta_i(\mathbf{g}) \geq 1$. If i deletes all his links and forms a link to 1, then he gets at least $f(\hat{y}_1) - cx_i - k_1 > f(\hat{y}) - cx_i - k \geq \Pi_i(\mathbf{s})$, where the first inequality follows from $k_1 < k$ and $\hat{y}_1 > \hat{y}$, while the second inequality follows because $\eta_i(\mathbf{g}) \geq 1$. This contradicts the hypothesis that $\mathbf{s}$ is equilibrium. The case where 1 belongs to a non-singleton component can be ruled out using analogous arguments, the details of which are omitted. Hence, $\mathbf{g}$ is connected.

Since the network $\mathbf{g}$ is minimally connected, claim 5 implies that aggregate information is $\hat{y}_1$. If $x_i > 0$, $i \neq 1$, then Lemma 1 implies that $x_i + y_i(\bar{\mathbf{g}}) = \hat{y} < \hat{y}_1$, which is a contradiction. Thus, $x_i = 0$ for all $i \neq 1$ and therefore $x_1 = \hat{y}_1$ and player 1 does not form any links. This implies that if $g_{ij'} = 1$ and $j' \neq 1$, then i must access 1 via j'; for otherwise player i would not access any information via j' and therefore i would strictly gain by setting $g_{ij'} = 0$. But then the payoff of player i is $\Pi(\mathbf{s}) = f(\hat{y}_1) - k\eta_i(\mathbf{g}) < f(\hat{y}_1) - k_1$, where the last expression is the payoff to i by deleting all his links and sponsoring a link to player 1. This contradicts the hypothesis that $\mathbf{s}$ is equilibrium. Hence, $g_{ij} = 0$ for all $i, j \neq 1$, which implies that $\mathbf{g}$ is a star, the hub is player 1 and each spoke forms one link with the hub. To conclude the proof of Part 1, note that the payoffs to player $i \neq 1$ are $\Pi_i(\mathbf{s}) = f(\hat{y}_1) - k_1$ and, given the assumption on k_1, $\Pi_i(\mathbf{s}) > f(\hat{y}) - c\hat{y}$, where the last expression is the best payoff that i can earn should he delete his link and become isolated.

Proof of part 2: Suppose $j \neq 1$, $k < f(\hat{y}_1) - f(\hat{y}) + c\hat{y}$ and $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ is equilibrium. Without loss of generality, set $j = 2$. From standard arguments we can establish that $\mathbf{g}$ is connected and from the assumption of frictionless flow we conclude that $\mathbf{g}$ is minimally connected. Claim 5 implies that aggregate information equals $\hat{y}_1$. Therefore, $x_i = 0$ for all $i \neq 1$ and $x_1 = \hat{y}_1$. There are three facts that follow. Fact 1: player 1 does not form any links. Fact 2: for all $i \neq 1$, if $g_{ii'} = 1$, then i accesses player 1 via i' (including the possibility that $i' = 1$); for otherwise, i strictly gains by deleting the link to i'. Fact 3: for all $i \neq 1$, with $g_{ii'} = 1$ and $i' \neq 2$, if i accesses l via i', then $l \neq 2$; for otherwise player i strictly gains by switching from i' to 2. These three facts immediately imply that player 1 is an end-agent. We then have two possibilities. One, $g_{21} = 1$; in this case, fact 2 and 3 imply that $\mathbf{g}$ is a star, player 2 is the hub, and $g_{i2} = 1$ for all $i \neq 1$. Two, $g_{21} = 0$. Consider the path between 2 and 1: $\bar{g}_{2j_1} = \bar{g}_{j_1 j_2} = ... = \bar{g}_{j_d 1} = 1$. Fact 2 implies that $g_{2j_1} = g_{j_1 j_2} = ... = g_{j_d 1} = 1$; fact 3 implies that players $2, j_1, ..., j_d$ do not sponsor additional links. Let $i \neq \{2, j_1, ..., j_d, 1\}$, fact 3 implies that player i only links with player 2, i.e., $g_{i2} = 1$. This concludes the characterization of equilibria. Finally, it is easy to verify that every non-star equilibrium network involves a path such as $g_{2j_1} = g_{j_1 j_2} = ... = g_{j_d 1} = 1$; and player 2 is indifferent between a link with j_1 and 1; so such paths are not sustainable in a strict equilibrium. This concludes the proof of the proposition.

2 Alternative formulations of model

We analyze three alternative formulation of the model, involving discrete information acquisition, one-way flow of information in communication and investment in strength of communication links.

The Best Shot Game. We now study a model in which players have a choice between two actions 0 and 1, where we interpret 1 as acquire information and 0 as not acquire information. The main point we wish to bring out is that every equilibrium exhibits the law of the few and that equilibria are efficient. We focus on the case where players only access the information personally acquired by their direct neighbors.

Formally, a player either acquires information at a cost c or he does not acquire any information personally, i.e. $X = \{0, 1\}$. The returns to a player from acquiring information are $f(x_i + y_i(\bar{\mathbf{g}})) = 1$ if $x_i + y_i(\bar{\mathbf{g}}) \geq 1$, otherwise $f(x_i + y_i(\bar{\mathbf{g}})) = 0$. We assume that $c < 1$. This specification resembles the best shot game which has been widely studied in economics. The best-shot game is a good metaphor for situations in which there are significant externalities between players' information acquisition efforts.[1] The following proposition characterizes the equilibria in the best shot game.

PROPOSITION B: *Suppose $X = \{0, 1\}$. If $k < c$ then every equilibrium has a periphery-sponsored star architecture, the hub chooses* 1 *and every spoke chooses* 0*. If $k > c$ then there exists a unique equilibrium: every player chooses* 1 *and no one forms any links.*

PROOF: Suppose $k < c$ and let $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ be an equilibrium. We claim that there exists an $i \in N$ such that $x_i = 1$ and that $x_j = 0$, $\forall j \neq i$. First, since $k < c$, there must be at least a player who acquires information 1. Second, suppose both i and j choose 1. Then, it must be the case that $x_{i'} = 0$, $\forall i' \in N_i(\bar{\mathbf{g}})$; for if a neighbor of i chooses 1, player i strictly gains by choosing 0. Since $x_{i'} = 0$, $\forall i' \in N_i(\bar{\mathbf{g}})$, then $g_{il} = 0$ for all l. Hence, player i's payoffs in equilibrium $\mathbf{s}$ are $1 - c$. If player i chooses 0 and forms a link with j then he obtains $1 - k$. Since $k < c$, $1 - k > 1 - c$ and therefore s cannot be an equilibrium. Next, let $x_i = 1$ and $x_j = 0$, $\forall j \neq i$. Trivially, $g_{j'j} = 0$, $\forall j' \in N$, $j \neq i$, and, since $k < c$, every player $j \neq i$ has a link with i. This completes the proof for the case $k < c$. The proof for the case $k > c$ is trivial and therefore omitted.

We finally note that in the best shot game every equilibrium is efficient. This is in sharp contrast to the case in which information is a continuous variable.

PROPOSITION C: *Suppose $X = \{0, 1\}$. If $k < c$, then the socially optimal outcome is a star network, the hub chooses* 1 *and every spoke chooses* 0*. If $k > c$, then in the socially optimal outcome every player chooses* 1 *and no one forms links.*

PROOF: Suppose $k < c$ and suppose that $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ is efficient. It is easy to see that the only links in $\mathbf{g}$ are between pair of players (i, j) with $x_i \neq x_j$. Also, if player i chooses 0 then player i has only one link with a player choosing 1. Indeed, if player i had two distinct links with two players choosing 1, then welfare can be made strictly higher by deleting one of the link. Hence, the total number of links are $(n - m)$, where m is the number of players choosing 1, and each player gets returns of 1. Then the social welfare is $n - mc - (n - m)k$. If $k > c$, this expression decreases with m and therefore $m = 1$, which implies the result. Suppose now that $k > c$. The above arguments show that if there are $m < n$ players choosing 1, and

[1] For a discussion of best-shot games within the context of public good games see, e.g., Jack Hirshleifer 1983 and Glenn W. Harrison and Jack Hirshleifer 1989.

$\mathbf{s}$ is efficient then the social welfare is $n - mc - (n-m)k$, but then welfare can be increased by setting $m = n$, which implies the result. This concludes the proof.

One-way information transmission: We have assumed so far that if i forms a link with j then i accesses the information of j and *vice-versa*. An alternative model is one where a link $g_{ij} = 1$ allows only player i to access the information acquired by player j, i.e., one-way information flow.[2] This is a reasonable model for links on the web, or for citations. Let $N_i(\mathbf{g}) = \{j \in N : g_{ij} = 1\}$; the payoffs to player i in a strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ are then given by

$$\Pi_i(\mathbf{s}) = f\left(x_i + \sum_{j \in N_i(\mathbf{g})} x_j\right) - cx_i - \eta_i(\mathbf{g})k. \tag{1}$$

We observe that some of our main results carry over to the one-way information flow formulation. To illustrate this, consider the equilibria described in Proposition 2. Recall that at equilibrium $\mathbf{s} = (\mathbf{x}, \mathbf{g})$, aggregate information equals $\hat{y}$, $\mathbf{g}$ is a core-periphery network, hubs acquire information and spokes do not acquire any information personally. Let $\mathbf{g}'$ be a network in which if $g_{ij} = 1$ then $g'_{ij} = 1$ and if $g_{ij} = 0$ and $x_i, x_j > 0$, then $g'_{ij} = 1$. It is easy to see that $\mathbf{s}' = (\mathbf{x}, \mathbf{g}')$ is an equilibrium under the one-way flow formulation (for appropriately chosen k). In particular, in the one way flow formulation, the periphery-sponsored star, with the hub acquiring all the information is an equilibrium whenever $k < c\hat{y}$. An analogous argument can be used to extend the results of Proposition 3. Indeed, it is easy to see that a periphery-sponsored star, where the player with the lowest cost of information acquisition is the hub, and personally acquires information while all other players do not acquire information personally is an equilibrium under the assumption of one-way information flow as well.

Investments in link strength: We consider an alternative model where the investment in linking is also a continuous choice. A strategy of player i is then an investment in information acquisition $x_i \in X$ and an investment in linking summarized by a vector $\mathbf{v}_i = \{v_{i1}, ..., v_{in}\}$, where $v_{ij} \in [0,1]$ and $v_{ii} = 0$. A profile $\mathbf{v}$ generates a (undirected) symmetric value network $\mathbf{g}$, where the strength of an arbitrary link $g_{ij} = g_{ji} = \max[v_{ij} + v_{ji}, 1]$.[3] The payoffs to player i in a strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{v})$ are then given by:

$$\Pi_i(\mathbf{s}) = f\left(x_i + \sum_{j \in N} g_{ij} x_j\right) - cx_i - k \sum_{j \in N} v_{ij}. \tag{2}$$

So, the information that i accesses from j is the information that j has acquired personally weighted by the strength of the link between i and j. We can adapt the definition of a core-periphery network as follows. In a core-periphery network there are two groups of players $\hat{N}_1(\mathbf{g})$ and $\hat{N}_2(\mathbf{g})$ with the feature that: (1.) for all $i, j \in \hat{N}_1(\mathbf{g})$, $g_{ij} = 1$, (2.) for all $i \in \hat{N}_1(\mathbf{g})$, $v_{ij} = 1$ for all $j \in \hat{N}_2(\mathbf{g})$ and (3.) all other links have zero strength. The following

[2] For a general analysis of a game of pure network formation with one-way information flow, see Bala and Goyal (2000) and Andrea Galeotti (2006).

[3] For a general analysis of a game of pure network formation when link strength is endogenous, see Francis Bloch and Bhaskar Dutta 2008.

proposition shows that the results presented in our basic model are robust to this extension.

PROPOSITION D: *Suppose payoffs are given by (2). If $k > c\hat{y}$ then there exists a unique equilibrium in which every player acquires information $\hat{y}$ and no one invests in any links. If $k < c\hat{y}$ then in every strict equilibrium $\mathbf{s}^* = (\mathbf{x}^*, \mathbf{v}^*)$: (1) $\sum_{i \in N} x_i^* = \hat{y}$; (2) every equilibrium network has a core-periphery architecture, hubs acquire information personally and spokes acquire no information personally; (3) for given c and k, with $k < c\hat{y}$ the ratio $|I(\mathbf{s}^*)|/n \to 0$ as $n \to \infty$.*

The new insight of the analysis is that if two players who acquire information have a link then it must be a full strength link. To see why this is true, suppose player i acquires information personally and also invests effort $v_{ij} > 0$ in forming a link with another player j but that $v_{ij} + v_{ji} < 1$. This is strictly profitable only if the costs of connecting are lower than the costs of personal information acquisition; in other words, if $k < cx_j$. But then it is strictly profitable to lower personal information acquisition and increase the strength of the link with player j. Thus it is never optimal to have a link of less than full strength. Once we have proved that links between positive effort players are full strength, we can exploit the endogeneity of links to show, as in Propositions 1-2, that aggregate information acquisition in a strict equilibrium must equal $\hat{y}$; the rest of the result then follows.

PROOF OF PROPOSITION:

First, the proof for the case in which $k > c\hat{y}$ is straightforward and therefore it is omitted. Second, suppose that $k < c\hat{y}$. The following two claims are useful to prove the second part of the proposition.

CLAIM 6: *Assume that $k < c\hat{y}$. If $\mathbf{s}^*$ is equilibrium then $v_{ij}^* + v_{ji}^* \leq 1$. Moreover, in every strict equilibrium $\mathbf{s}^*$ if $i, j \in I(\mathbf{s}^*)$ and $v_{ij}^* + v_{ji}^* > 0$ then $v_{ij}^* + v_{ji}^* = 1$, i.e. the link between i and j has full strength.*

PROOF OF CLAIM 6: The first part of the claim is straightforward. To see the second part, assume that $i, j \in I(\mathbf{s}^*)$ and $v_{ij}^* + v_{ji}^* \in (0, 1)$; suppose, without loss of generality, that $v_{ij}^* > 0$. Since $x_i^* > 0$, Lemma 1 implies that $x_i^* + y_i^* = \hat{y}$. So the profits of i in s^* are $\Pi_i(\mathbf{s}^*) = f(\hat{y}) - cx_i^* - \sum_{j' \neq j} v_{ij'}^* k - v_{ij}^* k$. Consider the deviation in which player i chooses $v_{ij}' = 0$ and an effort $x_i' = x_i^* + v_{ij}^* x_j^*$. Note that, under this deviation we have that $x_i' + y_i' = x_i^* + v_{ij}^* x_j^* + y_i^* - v_{ij}^* x_j^* = \hat{y}$ and player i's payoffs are $\Pi_i(\mathbf{s}_i', \mathbf{s}_{-i}^*) = f(\hat{y}) - c[x_i^* + v_{ij}^* x_j^*] - \sum_{j' \neq j} v_{ij'}^* k$. Since $\mathbf{s}^*$ is a strict equilibrium $\Pi_i(\mathbf{s}^*) > \Pi_i(\mathbf{s}_i', \mathbf{s}_{-i}^*)$ and so $k < cx_j^*$. However, given that $k < cx_j^*$, player i strictly gains by increasing the strength of the link with j by some small and positive ϵ (and so i increases his costs of linking by $k\epsilon$) and by reducing his own information acquisition by ϵx_j^* (and so i decreases his costs of information acquisition by $c\epsilon x_j^*$). This means that $\mathbf{s}^*$ is not an equilibrium, a contradiction which completes the proof of Claim 6.

CLAIM 7: *Assume $k < c\hat{y}$ and suppose that $\mathbf{s}^*$ is a strict equilibrium. If $\sum_{i \in N} x_i^* > \hat{y}$ then $x_i^* = x$ for all $i \in I(\mathbf{s})$.*

PROOF OF CLAIM 7: First note that since $\sum_{i\in N} x_i^* > \hat{y}$, Lemma 1 implies that $|I(\mathbf{s}^*)| > 1$. Second, in contradiction with the claim suppose that $x_j^* \neq x_{j'}^*$ for some $j, j' \in I(\mathbf{s}^*)$. In particular, select player $i \in I(\mathbf{s}^*)$ such that $x_i^* = \underline{x} = \min_{j'\in I(\mathbf{s}^*)}\{x_{j'}^*\}$ and select player $j \in I(\mathbf{s}^*)$ such that $x_j^* = \bar{x} = \max_{j'\in I(\mathbf{s}^*)}\{x_{j'}\}$; hence, $\underline{x} < \bar{x}$. Lemma 1 implies that $\underline{x} + y_i^* = \bar{x} + y_j^* = \hat{y}$. Since $\underline{x} < \bar{x}$, it follows that $y_i^* > y_j^*$. We need to analyze two cases.
Case 1. There exists a player $l \in I(\mathbf{s}^*)$ such that $g_{lj}^* = 0$ and $g_{li}^* = 1$; note that $\underline{x} \leq x_l^* \leq \bar{x}$. If $v_{li}^* > 0$, then player l strictly gains by setting $v'_{li} = 0$ and choosing $v'_{lj} = v_{li}^*$. In fact, under this deviation player l faces the same costs of linking and the same costs of effort, but he accesses an extra effort $v_{li}^*(\bar{x} - \underline{x}) > 0$. So, suppose $v_{li}^* = 0$; in this case claim 6 implies that $v_{il}^* = 1$. Since $\mathbf{s}^*$ is a strict equilibrium and $v_{il}^* = 1$ we have that $k < cx_l^*$. Since $g_{jl}^* = 0$, $x_l^* \leq \bar{x}$ and $k < cx_l^*$, player j strict gains by setting $x'_j = \bar{x} - x_l^*$ and $v'_{jl} = 1$.
Case 2. Suppose that for all $l \in I(\mathbf{s}^*)$ if $g_{li}^* = 1$ then $g_{lj}^* = 1$. In this case, since $y_i^* > y_j^*$, for all $l \in I(\mathbf{s}^*)$ such that $g_{lj}^* = 1$ it must be that $g_{li}^* = 1$. That is, i and j share the same positive information neighbors. Furthermore, since $\underline{x} + y_i^* = \bar{x} + y_j^* = \hat{y}$ and $\underline{x} < \bar{x}$, it must be the case that $g_{ij}^* = 1$. These observations together with our initial hypothesis that aggregate information exceeds $\hat{y}$ imply that there exists a player $l \in I(\mathbf{s}^*)$ such that $g_{il}^* = g_{jl}^* = 0$. In a strict equilibrium, player l must set $v_{ll'}^* = 0$ for all $l' \in I(\mathbf{s}^*)$; in fact, if $v_{ll'}^* > 0$ for some $l' \in I(\mathbf{s}^*) \setminus \{j\}$ then player l will weakly gain by setting $v'_{ll'} = 0$ and $v'_{lj} = v_{ll'}^*$. Thus, for each $g_{ll'} = 1$, $v_{l'l} = 1$; this implies that in a strict equilibrium $k < cx_l^*$. But then player j strictly gains by setting $v_{jl}^* = 1$ and choosing effort $x'_j = \bar{x} - x_l^* \geq 0$, where the inequality follows because by construction $\bar{x} \geq x_l^*$.

Hence, in a strict equilibrium where aggregate information is strictly higher than $\hat{y}$, all positive effort players must exert the same effort. This concludes the proof of Claim 7.

Final Step. An immediate consequence of Claim 6 and Claim 7 is that in a strict equilibrium where aggregate information exceeds $\hat{y}$ the network among positive effort players must be regular (with each link having strength 1). We now show that any such strategy profile cannot constitute a strict equilibrium. Since aggregate information exceeds $\hat{y}$ and $k < c\hat{y}$, there exists a player $i \in I(\mathbf{s}^*)$ such that $g_{ij}^* = 0$ for some $j \in I(\mathbf{s}^*)$ and $g_{ij'}^* = 1$ for some $j' \in I(\mathbf{s}^*)$ and $v_{ij'}^* > 0$. Since $x_j^* = x_{j'}^* = x_i^*$, it is easy to check that player i is indifferent between playing his current strategy and deviating by setting $v'_{ij'} = 0$ and $v'_{ij} = v_{ij'}^*$. Hence, in every strict equilibrium aggregate information cannot exceed $\hat{y}$. This fact and Lemma 1 imply that in equilibrium aggregate information must be exactly $\hat{y}$. Recall that Claim 6 says that links among positive effort players have full strength. So, since aggregate effort equals $\hat{y}$ positive effort players must form a clique. It is now clear that if $i \notin I(\mathbf{s}^*)$ then $v_{ij}^* = 1$ for all $j \in I(\mathbf{s}^*)$; it is also straightforward to see that for all $i, j \notin I(\mathbf{s}^*)$, $v_{ij}^* = 0$. So the network has a core-periphery architecture, hubs acquire information personally, while spokes acquire no information personally. Part (3) of the proposition now follows by replicating the arguments used in the proof of Proposition 2. This concludes the proof of Proposition 8.