

Mathematical Appendix

PROOF OF LEMMA 1:

The steps of the proof are exercised in detail only for the seller's payoff function, the result for the buyer can then be derived in a similar way. In a first step, we calculate the derivative of the seller's expected profit. For this, note that as a direct application of the envelope theorem (for constrained maximization) we get for all $\theta \in \Theta$

$$(A1) \quad \frac{\partial}{\partial \sigma} W(\beta, \sigma, \theta, Q^*(\beta, \sigma, \theta)) = -C_\sigma(\sigma, \theta, Q^*(\beta, \sigma, \theta)),$$

and

$$(A2) \quad \frac{\partial}{\partial \sigma} S(\sigma, \theta, \hat{Q}_S(\sigma, \theta)) = -C_\sigma(\sigma, \theta, \hat{Q}_S(\sigma, \theta)).$$

Next, to calculate the derivative $\frac{\partial}{\partial \sigma} s(\beta, \sigma)$, note that for each θ the integrand in s is the piecewise defined function

$$(A3) \quad \sigma \mapsto \begin{cases} (1 - \gamma)S(\sigma, \theta, \bar{q}) + \gamma W(\beta, \sigma, \theta, Q^*) - \gamma B(\beta, \theta, \bar{q}) & \text{if } Q^* > \bar{q} \\ S(\sigma, \theta, \hat{Q}_S) & \text{if } Q^* \leq \hat{Q}_S \\ W(\beta, \sigma, \theta, Q^*) - B(\beta, \theta, \hat{Q}_B) & \text{if } Q^* \leq \hat{Q}_B \end{cases}$$

It turns out that the piecewise defined derivative of this function is continuous, i.e. the pieces of this function are joined smoothly. We assume integrability of C_σ , so that we can interchange integration and differentiation, and get:

$$(A4) \quad \begin{aligned} \frac{\partial}{\partial \sigma} s(\beta, \sigma) &= - \int_{[Q^* > \bar{q}]} ((1 - \gamma)C_\sigma(\sigma, \theta, \bar{q}) + \gamma C_\sigma(\sigma, \theta, Q^*)) dF - 1 \\ &\quad - \int_{[\hat{Q}_S \geq Q^*]} C_\sigma(\sigma, \theta, \hat{Q}_S) dF - \int_{[\hat{Q}_B \geq Q^*]} C_\sigma(\sigma, \theta, Q^*) dF \\ &= -(1 - \gamma) \int_{[Q^* > \bar{q}]} \Delta_\sigma(\beta, \sigma, \theta, \bar{q}) dF - \int_{[\hat{Q}_S \geq Q^*]} \Delta_\sigma(\beta, \sigma, \theta, \hat{Q}_S) dF \\ &\quad - \int C_\sigma(\sigma, \theta, Q^*) dF - 1 \end{aligned}$$

Because we already know that for $\beta = \beta^*$ the expected joint surplus is uniquely maximized at σ^* , we will study the function

$$(A5) \quad \tilde{s}(\sigma) := s(\beta^*, \sigma) - \left(\int W(\beta^*, \sigma, \theta, Q^*(\beta^*, \sigma, \theta)) dF - \sigma \right).$$

which has derivative

$$(A6) \quad \tilde{s}'(\sigma) = -(1 - \gamma) \int_{[Q^* > \bar{q}]} \Delta_\sigma(\beta^*, \sigma, \theta, \bar{q}) dF - \int_{[\hat{Q}_S \geq Q^*]} \Delta_\sigma(\beta^*, \sigma, \theta, \hat{Q}_S) dF.$$

By exploiting $C_{\sigma q} \leq 0$, it is straightforward to see that $\Delta_\sigma(\beta^*, \sigma, \theta, q)$ is weakly decreasing in q , and that the first term in $\tilde{s}'(\sigma)$ is negative and the second is positive (if they do

not vanish). The first term is the derivative of what ER call the “hold-up tax”, this term is responsible for any potential underinvestment, and the second term is the derivative of the seller’s “breach subsidy”, this term may create overinvestment.

Now, in order to prove the lemma, consider first $\bar{q} = q_L$. In this case, for all $\sigma \geq \sigma^*$, since Q^* is nondecreasing in σ , the event $[Q^* > q_L]$ is equal to Θ and

$$(A7) \quad \tilde{s}'(\sigma) = -(1 - \gamma) \int \Delta_\sigma(\beta^*, \sigma, \theta, q_L) dF \leq 0.$$

Hence, \tilde{s} is a monotonically decreasing function in this range. All $\sigma > \sigma^*$ then lead to a lower payoff than σ^* , hence $\max \sigma_S(q_L, p) \leq \sigma^*$. For a contract over q_H the first term in \tilde{s}' vanishes for $\sigma \leq \sigma^*$, i.e. \tilde{s} is a weakly increasing function. Therefore, at q_H all $\sigma < \sigma^*$ are dominated by σ^* , and $\min \sigma_S(q_H, p) \geq \sigma^*$. Finally, consider q_H and a low price p_L . By definition of p_L it holds that $\hat{Q}_S(\sigma, \theta) \leq Q^*(\beta^*, \sigma, \theta)$ for all $\theta \in \Theta$ and σ . Therefore, the function \tilde{s} is weakly decreasing for $\sigma \geq \sigma^*$, hence $\sigma_S(q_H, p_L) = \{\sigma^*\}$. For the buyer, the corresponding claims follow from the assumption that $V_{\beta q} \geq 0$.

PROOF OF LEMMA 4:

Again, we prove the claim only for the seller. First, let us state the required conditions more precisely. For each θ , whenever $Q^*(\beta^*, \sigma, \theta) \leq \hat{Q}_S(\sigma, \theta)$ we need that $S(\sigma, \theta, \hat{Q}_S)$ is concave in σ , i.e.

$$(A8) \quad C_{\sigma\sigma}(\sigma, \theta, \hat{Q}_S) - \frac{C_{q\sigma}(\sigma, \theta, \hat{Q}_S)^2}{C_{qq}(\sigma, \theta, \hat{Q}_S)} \geq 0.$$

This condition follows from Assumption 3, because the determinant of the Hessian matrix of $(\sigma, q) \mapsto C(\sigma, \theta, q)$ is positive at $q = \hat{Q}_S$. One can see here why a linear cost function might be a problem: as C_{qq} becomes small, this condition becomes harder to fulfill. Furthermore we need the condition that $W(\beta^*, \sigma, \theta, Q^*)$ is concave, meaning that

$$(A9) \quad C_{\sigma\sigma}(\sigma, \theta, Q^*) + \frac{C_{\sigma q}(\sigma, \theta, Q^*)^2}{W_{qq}(\beta^*, \sigma, \theta, Q^*)} \geq 0,$$

which also follows from Assumption 3. Last, we need the condition $C_{\sigma\sigma}(\sigma, \theta, \bar{q}) \geq 0$, which is also implied by convexity of C in both variables.

Since s is continuous in q , p and σ (which is straightforward to check), according to Berge’s theorem, the argmax correspondence $\sigma_S(q, p)$ is upper hemicontinuous. Since upper hemicontinuity coincides with continuity if the correspondences are functions, for Assumption 2 to hold it suffices that the function $\sigma \mapsto s(\sigma, \beta^*)$ has a unique maximizer for all q and p . We therefore show that s is strictly concave, given that Assumption 3 holds. For this we need that the derivative (see equation A4) is decreasing in σ . It suffices to show that the continuous integrand is piecewise decreasing, which can be done by calculating the piecewise derivatives and using the above conditions.

PROOF OF PROPOSITION 5:

Since because of Assumption 2 the best responses have a continuous selection, we may assume that $\sigma_S(q, p)$ and $\beta_B(q, p)$ are continuous functions. For all $p \in [p_L, p_H]$, define

$$(A10) \quad \bar{q}_S(p) := \{q \in [q_L, q_H] : \sigma_S(q, p) = \sigma^*\}$$

and

$$(A11) \quad \bar{q}_B(p) := \{q \in [q_L, q_H] : \beta_B(q, p) = \beta^*\}.$$

From Lemma 1 and the intermediate value theorem it follows that these sets are nonempty for each p . Since the derivatives of the parties' payoff functions are weakly increasing in q (go back to equation (A4) to see that this holds for $s'(\beta^*, \sigma)$), these sets must also be convex, i.e. \bar{q}_S and \bar{q}_B are compact and convex valued upper hemicontinuous correspondences. Consider first the case that they are functions.¹ Lemma 1 tells us that $\bar{q}_S(p_L) = q_H \geq \bar{q}_B(p_L)$ and $\bar{q}_B(p_H) = q_H \geq \bar{q}_S(p_H)$. Applying the intermediate value theorem again yields existence of a \bar{p} such that $\bar{q}_S(\bar{p}) = \bar{q}_B(\bar{p}) =: \bar{q}$. This contract (\bar{q}, \bar{p}) thus leads to β^* as a best response to σ^* and σ^* as a best response to β^* .

If the correspondences \bar{q}_S and \bar{q}_B are not single-valued, their graphs are still pathwise connected and a similar argument applies: Since \bar{q}_S and \bar{q}_B are compact and convex valued upperhemicontinuous correspondences, the same is true for $d := \bar{q}_S - \bar{q}_B$. We have to show that there exists a \bar{p} with $0 \in d(\bar{p})$. Since $d(p_L)$ contains nonnegative elements, and d is upper hemicontinuous, the set $\{p \in [p_L, p_H] : d(p) \cap [0, q_H] \neq \emptyset\}$ is nonempty and closed. But the same holds for $\{p \in [p_L, p_H] : d(p) \cap [-q_H, 0] \neq \emptyset\}$, and since the union of these sets is $[p_L, p_H]$, there must exist a \bar{p} in their intersection. Convexity of $d(\bar{p})$ then implies that $0 \in d(\bar{p})$.

PROOF OF COROLLARY 7:

The derivative of $\tilde{s}(\sigma)$, as calculated in the proof of Lemma 1 (equation A6), evaluated at σ^* , must vanish at the optimal contract. The corollary follows since for the kind of functions defined in Assumption 6 it holds that

$$(A12) \quad \Delta_\sigma(q) = -C'_1(\sigma)(Q^* - q) \text{ and } \Delta_\beta(q) = V'_1(\beta)(Q^* - q).$$

PROOF OF PROPOSITION 8:

When the price is p_L , the buyer makes a profit on each unit, i.e. $\hat{Q}_B = \bar{q}$ for all θ . When price is p_H , it holds that $\hat{Q}_S = \bar{q}$ for all θ . Expected payoff is analogous to the case with an intermediate price and can be rearranged to look as follows (again only for the seller):

$$(A13) s(\sigma, \beta) = \int W(\beta, \sigma, \theta, Q^*) dF - \sigma - \int B(\beta, \theta, \bar{q}) dF \\ - (1 - \gamma) \int_{[Q^* > \bar{q}]} \Delta(\beta, \sigma, \theta, \bar{q}) dF - (1 - \lambda) \int_{[Q^* \leq \bar{q}]} \Delta(\beta, \sigma, \theta, \bar{q}) dF$$

with $p = \lambda p_L + (1 - \lambda) p_H$. The claim can now be proved following the same steps as in the proof of Proposition 5, the role of the price being played by λ .

PROOF OF PROPOSITION 9:

We prove this result independently of previous results in this paper, because it holds without Assumption 2, and would hold also for arbitrary investment decisions and linear functions. For $\lambda = \gamma$, the seller's expected payoff functions as stated in equation (A13)

¹This holds for example if the inequalities $C_{\sigma q} \leq 0$ and $V_{\beta q} \geq 0$ hold strictly everywhere, Q^* is continuous in θ , $\gamma \in (0, 1)$, and σ^* and β^* are interior solutions.

equals

$$(A14) \quad s(\beta, \sigma) = \bar{p}\bar{q} + (1 - \gamma) \left(\int -C(\sigma, \theta, \bar{q})dF - \sigma \right)$$

$$(A15) \quad + \gamma \left(\int W(\beta, \sigma, \theta, Q^*)dF - \sigma \right) - \gamma \int V(\beta, \theta, \bar{q})dF$$

with $\bar{p} = \gamma p_L + (1 - \gamma)p_H$. In this case, the payoff functions are identical to the ones that result from specific performance in ER. Next, consider the defining equation of σ^* , which is that for all other σ

$$(A16) \quad \int W(\beta^*, \sigma^*, \theta, Q^*(\sigma^*, \beta^*, \theta))dF - \sigma^* \geq \int W(\beta^*, \sigma, \theta, Q^*(\sigma, \beta^*, \theta))dF - \sigma.$$

Furthermore, from the definition of Q^* we know that

$$(A17) \quad W(\beta^*, \sigma, \theta, Q^*(\sigma, \beta^*, \theta)) \geq W(\beta^*, \sigma, \theta, Q^*(\sigma^*, \beta^*, \theta)) \quad \text{for all } \sigma, \theta.$$

From these two equations, it follows that

$$(A18) \quad \sigma^* \in \arg \max_{\sigma} \int -C(\sigma, \theta, Q^*(\sigma^*, \beta^*, \theta))dF - \sigma$$

Since we assumed the special payoff functions defined in Assumption 6 it follows that with $\bar{q} = \int Q^*(\beta^*, \sigma^*, \theta)dF$

$$(A19) \quad \sigma^* \in \arg \max_{\sigma} \int -C(\sigma, \theta, \bar{q})dF - \sigma.$$

Hence, when $\beta = \beta^*$, all terms in the seller's payoff function are maximized at σ^* , and it is straightforward to show that the same holds symmetrically for the buyer.

Examples

In this appendix we compute two examples, to explore for which type of functions Assumption 2 is likely to hold. In the first example P^* is deterministic, such that the concavity assumption becomes very important. The second example shows that the first best can also sometimes be reached although the cost function is linear, as long as there is enough variance in P^* . Let $\gamma = \frac{1}{2}$ and

$$(B1) \quad C(\sigma, \theta, q) = \frac{1}{2\sigma}q + c\frac{q^2}{2\theta},$$

$$(B2) \quad V(\beta, \theta, q) = \left(\frac{4}{3}c + \frac{7}{3} - \frac{1}{2\beta} \right) q - \frac{q^2}{2\theta}.$$

In the specification of the model the investment cost was normalized to be linear, but it can as well be any convex function. For this example, we take $\sigma^2/2$ to be the cost of investment σ . The uncertainty parameter θ is assumed to be uniformly distributed on

the interval $[1, 2]$. The efficient quantity is

$$(B3) \quad Q^*(\beta, \sigma, \theta) = \left(\frac{4}{3}c + \frac{7}{3} - \frac{1}{2\beta} - \frac{1}{2\sigma} \right) \frac{\theta}{1+c}.$$

Calculations reveal that $\sigma^* = \beta^* = 1$. Since the equilibrium price

$$(B4) \quad P^*(\beta, \sigma, \theta) = \frac{c}{1+c} \left(\frac{4}{3}c + \frac{7}{3} - \frac{1}{2\beta} - \frac{1}{2\sigma} \right) + \frac{1}{2\sigma}.$$

does not depend on θ , the only candidate for an efficient contract is $\bar{q} = \frac{8}{3}$ and $\bar{p} = \frac{4}{3}c + \frac{1}{2}$. The sufficient condition in Assumption 3 is fulfilled if $c > 3/16$.² For very low c , this contract leads to a saddle point instead of a maximum of the seller's payoff function at σ^* . This can be seen by calculating the second derivative for $\sigma \geq \sigma^*$: for small c it becomes positive.

This example is one in which, once investment is sunk, only one party breaches the contract. Nevertheless, since the overinvesting party faces hold-up and non-breach contingencies, the equilibrium is efficient if the payoff functions are sufficiently concave. As the cost function approaches a linear and deterministic one, the first best ceases to be attainable.

It is not the linearity alone that prevents the price-quantity contracts with expectation damages from being optimal. If there is a random element in the linear term, such that always both parties face the risk of breaching, the first best may also be attainable. Consider the following variant of the preceding example:

$$(B5) \quad C(\sigma, \theta, q) = \left(\frac{1}{2\sigma} + \theta_1 \right) q$$

$$(B6) \quad V(\beta, \theta, q) = \left(\frac{7}{3} - \frac{1}{2\beta} + \theta_1 \right) q - \frac{q^2}{2\theta_2}.$$

That is, we set $c = 0$ and to the contingency we add a new component which makes marginal cost volatile. The part θ_1 is a market shock which affects both the buyer's valuation and the seller's cost (which could be opportunity cost). The part θ_2 only affects the buyer, and is again uniformly distributed on $[1, 2]$. With regard to θ_1 , we assume that it is uniformly distributed on $[0, 1]$. The efficient quantity is now

$$(B7) \quad Q^*(\beta, \sigma, \theta) = \left(\frac{7}{3} - \frac{1}{2\beta} - \frac{1}{2\sigma} \right) \theta_2.$$

Looking for the optimal contract, we get the following equation from the seller's maximization problem:

$$(B8) \quad \int_{[Q^* \leq \bar{q}]} \left(\bar{p} - \frac{1}{2} \right) (\bar{q} - Q^*) d\theta_2 = \int_{[Q^* > \bar{q}]} \frac{1}{2} (Q^* - \bar{q}) d\theta_2$$

²This bound is even lower if the convex investment cost is taken into account.

One obvious solution is $\bar{q} = 2$ and $\bar{p} = 1$. All solutions are characterized by

$$(B9) \quad \bar{p}_S(q) = \frac{1}{2} + \frac{(\frac{3}{4}\bar{q} - 2)^2}{2(\frac{3}{4}\bar{q} - 1)^2}$$

for all $q_H = \frac{8}{3} > \bar{q} > q_L = \frac{4}{3}$. The buyer's payoff fulfills all assumptions. Unfortunately, the condition that characterizes the optimal contract for the buyer becomes quite complex. As numerical solutions of the two equations we get $\bar{q} = 2.039$ and $\bar{p} = 0.8956$.