Technical Appendix
Speculative Growth: Hints from the US Economy

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Proof of Propositions in Section II

Fiscal

We can rewrite the capital market equilibrium condition as

$$s\left((1-\overline{\tau})w(k_t), r_t\right) + \overline{\tau}w(k_t) - \overline{g}_t = \left(q_t + x_t + \frac{1}{2}\theta^{-1}(x_t-\gamma)^2\right)k_t + d_t,$$

where

$$\overline{g}_t = \overline{g} + \alpha(w_t - w^n).$$

Let us define a net saving function

$$\widetilde{s}(w(k_t), r_t) = s\left((1-\overline{\tau})w(k_t), r_t\right) + \overline{\tau}w(k_t) - \overline{g}_t.$$

In this proposition, we perform a comparative statics exercise varying α (or equivalently $\overline{g}_t$) and keeping the normal steady state fixed. Therefore, let us denote by

$$s_k = \frac{\partial s\left((1-\overline{\tau})w(k_t), r_t\right) + \overline{\tau}w(k_t) - \overline{g}}{\partial k_t},$$

and

$$s_r = \frac{\partial s\left((1-\overline{\tau})w(k_t), r_t\right) + \overline{\tau}w(k_t) - \overline{g}}{\partial r_t}.$$

These numbers do not depend on α.

Note that a first order approximation of wages around the normal steady state is:

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$$w(k_t) = w^n + \pi_1 k^n \widehat{k}_t.$$

We therefore have the following first order approximation for the net saving function

$$\widetilde{s}(w(k_t), r_t) = \begin{cases} s_k^\alpha k_t + s_r r_t + s_0^\alpha, & k < k^o; \\ s_k^\alpha k_t + s_r r_t + s_0^\alpha + \delta, & k \geq k^o, \end{cases}$$

where

$$s_k^\alpha = s_k - \alpha \pi_1 k^n, \quad \text{and} \quad s_0^\alpha = s_0 + \frac{\alpha \pi_1 (k^n)^2}{s_r}.$$

Assumption 4 guarantees that the normal steady state is in a dynamically efficient region, while the speculative steady state, if it exists, is in a dynamically inefficient region.

The speculative steady state, if it exists, is characterized by the following equations

$$k^s > k^0,$$

$$\widetilde{s}(w(k^s), r^s) = d^s + (1+\gamma) k^s,$$

$$r^s = \pi_0 - \pi_1 k^s,$$

$$d^s = \frac{1+r^s}{\gamma - r^s} (\overline{g} + \alpha (w^s - w^n) - \overline{\tau} w^s).$$

Similarly, in the normal steady state, we have

$$\widetilde{s}(w(k^n), r^n) = (1+\gamma) k^n,$$

$$r^n = \pi_0 - \pi_1 k^n,$$

$$\overline{g} = \overline{\tau} w^n.$$

Using these equations, we can rewrite the conditions characterizing the speculative steady state as

$$k^s > k^0,$$

$$s_k^\alpha (k^s - k^n) + s_r (r^s - r^n) + \delta = d^s + (1+\gamma)(k^s - k^n),$$

$$r^s - r^n = -\pi_1(k^s - k^n),$$

$$d^s = \frac{1+r^s}{\gamma - r^s}(\alpha - \overline{\tau})(w^s - w^n),$$

$$w^s - w^n = \pi_1 k^n (k^s - k^n).$$

We can solve these equations for k^s, w^s, r^s and d^s.

In particular, we get

$$\left[s_k^\alpha - \pi_1 s_r - (1+\gamma)\right](k^s - k^n) + \delta = \frac{1 + r^n - \pi_1(k^s - k^n)}{\gamma - r^n + \pi_1(k^s - k^n)}(\alpha - \overline{\tau})\pi_1 k^n (k^s - k^n).$$

We can rewrite this equation as

$$\begin{aligned} \delta \; = \; & \left[(1+\gamma) - s_k + \pi_1 s_r\right](k^s - k^n) + \alpha \pi_1 k^n (k^s - k^n) \\ & + (\alpha - \overline{\tau}) \frac{\left[1 + r^n - \pi_1(k^s - k^n)\right] \pi_1 k^n (k^s - k^n)}{\gamma - r^n + \pi_1(k^s - k^n)}. \end{aligned} \tag{1}$$

Using the fact that $1 + r^s = 1 + r^n - \pi_1(k^s - k^n) > 0$, $(1+\gamma) - s_k - \pi_1 s_r > 0$ and $k^s > k^0$, we see that for $\alpha \geq \overline{\tau}$, the right hand side of this equation is greater than

$$\left[(1+\gamma) - s_k + \pi_1 s_r\right](k^0 - k^n) + \alpha \pi_1 k^n (k^0 - k^n).$$

It is clear that

$$\lim_{\alpha \to +\infty} \left\{ \left[(1+\gamma) - s_k + \pi_1 s_r\right](k^0 - k^n) + \alpha \pi_1 k^n (k^0 - k^n) \right\} = +\infty.$$

This proves that for α high enough (1), has no solution.

We have the stronger result that a sufficient condition for (1) not to have a solution is

$$\alpha \geq \max \left\{ \frac{\Delta}{\pi_1 k^n} \frac{\frac{\delta}{\Delta}}{(k^0 - k^n)} \left(1 - \frac{k^0 - k^n}{\frac{\delta}{\Delta}}\right), \overline{\tau} \right\}.$$

Therefore if $\frac{\Delta}{\pi_1 k^n} \frac{\frac{\delta}{\Delta}}{(k^0 - k^n)}(1 - \frac{k^0 - k^n}{\frac{\delta}{\Delta}}) < 1$, it can be the case that there is no solution for $\alpha \geq \overline{\alpha}$ where $\overline{\alpha} < 1$.

External

We continue to assume that δ is small so that we can make a first order approximation around $q = 1$ and $k = k^n$ for the transition from k^n to k^s, if such transition is possible. Let $\widehat{q_t} \equiv (q_t - 1)$, $\widehat{k_t} \equiv (k_t - k^n)$, and $z_t = (z_t - z_t^n)$ then the linearized dynamic system can be written as:

$$\begin{aligned} \widehat{q_{t+1}} \quad &= \quad \left[\frac{1+r^n}{1+\gamma} + \left(\pi_1 \frac{\theta}{1+\gamma} + \widetilde{\pi_1}\left(1+\frac{\theta}{1+\gamma}\right)\right)\frac{k^n}{1+\gamma}\right]\widehat{q_t} \\ &+ \left[\pi_1 + \widetilde{\pi_1}\frac{1+\gamma-s_k}{1+\gamma}\right]\widehat{k_t} - 1\left\{\widehat{k_t} \geq \widehat{k^o}\right\}\frac{\widetilde{\pi_1}\delta}{1+\gamma}, \end{aligned} \tag{2}$$

$$\widehat{k_{t+1}} = \frac{\theta k^n}{(1+\gamma)^2}\widehat{q_t} + \widehat{k_t}. \tag{3}$$

The analysis and proofs that follow refer to the dynamic system described by (2) and (3).

The speculative steady state is characterized by $\widehat{k^s} = \frac{\widetilde{\pi_1}\delta}{\pi_1(1+\gamma)+\widetilde{\pi_1}(1+\gamma-s_k)} > 0$ and $\widehat{z^s} = \frac{\pi_1\delta}{\pi_1(1+\gamma)+\widetilde{\pi_1}(1+\gamma-s_k)} > 0$.

For a speculative growth path to exist, it must be the case that the unstable path of the normal steady state intersects the k^o line *below* the intersection of the latter and the saddle path of the speculative steady state. We can express this condition as

$$\Lambda^e \equiv \frac{\lambda^{e+} - 1}{1 - \lambda^{e-}} < \frac{k^s - k^0}{k^0 - k^n}, \tag{4}$$

where the superscript e stands for external and

$$\lambda^{e+} = 1 + \frac{1}{2}\left[\begin{array}{c} -1 + \frac{1+r^n}{1+\gamma} + \frac{\widetilde{\pi_1}(1+\frac{\theta}{1+\gamma})k^n}{1+\gamma} + \frac{\pi_1\theta k^n}{(1+\gamma)^2} \\ +\sqrt{\left(-1 + \frac{1+r^n}{1+\gamma} + \frac{\widetilde{\pi_1}(1+\frac{\theta}{1+\gamma})k^n}{1+\gamma} + \frac{\pi_1\theta k^n}{(1+\gamma)^2}\right)^2 + 4\frac{\widetilde{\pi_1}\theta k^n(1-\frac{s_k}{1+\gamma})}{(1+\gamma)^2} + 4\frac{\pi_1\theta k^n}{(1+\gamma)^2}} \end{array}\right]$$

and

$$\lambda^{e-} = 1 + \frac{1}{2}\left[\begin{array}{c} -1 + \frac{1+r^n}{1+\gamma} + \frac{\widetilde{\pi_1}(1+\frac{\theta}{1+\gamma})k^n}{1+\gamma} + \frac{\pi_1\theta k^n}{(1+\gamma)^2} \\ -\sqrt{\left(-1 + \frac{1+r^n}{1+\gamma} + \frac{\widetilde{\pi_1}(1+\frac{\theta}{1+\gamma})k^n}{1+\gamma} + \frac{\pi_1\theta k^n}{(1+\gamma)^2}\right)^2 + 4\frac{\widetilde{\pi_1}\theta k^n(1-\frac{s_k}{1+\gamma})}{(1+\gamma)^2} + 4\frac{\pi_1\theta k^n}{(1+\gamma)^2}} \end{array}\right].$$

We have

$$\Lambda^e = -1 + 2\frac{1}{1-P},$$

where

$$P = \left[1 + \frac{4\frac{\widetilde{\pi_1}\theta k^n(1-\frac{s_k}{1+\gamma})}{(1+\gamma)^2} + 4\frac{\pi_1\theta k^n}{(1+\gamma)^2}}{C}\right]^{-1/2},$$

and

$$C = \left[-1 + \frac{1+r^n}{1+\gamma} + \frac{\widetilde{\pi_1}(1+\frac{\theta}{1+\gamma})k^n}{1+\gamma} + \frac{\pi_1\theta k^n}{(1+\gamma)^2}\right]^2$$

This is true because we have assumed that Assumption 0' holds, which implies that $-1 + \frac{1+r^n}{1+\gamma} + \frac{\widetilde{\pi_1}k^n}{1+\gamma} > 0$ and therefore

$$-1 + \frac{1+r^n}{1+\gamma} + \frac{\widetilde{\pi_1}(1+\frac{\theta}{1+\gamma})k^n}{1+\gamma} + \frac{\pi_1\theta k^n}{(1+\gamma)^2} > 0.$$

Λ^e is a decreasing function of

$$\Upsilon^e \equiv \frac{4\frac{\widetilde{\pi_1}\theta k^n(1-\frac{s_k}{1+\gamma})}{(1+\gamma)^2} + 4\frac{\pi_1\theta k^n}{(1+\gamma)^2}}{C},$$

where C is defined as before and Υ^e is non-monotonic in θ. It is increasing on $(0, \widetilde{\theta}^e)$ and decreasing on $(\widetilde{\theta}^e, +\infty)$, where

$$\widetilde{\theta}^e = \frac{-1 + \frac{1+r^n}{1+\gamma} + \frac{\widetilde{\pi_1}k^n}{1+\gamma}}{\frac{(\pi_1+\widetilde{\pi_1})k^n}{(1+\gamma)^2}}.$$

In addition $\lim_{\theta\to 0}\Upsilon^e = 0$, and $\lim_{\theta\to+\infty}\Upsilon^e = 0$.

As a result, Λ^e is non-monotonic in θ. It is decreasing on $(0, \widetilde{\theta}^e)$ and increasing on $(\widetilde{\theta}^e, +\infty)$, and reaches a minimum for $\theta = \widetilde{\theta}^e$. We also have $\lim_{\theta\to 0}\Lambda^e = +\infty$, and $\lim_{\theta\to+\infty}\Lambda^e = +\infty$.

Proof of Claims in Section III

Technological progress

Let us assume that $\delta_p = 1 - p$ is small so that we can make a first order approximation around $q = 1$ and $k = k^n$ for the transition from k^n to k^s, if such transition is possible. Let $\widehat{q_t} \equiv (q_t - 1)$ and $\widehat{k_t} \equiv (k_t - k^n)$, then the linearized dynamic system can be written as:

$$\widehat{q_{t+1}} = \begin{cases} \left[\frac{1+r^n}{1+\gamma} + \left(\frac{\pi_1\frac{\theta}{1+\gamma}}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r}\right)k^n\right]\widehat{q_t} + \frac{\Delta}{s_r}\widehat{k_t}, & k_t < k^o; \\ \left[\frac{1+r^n}{1+\gamma} + \left(\frac{\pi_1\frac{\theta}{1+\gamma}}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r}\right)k^n\right]\widehat{q_t} + \frac{\Delta}{s_r}\widehat{k_t} - \left[k^n\frac{1+\gamma}{s_r} + r^n\right]\delta_p, & k \geq k^o, \end{cases} \quad (5)$$

$$\widehat{k_{t+1}} = \frac{\theta k^n}{(1+\gamma)^2}\widehat{q_t} + \widehat{k_t} \tag{6}$$

The analysis and proofs that follow refer to the dynamic system described by (5) and (6).

Assumption 5 *(Minimum technological progress)*

$$\delta_p > \underline{\delta_p} = \frac{\frac{\Delta}{s_r}}{r^n + \frac{1+\gamma}{s_r}k^n}(k^0 - k^n).$$

Proposition 9: ***(Multiple Steady States)***

If Assumptions 0, 0' and 5 are satisfied, the economy has two non-degenerate steady states, k^n *and* k^s*, with:*

$$k^n = \frac{s_0 + \pi_0}{\Delta} < k^o < k^n + \frac{1 + r^n + (1+\gamma)k^n/s_r}{\Delta}\delta_p = k^s, \tag{7}$$

where $\Delta \equiv \pi_1 + (1+\gamma - s_k)/s_r > 0$*, and the superscripts "n" and "s" stand for* normal *and* speculative*, respectively.*

Assumption 6 *(Speculative adjustment costs region)*

$$\frac{\lambda^+ - 1 + \frac{\frac{\Delta}{s_r}\frac{k^n}{(1+\gamma)^2}}{r^n + \frac{1+\gamma}{s_r}k^n}\theta}{1 - \lambda^- - \frac{\frac{\Delta}{s_r}\frac{k^n}{(1+\gamma)^2}}{r^n + \frac{1+\gamma}{s_r}k^n}\theta} < \frac{k^s - k^0}{k^0 - k^n}.$$

Proposition 10: ***(Multiple Equilibria and Speculative Growth)***

If Assumptions 0, 0', 5 and 6 hold, there is a speculative growth path that takes the economy from k^n *to* k^s*.*

Proof: Imposing $\widehat{q_{t+1}} = \widehat{q_t}$ and $\widehat{k_{t+1}} = \widehat{k_t}$ and solving for $\widehat{k^s}$ and $\widehat{q^s}$ in

$$\begin{aligned}\widehat{q_{t+1}} &= \left[\frac{1+r^n}{1+\gamma} + \left(\frac{\pi_1\frac{\theta}{1+\gamma}}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r}\right)k^n\right]\widehat{q_t} \\ &\quad + \frac{\Delta}{s_r}\widehat{k_t} - \left[k^n\frac{1+\gamma}{s_r} + r^n\right]\delta_p\end{aligned}$$

and

$$\widehat{k_{t+1}} = \frac{\theta k^n}{(1+\gamma)^2}\widehat{q_t} + \widehat{k_t},$$

we find that $\widehat{k^s} = \frac{r^n + \frac{1+\gamma}{s_r}k^n}{\frac{\Delta}{s_r}}\delta_p$ and $\widehat{q^s} = 0$. This is a steady state if and only if $\widehat{k^s} > \widehat{k^0} = k^0 - k^n$, which we can rewrite as

$$\delta_p > \underline{\delta_p} = \frac{\frac{\Delta}{s_r}}{r^n + \frac{1+\gamma}{s_r}k^n}\left(k^0 - k^n\right).$$

Let us now assume that this inequality holds. We now ask under what conditions a transition from the normal steady state to the speculative steady state is possible. For a speculative growth path to exist, it must be the case that the unstable path of the normal steady state intersects the k^o line *below* the intersection of the latter and the saddle path of the speculative steady state. We can express this condition as

$$\left[\left(\lambda^{+}-1\right)\left(\frac{\theta k^{n}}{(1+\gamma)^{2}}\right)^{-1}\right]\left(k^{0}-k^{n}\right)<\left[\left(\lambda^{+}-1\right)\left(\frac{\theta k^{n}}{(1+\gamma)^{2}}\right)^{-1}\right]\left(k^{s}-k^{0}\right)-\delta_{p}. \tag{8}$$

We can rewrite (8) as

$$\left(\lambda^{+}-1+\frac{\frac{\Delta}{s_{r}}\frac{k^{n}}{(1+\gamma)^{2}}}{r^{n}+\frac{1+\gamma}{s_{r}}k^{n}}\theta\right)\left(1-\lambda^{-}-\frac{\frac{\Delta}{s_{r}}\frac{k^{n}}{(1+\gamma)^{2}}}{r^{n}+\frac{1+\gamma}{s_{r}}k^{n}}\theta\right)^{-1}<\frac{k^{s}-k^{0}}{k^{0}-k^{n}}.$$

Bubbles

Proof of claims in Section III.C

In principle, there could be a third steady state once bubbles are feasible: $(k^b, 1, b)$, where $\pi_0 - \pi_1 k^b = \gamma$ and $b > 0$. Assumption 4 guarantees that this steady state does not exist. To see why, assume the contrary. It is clear from assumption 3 that $k^b < k^0$. But the associated bubble in this steady state would be negative, since $b = s_0 + s_r\pi_0 - \Delta k^b < s_0 + s_r\pi_0 - \Delta k^n = 0$, which is a contradiction.

In the proof of this proposition, we have to perform two linearizations, one around the normal steady state and one around the speculative steady state. This is because the characteristics of the dynamic system are different around the two steady states. The normal steady state has a two-dimensional unstable manifold and a one-dimensional stable manifold, whereas the speculative steady state has a one-dimensional unstable manifold and a two-dimensional stable manifold.

We consider the possibility of a small initial bubbles, $0 < b_0 < M_b(\delta_p)\,\delta_p$ for some function $M_b(\delta_p) > 0$ such that

$$\lim_{\delta_p \to 0} M_b(\delta_p) = 0$$

and

$$\lim_{\delta_p \to 0} \frac{M_b(\delta_p)}{\delta_p} = +\infty$$

and show that if

$$\left(\lambda^{+}-1+\frac{\frac{\Delta}{s_r}\frac{k^n}{(1+r^n)^2}}{r^n+\frac{1+r^n}{s_r}k^n}\theta\right)\left(1-\lambda^{-}-\frac{\frac{\Delta}{s_r}\frac{k^n}{(1+r^n)^2}}{r^n+\frac{1+r^n}{s_r}k^n}\theta\right)^{-1}<\frac{k^s-k^0}{k^0-k^n}$$

then there exists a function $m_b(\delta_p)>0$ such that a transition is possible for δ_p small enough and $0<b_0<m_b(\delta_p)\,\delta_p$, and that if

$$\left(\lambda^{+}-1+\frac{\frac{\Delta}{s_r}\frac{k^n}{(1+r^n)^2}}{r^n+\frac{1+r^n}{s_r}k^n}\theta\right)\left(1-\lambda^{-}-\frac{\frac{\Delta}{s_r}\frac{k^n}{(1+r^n)^2}}{r^n+\frac{1+r^n}{s_r}k^n}\theta\right)^{-1}>\frac{k^s-k^0}{k^0-k^n},$$

then for δ_p small enough, for every $\widetilde{m_b}>0$, there exists $0<b_0<\widetilde{m_b}\delta_p$ such that no transition is possible with initial bubble b_0.

Let us first linearize the system around the normal steady state. Up to second order terms in $\widehat{q_t}=(q_t-1)$, $\widehat{k_t}=(k_t-k^n)$, $\widehat{r_t}=(r_t-r^n)$, $\widehat{b_t}=b_t$ and δ_p we have that

$$\begin{aligned}
\widehat{r_t} &= \frac{\frac{\theta}{1+\gamma}+1}{s_r}k^n\widehat{q_t}+\frac{1+\gamma-s_k}{s_r}\widehat{k_t}+\frac{1}{s_r}\widehat{b_t},\\
\widehat{q_{t+1}} &= \left[\frac{1+r^n}{1+\gamma}+\frac{\pi_1\frac{\theta}{1+\gamma}k^n}{1+\gamma}+\frac{\frac{\theta}{1+\gamma}+1}{s_r}k^n\right]\widehat{q_t}+\left[\pi_1+\frac{1+\gamma-s_k}{s_r}\right]\widehat{k_t}+\frac{1}{s_r}\widehat{b_t},\\
\widehat{k_{t+1}} &= \frac{\theta k^n}{(1+\gamma)^2}\widehat{q_t}+\widehat{k_t},\\
\widehat{b_{t+1}} &= \frac{1+r^n}{1+\gamma}\widehat{b_t}.
\end{aligned}$$

Similarly, we linearize the system around the speculative steady state. Up to second order terms in $\widehat{q_t}=(q_t-1)$, $\widehat{k_t}=(k_t-k^s)$, $\widehat{r_t}=(r_t-r^s)$, $\widehat{b_t}=b_t$ and δ_p, we have that

$$\begin{aligned}
\widehat{r_t} &= \frac{\frac{\theta}{1+\gamma}+1}{s_r}k^s\widehat{q_t}+\frac{1+\gamma-s_k}{s_r}\widehat{k_t}+\frac{1}{s_r}\widehat{b_t},\\
\widehat{q_{t+1}} &= \left[\frac{1+r^s}{1+\gamma}+\frac{\pi_1\frac{\theta}{1+\gamma}k^s}{1+\gamma}+\frac{\frac{\theta}{1+\gamma}+1}{s_r}k^s\right]\widehat{q_t}\\
&\quad+\left[\pi_1+\frac{1+\gamma-s_k}{s_r}\right]\widehat{k_t}+\frac{1}{s_r}\widehat{b_t},\\
\widehat{k_{t+1}} &= \frac{\theta k^s}{(1+\gamma)^2}\widehat{q_t}+\widehat{k_t},\\
\widehat{b_{t+1}} &= \frac{1+r^s}{1+\gamma}\widehat{b_t}.
\end{aligned}$$

The dynamic system around the normal steady state is characterized by the matrix

$$\Omega^n = \begin{bmatrix} \frac{1+r^n}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^n}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^n & \pi_1 + \frac{1+\gamma-s_k}{s_r} & \frac{1}{s_r} \\ \frac{\theta k^n}{(1+\gamma)^2} & 1 & 0 \\ 0 & 0 & \frac{1+r^n}{1+\gamma} \end{bmatrix}.$$

Similarly, the dynamic system around the speculative steady state is characterized by the matrix

$$\Omega^s = \begin{bmatrix} \frac{1+r^s}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^s}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^s & \pi_1 + \frac{1+\gamma-s_k}{s_r} & \frac{1}{s_r} \\ \frac{\theta k^s}{(1+\gamma)^2} & 1 & 0 \\ 0 & 0 & \frac{1+r^s}{1+\gamma} \end{bmatrix}.$$

Keeping the same notations as above, it is clear that the eigenvalues of Ω^n are $\lambda^{n+} > 1$, $\lambda^{n-} < 1$ and $\frac{1+r^n}{1+\gamma} > 1$, with corresponding eigenvectors $(x^{n+}, 1, 0)'$, $(x^{n-}, 1, 0)'$ and z_b^n where

$$\lambda^{n+} = 1 + \frac{1}{2} \left[\begin{array}{c} \frac{1+r^n}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^n}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^n - 1 \\ + \sqrt{\left[\frac{1+r^n}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^n}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^n - 1 \right]^2 + 4(\pi_1 + \frac{1+\gamma-s_k}{s_r}) \frac{\theta k^n}{(1+\gamma)^2}} \end{array} \right]$$

and

$$\lambda^{n-} = 1 + \frac{1}{2} \left[\begin{array}{c} \frac{1+r^n}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^n}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^n - 1 \\ - \sqrt{\left[\frac{1+r^n}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^n}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^n - 1 \right]^2 + 4(\pi_1 + \frac{1+\gamma-s_k}{s_r}) \frac{\theta k^n}{(1+\gamma)^2}} \end{array} \right].$$

Therefore, the tangent plane to the unstable manifold of the normal steady state is the plane through the normal steady state with directing vectors $(x^{n+}, 1, 0)'$ and z_b^n, and the tangent line to its stable manifold is the line through the normal steady state with directing vector $(x^{n-}, 1, 0)'$.

Similarly, the eigenvalues of Ω^s are $\lambda^{s+} > 1$, $\lambda^{s-} < 1$ and $\frac{1+r^s}{1+\gamma} < 1$, with corresponding eigenvectors $(x^{s+}, 1, 0)'$, $(x^{s-}, 1, 0)'$ and z_b^s where

$$\lambda^{s+} = 1 + \frac{1}{2} \left[\begin{array}{c} \frac{1+r^s}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^s}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^s - 1 \\ + \sqrt{\left[\frac{1+r^s}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^s}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^s - 1 \right]^2 + 4(\pi_1 + \frac{1+\gamma-s_k}{s_r}) \frac{\theta k^s}{(1+\gamma)^2}} \end{array} \right]$$

and

$$\lambda^{s-} = 1 + \frac{1}{2} \left[\begin{array}{c} \frac{1+r^s}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^s}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^s - 1 \\ - \sqrt{\left[\frac{1+r^s}{1+\gamma} + \frac{\pi_1 \frac{\theta}{1+\gamma} k^s}{1+\gamma} + \frac{\frac{\theta}{1+\gamma}+1}{s_r} k^s - 1 \right]^2 + 4(\pi_1 + \frac{1+\gamma-s_k}{s_r}) \frac{\theta k^s}{(1+\gamma)^2}} \end{array} \right].$$

Therefore, the tangent line to the unstable manifold of the speculative steady state is the line through the speculative steady state with directing vectors $(x^{s+},1,0)'$, and the tangent plane to its stable manifold is the plane through the speculative steady state with directing vectors $(x^{s-},1,0)'$ and z_b^s.

Let us now vary δ_p, $k^0 = k^0(\delta_p)$ $\gamma = \gamma(\delta_p)$ and $s^0 = s^0(\delta_p)$ in such a way that $k^n = (s_0 + s_r\pi_0)/(s_r\pi_1 + (1+\gamma - s_k))$ is fixed, assumptions 0,1 and 4 are verified and $|k^0 - k^n| > \varepsilon\delta_p$ for some $\varepsilon > 0$. Note that $\gamma(\delta_p) < r^n = \pi_0 - \pi_1 k^n$ for $\delta_p > 0$ and $\lim_{\delta_p\to 0}\gamma(\delta_p) = r^n = \pi_0 - \pi_1 k^n$. Also $k^0(\delta_p) > k^n$ for $\delta_p > 0$ and $\lim_{\delta_p\to 0} k^0(\delta_p) = k^n$.

As we do this, k^s, x^{n+}, x^{n-}, λ^{n+},λ^{n-}, x^{s+}, x^{s-}, λ^{s+} and λ^{s-} also vary. These variables are all continuous functions of δ_p. Let us denote them by $k^s(\delta_p)$, $x^{n+}(\delta_p)$, $x^{n-}(\delta_p)$, $\lambda^{n+}(\delta_p)$,$\lambda^{n-}(\delta_p)$, $x^{s+}(\delta_p)$, $x^{s-}(\delta_p)$, $\lambda^{s+}(\delta_p)$ and $\lambda^{s-}(\delta_p)$. It is clear that $k^s(\delta_p) = k^n + O(\delta_p)$, $x^{n+}(\delta_p) = x^+ + O(\delta_p)$, $x^{n-}(\delta_p) = x^- + O(\delta_p)$, $\lambda^{n+}(\delta_p) = \lambda^+ + O(\delta_p)$, $\lambda^{n-}(\delta_p) = \lambda^- + O(\delta_p)$, $x^{s+}(\delta_p) = x^+ + O(\delta_p)$, $x^{s-}(\delta_p) = x^- + O(\delta_p)$, $\lambda^{s+}(\delta_p) = \lambda^+ + O(\delta_p)$ and $\lambda^{s-}(\delta_p) = \lambda^- + O(\delta_p)$, with

$$\lambda^+ = 1 + \frac{1}{2}\left[\begin{array}{c} \frac{1+r^n}{1-r^n} + \frac{\pi_1 \frac{\theta}{1+r^n}k^n}{1+r^n} + \frac{\frac{\theta}{1+r^n}+1}{s_r}k^n - 1 \\ + \sqrt{\left[\frac{1+r^n}{1-r^n} + \frac{\pi_1 \frac{\theta}{1+r^n}k^n}{1+r^n} + \frac{\frac{\theta}{1+r^n}+1}{s_r}k^n - 1\right]^2 + 4(\pi_1 + \frac{1+r^n-s_k}{s_r})\frac{\theta k^n}{(1+r^n)^2}} \end{array}\right],$$

$$\lambda^- = 1 + \frac{1}{2}\left[\begin{array}{c} \frac{1+r^n}{1-r^n} + \frac{\pi_1 \frac{\theta}{1+r^n}k^n}{1+r^n} + \frac{\frac{\theta}{1+r^n}+1}{s_r}k^n - 1 \\ - \sqrt{\left[\frac{1+r^n}{1-r^n} + \frac{\pi_1 \frac{\theta}{1+r^n}k^n}{1+r^n} + \frac{\frac{\theta}{1+r^n}+1}{s_r}k^n - 1\right]^2 + 4(\pi_1 + \frac{1+r^n-s_k}{s_r})\frac{\theta k^n}{(1+r^n)^2}} \end{array}\right],$$

where

$$x^+ = \frac{\lambda^+ - 1}{\frac{\theta k^n}{(1+r^n)^2}} > 0, \quad x^- = \frac{\lambda^- - 1}{\frac{\theta k^n}{(1+r^n)^2}} < 0.$$

Because the dynamic system we consider is twice continuously differentiable in the regions $k < k^0$ and $k > k^0$, the following results are true:

- there exists $\overline{\delta_p} > 0$ such that for all $\delta_p < \overline{\delta_p}$, there exist continuously differentiable functions $q_{\delta_p}^{s-} : [k^0, k^s] \times [0, M_b(\delta_p)\delta_p] \to [1,+\infty[$ and $q^{n+} : [k^n, k^0] \times [0, M_b(\delta_p)\delta_p] \to [1,+\infty[$ such that the stable manifold of $(k^s(\delta_p),1,0)$ is parameterized by $(k,b,q_{\delta_p}^{s-}(k,b))$ in the region $(k,b) \in [k^0, k^s(\delta_p)] \times [0, M_b(\delta_p)\delta_p]$, and the unstable manifold of $(k^n,1,0)$ is parameterized by $(k,b,q_{\delta_p}^{n+}(k,b))$ in the region $(k,b) \in [k^n,k^0] \times [0, M_b(\delta_p)\delta_p]$.
- there exists $\overline{\delta_p} > \overline{\overline{\delta_p}} > 0$ and $m^n > 0$ such that for all $\delta_p < \overline{\overline{\delta_p}}$, $(k,b) \in [k^0, k^s(\delta_p)] \times [0, M_b(\delta_p)\delta_p]$, and $(k',b') \in [k^n,k^0] \times [0, M_b(\delta_p)\delta_p]$,

$$|q_{\delta_p}^{s-}(k,b) - 1 - x^{s-}(\delta_p)(k - k^s(\delta_p))| < m^n M_b(\delta_p)\delta_p$$

and

$$|q^{n+}_{\delta_p}(k',b') - 1 - x^{n+}(\delta_p)(k-k^n)| < m^n M_b(\delta_p)\,\delta_p.$$

Using the fact that $k^s(\delta_p) = k^n + O(\delta_p)$, $x^{s-}(\delta_p) = x^- + O(\delta_p)$, $x^{n+}(\delta_p) = x^+ + O(\delta_p)$, $\lambda^{s-}(\delta_p) = \lambda^- + O(\delta_p)$ and $\lambda^{n+}(\delta_p) = \lambda^+ + O(\delta_p)$, we see that there exists $\overline{\overline{\delta_p}} > \overline{\overline{\overline{\delta_p}}} > 0$ and $m^s > 0$ such that for all $\delta_p < \overline{\overline{\overline{\delta_p}}}$, $(k,b) \in [k^0, k^s(\delta_p)] \times [0, M_b(\delta_p)\,\delta_p]$, and $(k',b') \in [k^n, k^0] \times [0, M_b(\delta_p)\,\delta_p]$,

$$|q^{s-}_{\delta_p}(k,b) - 1 - x^-(k - k^s(\delta_p))| < m^s M_b(\delta_p)\,\delta_p$$

and

$$|q^{n+}_{\delta_p}(k',b') - 1 - x^+(k-k^n)| < m^s M_b(\delta_p)\,\delta_p$$

Let $M = \max\{m^n, m^s\}$. The $q = 1 + x^+(k-k^n)$ plane intersects the $k = k^0$ plane along the line $k = k^0, q = 1 + x^+(k^0 - k^n)$. Similarly the $q = 1 + x^-(k - k^s(\delta_p))$ plane intersects the $k = k^0$ plane along the line $k = k^0, q = 1 + x^-(k^0 - k^s(\delta_p))$. Therefore, if $\delta_p < \overline{\overline{\overline{\delta_p}}}$, we have that

$$1 + x^-(k^0(\delta_p) - k^s(\delta_p)) + MM_b(\delta_p)\,\delta_p > 1 + x^+(k^0(\delta_p) - k^n) - \delta_p - MM_b(\delta_p)\,\delta_p \quad (9)$$

is a necessary condition for the intersection of the unstable manifold of the normal steady state with the vertical $k = k_0$ plane to lie below the intersection of stable manifold of the speculative steady state with the vertical $k = k_0$ plane in the region $b \in [0, M_b(\delta_p)]$.

Similarly, a sufficient condition is

$$1 + x^-(k^0(\delta_p) - k^s(\delta_p)) - MM_b(\delta_p)\,\delta_p > 1 + x^+(k^0(\delta_p) - k^n) - \delta_p + MM_b(\delta_p)\,\delta_p. \quad (10)$$

We can rewrite (9) and (10) as

$$\frac{\lambda^+ - 1 + G}{1 - \lambda^- - G} < \frac{k^s - k^0}{k^0 - k^n} + \frac{2MM_b(\delta_p)\,\delta_p}{(1 - \lambda^- - G)(k^0 - k^n)}$$

and

$$\frac{\lambda^+ - 1 + G}{1 - \lambda^- - G} < \frac{k^s - k^0}{k^0 - k^n} - \frac{2MM_b(\delta_p)\,\delta_p}{(1 - \lambda^- - G)(k^0 - k^n)},$$

where

$$G = \frac{\frac{\Delta}{s_r}\frac{k^n}{(1+r^n)^2}}{r^n + \frac{1+r^n}{s_r}k^n}\theta$$

Using the fact that $k^0 - k^n > \varepsilon\delta_p$, we see that a necessary condition for a transition when $\delta_p < \overline{\overline{\overline{\delta_p}}}$ is

$$\frac{\lambda^+ - 1 + G}{1 - \lambda^- - G} < \frac{k^s - k^0}{k^0 - k^n} + \frac{2MM_b\left(\delta_p\right)}{\varepsilon\left(1 - \lambda^- - G\right)}$$

and that a sufficient condition for a transition is

$$\frac{\lambda^+ - 1 + G}{1 - \lambda^- - G} < \frac{k^s - k^0}{k^0 - k^n} - \frac{2MM_b\left(\delta_p\right)}{\varepsilon\left(1 - \lambda^- - G\right)}.$$

Notice that $\left(2MM_b\left(\delta_p\right)\right)/\varepsilon \to 0$ when $\delta_p \to 0$.This shows that if

$$\frac{\lambda^+ - 1 + G}{1 - \lambda^- - G} < \frac{k^s - k^0}{k^0 - k^n}$$

then there exists $m_b(\delta_p) > 0$ such that a transition is possible for δ_p small enough and $0 < b_0 < m_b(\delta_p)\delta_p$, and that if

$$\frac{\lambda^+ - 1 + G}{1 - \lambda^- - G} > \frac{k^s - k^0}{k^0 - k^n}$$

then for δ_p small enough, for every $\widetilde{m_b} > 0$, there exists $0 < b_0 < \widetilde{m_b}\delta_p$ such that no transition is possible with initial bubble b_0.

This condition holds if and only if

$$\Lambda = \frac{\lambda^+ - 1 + G}{1 - \lambda^- - G} < \frac{k^n - k^0}{k^0 - k^s},$$

which is exactly the same condition as without bubbles. The discussion of the possibility of a transition is therefore entirely similar.

Proof of proposition 7: The proof follows directly from the fact that in the neighborhood of the speculative steady state, the interest rate is smaller than the growth rate of the economy.