Online Appendix for Strategic Redistricting

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Lemma 1. *The district outcome function π is continuous, strictly concave on $[\frac{1}{2}, 1]$, and $\pi(\theta) = 1 - \pi(1-\theta)$ for all $\theta \in [0, 1]$.*

PROOF:
We will show $x(\cdot)$ is strictly concave on $[\frac{1}{2}, 1]$. Since L is strictly concave to $[0, 1]$, this shows that π is strictly concave on $[\frac{1}{2}, 1]$. Let

$$\theta I_1(x(\theta)) + (1-\theta) I_2(x(\theta)) = \tfrac{1}{2}$$

Note that, for $\theta \geq \frac{1}{2}$, $I_2(x(\theta)) = 1$ and therefore

$$I_1(x(\theta)) = 1 - \frac{1}{2\theta} \tag{A1}$$

To prove that $x(\cdot)$ is concave, let $\theta_1, \theta_3 \in [\frac{1}{2}, 1]$, $\delta \in (0, 1)$, and set $\theta_2 = \delta\theta_1 + (1-\delta)\theta_3$. Then, it is enough to show that

$$I_1(\delta x(\theta_1) + (1-\delta)x(\theta_3)) < 1 - \frac{1}{2\theta_2} \tag{A2}$$

The convexity of I_1 and $(A1)$ imply

$$I_1(\delta x(\theta_1) + (1-\delta)x(\theta_3)) \leq \delta I_1(x(\theta_1)) + (1-\delta) I_1(x(\theta_3)) = 1 - \delta\frac{1}{2\theta_1} - (1-\delta)\frac{1}{2\theta_3}$$

Then, the strict concavity of $-\frac{1}{z}$ yields $(A2)$ as desired.

Lemma 4. *For every $F \in \mathcal{F}$, the maximization problem $(A3)$ has a unique solution F^* such that $F^* = F^p$ for some $p \in [0, 1]$.*

PROOF:
First, we note that the set $\{\hat{F} \in \mathcal{F} \,|\, \hat{F} \succeq F\}$ is closed in the topology of weak convergence. Since $\mathcal{F}$ is compact, it follows that the constraint set of $(A3)$ is compact. Since D is continuous (Lemma 2), a solution exists. Next, we will show that this solution is unique and is a segregation plan.

Note that if $\bar{\omega}(F) + s \leq \frac{1}{2}$, then, since π is strictly convex on $[0, \frac{1}{2}]$, the unique solution to $(A3)$ is F^1. So we are done. If $\underline{\omega}(F) + s \geq \frac{1}{2}$, then, since π is strictly concave on $[\frac{1}{2}, 1]$,

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the unique solution to $(A3)$ is F^0, and again we are done. So, henceforth it is sufficient to consider F such that $\bar{\omega}(F) + s > \frac{1}{2} > \underline{\omega}(F) + s$.

Step 1. *Let h be an uhc correspondence from $[x, x']$ to nonempty, convex subsets of the reals. If there are $w \in h(x), w' \in h(x')$ such that $w \leq 0 \leq w'$, then there exists $x^* \in [x, x']$ such that $0 \in h(x^*)$.*

PROOF:
Follows from elementary arguments.

Step 2. *(i) $\omega^*(p) + s > \frac{1}{2}$ for all p such that $\frac{1}{2} - s \in \omega(p)$. (ii) Either $0 \in W(p^*)$ for some $p^* \in [0,1]$ or $\omega^*(0) + s > \phi(\underline{\omega}(F) + s)$ and not both. (iii) If p^* in (ii) exists, it is unique.*

PROOF:
Part (i) is immediate since $\bar{\omega}(F)+s > \frac{1}{2}$. If $\omega^*(0)+s > \phi(\underline{\omega}(F)+s)$, then $\min W(0) > 0$ and since W is increasing, $w > 0$ for all w, p such that $w \in W(p)$. If $\omega(0)+s \leq \phi(\underline{\omega}(F)+s)$, we have $w \leq 0$ for some $w \in W(0)$. Choose $\hat{p}$ such that $\frac{1}{2} - s \in \omega(\hat{p})$. Since $\phi(\frac{1}{2}) = \frac{1}{2}$, it follows from (i) that $\omega^*(\hat{p}) + s - \phi(\frac{1}{2}) > 0$ and therefore $\max W(\hat{p}) > 0$. Then, by Step 1, there exists p^* such that $0 \in W(p^*)$. This proves (ii). That p^* is unique is immediate.

For any F, let $p_F = 0$ if $\min W(0) > 0$ and $p_F = p^*$ (as defined in Step 2) otherwise. Similarly, let $\omega_F = \underline{\omega}(F)$ if $\min W(0) > 0$ and $\omega_F = \min\{\omega \in \omega(p^*) \,|\, \omega^*(p^*) + s = \phi(\omega + s)\}$ otherwise.

Step 3. *F^{p_F} is the unique optimal redistricting plan.*

PROOF:
Verifying that $F^{p_F} \succeq F$ is straightforward. Define,

$$\pi^*(\theta) = \frac{\pi(\omega^*(p_F) + s) - \pi(\omega_F + s)}{\omega^*(p_F) - \omega_F}(\theta - \omega_F - s) + \pi(\omega_F + s)$$

Hence, π^* is the line that runs through both $(\omega_F+s, \pi(\omega_F+s))$ and $(\omega^*(p_F)+s, \pi(\omega^*(p_F)+s))$. Note that

$$\text{(A4)} \qquad \pi(\theta) \begin{cases} > \pi^*(\theta) & \text{whenever } \theta < \omega_F + s \\ = \pi^*(\theta) & \text{if } \theta \in \{\omega_F + s, \omega^*(p_F) + s\} \\ < \pi^*(\theta) & \text{otherwise} \end{cases}$$

Consider any optimal F^*. First, we show that, for any $\omega < \omega_F$, $F^*(\omega) \leq F^{p_F}(\omega) = F(\omega)$. For any cumulative $\hat{F}$, define $h(\omega, \hat{F}) := \int_{\underline{\omega}}^{\omega} \hat{F}(z)dz$ and $g(\omega, \hat{F}) := h(\omega, F) - h(\omega, \hat{F})$ for all $\omega \in \mathbb{R}$. Note that a strategy, $\hat{F}$, is feasible if and only if it has the same mean as F, and $g(\omega, \hat{F}) \geq 0$ for all ω. That is, $\hat{F}$ is feasible if and only if F is a mean preserving spread of $\hat{F}$. Note also that $g(\cdot, \hat{F})$ is continuous for any $\hat{F}$.

Suppose $F^*(\omega_2) > F(\omega_2)$ for some $\omega_2 < \omega_F$. Since F and F^* are right continuous, we can choose ω_2 to be a continuity point of both F and F^*. Then, since F and F^* are continuous at ω_2 and $F^*(\omega_2) > F(\omega_2)$, we conclude that $0 \leq g(\omega_2 + \epsilon', F^*) < g(\omega_2, F^*)$ for some $\epsilon' > 0$. Let $\epsilon = g(\omega_2, F^*)/2 > 0$.

Since $g(\cdot, F^*)$ is continuous and $g(\omega, F^*) = 0$ for $\omega \leq \min \Omega$, by the Intermediate Value Theorem, there exists $\omega < \omega_2$ such that $g(\omega, F^*) = \epsilon$. Let

$$\omega_1 = \sup\{\omega \leq \omega_2 \,|\, g(\omega, F^*) \leq \epsilon\}$$

Since $g(\cdot, F^*)$ is continuous, $g(\omega_1, F^*) = \epsilon$ and therefore $\omega_1 < \omega_2$. By definition, $g(\omega, F^*) > \epsilon$ for all $\omega \in (\omega_1, \omega_2]$. Let $p_i = F^*(\omega_i)$ for $i = 1, 2$.

For $\omega \leq \min \Omega$, $F(\omega) = F^*(\omega) = 0$ and hence $g(\cdot, F^*) = 0$; for $\omega \geq \max \Omega$, $g(\cdot, F^*) = 0$ because F, F^* have the same mean. We conclude that $\omega_i \in \Omega$ for $i = 1, 2$ and therefore $\omega_2 - \omega_1 < 1$. Then,

$$\epsilon = g(\omega_2, F^*) - g(\omega_1, F^*) < (\omega_2 - \omega_1)(F^*(\omega_2) - F^*(\omega_1)) < p_2 - p_1$$

Then, define

$$G(\omega) = \begin{cases} 0 & \text{if } \omega < \omega_1 \\ \frac{F^* - F^*(\omega_1)}{p_2 - p_1} & \text{if } \omega \in [\omega_1, \omega_2] \\ 1 & \text{otherwise} \end{cases}$$

Hence, $F^* = p_1 F_-^{*p_1} + (p_2 - p_1)G + (1 - p_2)F_+^{*p_2}$. Since F^* is continuous at ω_2 and $p_2 > p_1$, $\mu(G)$, the mean of G, satisfies $\omega_1 < \mu(G) < \omega_2$. Choose α so that

$$\alpha\omega_1 + (1 - \alpha)\omega_2 = \mu(G)$$

and let H be the distribution that has support $\{\omega_1, \omega_2\}$ and satisfies $H(\omega_1) = \alpha$. Also, let

$$F^\epsilon = p_1 F_-^{*p_1} + (p_2 - p_1 - \epsilon)G + \epsilon H + (1 - p_2)F_+^{*p_2}$$

For all ω, we have

$$g(\omega, F^\epsilon) = g(\omega, F^*) + h(\omega, F^*) - h(\omega, F^\epsilon)$$

For $\omega \notin (\omega_1, \omega_2)$, $h(\omega, F^*) - h(\omega, F^\epsilon) = 0$ and hence $g(\omega, F^\epsilon) = g(\omega, F^*)$; for $\omega \in (\omega_1, \omega_2)$, $h(\omega, F^*) - h(\omega, F^\epsilon) \geq -\epsilon$ and $g(\omega, F^*) > \epsilon$. Hence, in both cases, $g(\omega, F^\epsilon) \geq 0$, proving that F is a mean preserving spread of F^ϵ and therefore F^ϵ is feasible. But since H is a mean preserving spread of G and π is convex on $[0, \omega_F + s]$, we have $D(F^\epsilon, s) > D(F^*, s)$ contradicting the optimality of F^*. This proves $F^*(\omega) \leq F(\omega)$ for $\omega \leq \omega_F$.

Next, note that

$$\begin{aligned} D(F^{p_F}, s) &= \int_{\omega < \omega_F} \pi(\omega + s) dF^{p_F}(\omega) + \int_{\omega \geq \omega_F} \pi(\omega + s) dF^{p_F}(\omega) \\ &= \int_{\omega < \omega_F} \pi(\omega + s) dF^{p_F}(\omega) + \int_{\omega \geq \omega_F} \pi^*(\omega + s) dF^{p_F}(\omega) \\ &= \int_{\omega < \omega_F} \pi(\omega + s) dF^{p_F}(\omega) + \int_{\omega \geq \omega_F} \pi^*(\omega + s) dF^{p_F}(\omega) \\ &\qquad + \int_\Omega \pi^*(\omega + s) d[F^* - F^{p_F}](\omega) \end{aligned}$$

The last equality follows from the fact that π^* is linear and $\mu(F^*) = \mu(F^{p_F})$. Hence, we have

$$D(F^{p_F}, s) = \int_{\omega < \omega_F} (\pi(\omega + s) - \pi^*(\omega + s)) dF^{p_F}(\omega) + \int_\Omega \pi^*(\omega + s) dF^*(\omega)$$

Note that $\pi - \pi^*$ is strictly decreasing and positive for $\omega < \omega_F$ since π is strictly convex. Since $F^* \leq F^{p_F}$ for $\omega < \omega_F$ it follows that

$$\int_{\omega<\omega_F} (\pi(\omega+s) - \pi^*(\omega+s))dF^{p_F}(\omega) \geq \int_{\omega<\omega_F} (\pi(\omega+s) - \pi^*(\omega+s))dF^*(\omega)$$

and therefore

$$\begin{aligned} D(F^{p_F}, s) &= \int_{\omega<\omega_F} (\pi(\omega+s) - \pi^*(\omega+s))dF^{p_F}(\omega) + \int_{\omega<\omega_F} \pi^*(\omega+s)dF^*(\omega) \\ &\qquad + \int_{\omega\geq\omega_F} \pi^*(\omega+s)dF^*(\omega) \\ &\geq \int_{\omega<\omega_F} \pi(\omega+s)dF^*(\omega) + \int_{\omega\geq\omega_F} \pi(\omega+s)dF^*(\omega) = D(F^*, s) \end{aligned}$$

Moreover, unless F^* assigns probability 0 to $(\omega_F, \omega^*(p_F)) \cup (\omega^*(p_F), \bar{\omega}(F)]$ and $F^* = F^{p_F}$ for $\omega < \omega_F$ the inequality above is strict, contradicting the optimality of F^*. Hence, $F^* = F^{p_F}$.

Lemma 5. *The value p_s is a decreasing function of s; it is strictly decreasing at s such that $p_s > 0$.*

PROOF:

Assume $p_s > 0$. In Lemma 4, we showed that $\omega_s \in \omega(p_s)$ and

$$\omega^*(p_s) + s - \phi(\omega_s + s) = 0$$

Since ϕ is strictly decreasing, $\hat{s} > s$ implies that $\phi(\omega_s + \hat{s}) < \phi(\omega_s + s)$ and therefore

$$\omega^*(p_s) + \hat{s} - \phi(\omega_s + \hat{s}) > 0$$

Then, since W is increasing, $\omega_{\hat{s}} \in \omega(p_{\hat{s}})$, and

$$\omega^*(p_{\hat{s}}) + \hat{s} - \phi(\omega_{\hat{s}} + \hat{s}) = 0$$

it follows that $p_s > p_{\hat{s}}$. The case where $p_s = 0$ follows from an analogous argument and is therefore omitted.

Lemma 6. *If $\hat{p} < p \leq p_s$, then $D(F^{\hat{p}}, s) \leq D(F^p, s)$.*

PROOF:

Let $\hat{p} < p \leq p_s$, $\omega \in \omega(p)$, and $\hat{\omega} \in \omega(\hat{p})$. Also, define

$$\pi^{**}(\theta) = \frac{\pi(\omega^*(p) + s) - \pi(\omega + s)}{\omega^*(p) - \omega}(\theta - \omega - s) + \pi(\omega + s)$$

Hence, π^{**} is the line that runs through both $(\omega+s, \pi(\omega+s))$ and $(\omega^*(p)+s, \pi(\omega^*(p)+s))$.

Note that $p \leq p_s$, and therefore $\omega \leq \omega_s$ and $\omega^*(p) \leq \omega^*(p_s)$. Hence,

$$\pi(\theta) \begin{cases} > \pi^{**}(\theta) & \text{whenever } \theta < \omega + s \\ = \pi^{**}(\theta) & \text{if } \theta \in \{\omega + s, \omega^*(p) + s\} \\ < \pi^{**}(\theta) & \text{if } \theta \in (\omega + s, \omega^*(p) + s) \end{cases} \tag{A5}$$

Since $\hat{p} < p$ and $\omega \in \omega(p)$, we have $d[F^{\hat{p}} - F^p](\omega') \leq 0$ for all $\omega' < \omega$. Moreover, $\omega \leq \omega^*(\hat{p}) \leq \omega^*(p)$.

Therefore,

$$\begin{aligned} D(F^p, s) &= \int_{\omega' < \omega} \pi(\omega' + s) dF^p(\omega') + \int_{\omega' \geq \omega} \pi^{**}(\omega' + s) dF^p(\omega') \\ &\qquad + \int_{\Omega} \pi^{**}(\omega' + s) d[F^{\hat{p}} - F^p](\omega') \\ &= \int_{\omega' < \omega} \pi(\omega' + s) dF^p(\omega') + \int_{\omega' < \omega} \pi^{**}(\omega' + s) d[F^{\hat{p}} - F^p](\omega') \\ &\qquad + \int_{\omega' \geq \omega} \pi^{**}(\omega' + s) dF^{\hat{p}}(\omega') \\ &\geq \int_{\omega' < \omega} \pi(\omega' + s) dF^p(\omega') + \int_{\omega' < \omega} \pi(\omega' + s) d[F^{\hat{p}} - F^p](\omega') \\ &\qquad + \int_{\omega' \geq \omega} \pi(\omega' + s) dF^{\hat{p}}(\omega') \\ &= \int_{\omega' < \omega} \pi(\omega' + s) dF^{\hat{p}}(\omega') + \int_{\omega' \geq \omega} \pi(\omega' + s) dF^{\hat{p}}(\omega') = D(F^{\hat{p}}, s) \end{aligned}$$

as desired.