Online Appendix for "Surplus Maximization and Optimality"

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Rogerson (1980) proves that expected surplus is *ex ante* Pareto Consistent—*ex ante* expected surplus rises whenever all consumers prefer the change—if and only if the indirect utility function of each consumer is additively separable in income any variables that are random (his Theorem 2). As mentioned in the text, the two main differences between his Theorem 2 and Theorem 1 in Schlee (forthcoming) is that I consider *interim* optimality, and I restrict prices to satisfy a single-crossing property. Here I extend my Theorem 1 to *ex ante* optimality. In this Online Appendix I confine myself to pure preference uncertainty, and to two important polar cases, a common-values model in which consumer demands are comonotonic, and an independent private-values model.[1] The common-values model might be apt when demand is unknown because of uncertainty about the quality of a newly introduced product; the private-values model might be apt when the uncertainty is not about a good's quality, but how well the good matches each consumer's tastes.

Consumer i's *ex ante* preference over policies are represented by

$$\alpha \mapsto \int V(\widetilde{p}(\alpha,\omega), m_i, \tau_i(\omega))dF(\omega),$$

and i's *ex ante* expected surplus is $\int_\Omega \left[\int_{\widetilde{p}(\alpha,\omega)}^{\bar{p}} D(p, m_i, \tau_i(\omega))dp + m_i\right] dF(\omega)$, where now $\tau_i : \Omega \to \Theta$. A policy $\alpha^* \in A$ is *ex ante Pareto Optimal* if there is no $\alpha' \in A$ such that every consumer $i \in \{1, ..., I\}$ *ex ante* weakly prefers α^* to α', and the preference is strict for at least one i.

In both results which follow I impose all the assumptions of Theorem 1 in Schlee (forthcoming) on *interim* Pareto Optimality. Recall from the discussion of my Lemma 1 that a crucial step in the argument for *interim* optimality is to show that the marginal utility of money is higher in states of the world with high prices than it is in states with low prices. From an *interim* perspective, after types are revealed, the condition follows from the single crossing property and the assumption that V_m is increasing in p for each type. From an *ex ante* perspective, a consumer is uncertain of its type as well as the price. The additional assumptions I impose ensure that the marginal utility of money is still higher in states of the world in which price is high.

In the next Proposition, demand uncertainty is common values: consumer types are comonotonic (implying that they are positively correlated), and demand is increasing in the types (Assumptions (*ii*) and (*iii*)). I also impose the assumption from my Theorem 2 that equilibrium price and

[1] But see my remark following Proposition OA1 on extending the Propositions to pure income uncertainty.

equilibrium aggregate consumption are comonotonic (Assumption (i)). The three assumptions together imply that each consumer's type and the equilibrium price are comonotonic. Finally the indirect utility function satisfies increasing differences in income and both the price and the type. Together these assumptions ensure that each consumer's marginal utility of money is higher in states of the world with high prices.

Proposition OA1 (Ex ante Optimality: Common Values)**.** *Let all the hypotheses of Theorem 1 in Schlee (forthcoming) hold. In addition suppose that*

(i) aggregate demand $\mathcal{D}(\cdot, \omega)$ is strictly decreasing in price on $[\underline{p}, \overline{p})$ and equilibrium aggregate consumption and equilibrium price are comonotonic;

(ii) consumer preference types are pairwise comonotonic: for every i, j in $\{1, ..., I\}$, and every ω, ω' in Ω, $[\tau_i(\omega) - \tau_i(\omega')] \cdot [\tau_j(\omega) - \tau_j(\omega')] \geq 0$;

and for each consumer $i \in \{1, ..., I\}$,

(iii) at each price $p \in P$, consumer demand $D(p, m_i, \cdot)$ is increasing on Θ; and

(iv) $V_m(p, m_i, \cdot)$ is increasing on Θ for every $p \in P$.

Then the expected surplus maximizing policy α^* on $\mathcal{A}$ is ***ex ante*** *Pareto Optimal.*

Proof: First, I show that Assumptions *(i)-(iii)* of Proposition OA1 imply that for each consumer $i \in \{1, ..., I\}$ and every policy $\alpha \in \mathcal{A}$, the equilibrium price and i's preference type are comonotonic: for every ω, ω' in Ω,

$$[\tau_i(\omega') - \tau_i(\omega)][\widetilde{p}(\alpha, \omega') - \widetilde{p}(\alpha, \omega)] \geq 0. \tag{1}$$

Fix a policy $\alpha \in \mathcal{A}$, a consumer $j \in \{1, ..., I\}$ and a pair of states ω and ω' in Ω. If the prices are equal, (1) holds. Suppose that $\widetilde{p}(\alpha, \omega') - \widetilde{p}(\alpha, \omega) > 0$. Since by *(i)*,

$$[\mathcal{D}(\widetilde{p}(\alpha, \omega'),\ \omega') - \mathcal{D}(\widetilde{p}(\alpha, \omega),\ \omega)] \times [\widetilde{p}(\alpha,\ \omega') - \widetilde{p}(\alpha,\ \omega)] \geq 0$$

it follows that $\mathcal{D}(\widetilde{p}(\alpha, \omega'),\ \omega') \geq \mathcal{D}(\widetilde{p}(\alpha, \omega),\ \omega)$. Since aggregate demand is strictly decreasing in price at each state and the price is higher in state ω', it must be that $\tau^j(\omega') > \tau^j(\omega)$ for at least one consumer $j \in \{1, ..., I\}$ (aggregate demand must be higher in state ω' than in state ω at price $\widetilde{p}(\alpha, \omega)$, and each consumer's demand is increasing in θ). But then by *(ii)*, $\tau_i(\omega') \geq \tau_i(\omega)$ for every $i \in \{1, ..., I\}$ and (1) holds. A similar argument establishes the remaining case, $\widetilde{p}(\alpha, \omega') - \widetilde{p}(\alpha, \omega) < 0$.

It will suffice for a proof of the Proposition to establish a version of Lemma 1: for each consumer $i \leq I_C$, if i's expected surplus at α^* is not less than i's expected surplus at α, then i weakly prefers α^* to α; and if the inequality is strict, then the preference is strict.

Fix a consumer $i \leq I_C$. As in the proof of Lemma 1, let $\Omega_+ = \{\omega \in \Omega \,|\, \widetilde{p}(\alpha, \omega) > \widetilde{p}(\alpha^*, \omega)\}$ and Ω_- be the set of states for which the inequality is reversed. It is almost immediate that if either Ω_- or Ω_+ has zero measure, then the conclusions in the last paragraph hold. For the rest of the proof suppose that Ω_+ and Ω_- have positive measure and let r be a crossing price for α and α^*. Consider any $\omega_+ \in \Omega_+$, $\omega_- \in \Omega_-$, $p^+ \in [\widetilde{p}(\alpha^*, \omega_+)\,,\ \widetilde{p}(\alpha, \omega_+)]$, and $p^- \in [\widetilde{p}(\alpha, \omega_-)\,,\ \widetilde{p}(\alpha^*, \omega_-)]$. I now show that

$$V_m(p^+, m_i, \tau_i(\omega_+)) \geq V_m(p_-, m_i, \tau_i(\omega_-)). \tag{2}$$

First, by the single-crossing property, $p^+ \geq r \geq p^-$. Second, $\widetilde{p}(\alpha, \omega_+) > r > \widetilde{p}(\alpha, \omega_-)$ by the single-crossing property and the definition of Ω_+ and Ω_-. Third, since $\mathcal{D}(\cdot, \omega)$ is strictly decreasing, individual demand is increasing in θ, and since equilibrium aggregate consumption and equilibrium price are comonotonic, $\tau_j(\omega_+) > \tau_j(\omega_-)$ for some consumer j. By (1), $\tau_i(\omega_+) \geq \tau_i(\omega_-)$. Since V_m is increasing in (p, θ), inequality (2) holds. Now set

$$K = \inf\{V_m(p, m_i, \tau_i(\omega)) \,|\, p \in [\widetilde{p}(\alpha^*, \omega)\,,\ \widetilde{p}(\alpha, \omega)],\, \omega \in \Omega_+\}$$

and use Roy's Identity and (2) to find that

$$\begin{aligned}
&\int \Big(V(\widetilde{p}(\alpha^*, \omega), m_i, \tau_i(\omega)) - V(\widetilde{p}(\alpha, \omega), m_i, \tau_i(\omega)) \Big) dF(\omega) \\
&= \int_\Omega \Big(\int_{\widetilde{p}(\alpha,\omega)}^{\widetilde{p}(\alpha^*\omega)} V_p(p, m_i, \tau_i(\omega)) dp \Big) dF(\omega) \\
&= \int_\Omega \Big(\int_{\widetilde{p}(\alpha^*,\omega)}^{\widetilde{p}(\alpha,\omega)} V_m(p, m_i, \tau_i(\omega))(D(p, m_i, \tau_i(\omega))) dp \Big) dF(\omega) \\
&\geq K \int_\Omega \left[\int_{\widetilde{p}(\alpha^*,\omega)}^{\widetilde{p}(\alpha,\omega)} D(p, m_i, \tau_i(\omega))\ dp \right] dF(\omega).
\end{aligned} \tag{3}$$

The inequality implies a version of my Lemma 1 for *ex ante* welfare: if for consumer $i \leq I_C$, α^* has *ex ante* expected surplus no lower than at α, then i weakly prefers α^* to α; and if the inequality is strict, the preference is strict. The conclusion now follows in the same way that my Theorem 1 follows from Lemma 1. ■

Rogerson's Theorem 4 deals with pure preference uncertainty (his case of $\ell = 1$ fits my partial equilibrium framework). He shows that if the marginal utility of money does not depend on the preference type and the demand for good 1 is strictly increasing in the preference type, then the marginal utility of money must depend on the price of good 1—violating the sufficient condition of his Theorem 1. Proposition OA1 shows that aggregate expected surplus can be justified as an *ex ante* guide to policies in the case of pure preference uncertainty if the Assumptions of my Theorems 1 and 2 hold, demand uncertainty is common values, and V_m is increasing in the preference type.

Remark. *Assumption (*iv*)—the marginal utility of money is increasing in the type—deserves comment. I discuss it more fully after Proposition OA2, but for now I note a difficulty in extending Proposition OA1 to the case in which income is* ex ante *uncertain. It is natural to consider the case in which demand is increasing in income—replacing monotonicity of demand in θ with monotonicity of demand in m in Assumption (ii). But then if we replace monotonicity of V_m in θ with monotonicity of V_m in m in Assumption (iv), V must be* convex *in income—which is incompatible with strict risk aversion. Although the conditions are merely sufficient, it is hard to imagine how they could be weakened in an economy in which good 1 is normal, consumer incomes are positively correlated, consumers are strictly risk averse, and equilibrium aggregate consumption and price are positively* *correlated. An important example of* ex ante *income uncertainty as a source of demand uncertainty with common values is business cycle risk. I conclude:* if the main source of demand uncertainty is business cycle fluctuations, and policies change less often than the fluctuations, then aggregate expected surplus has little justification in an economy with risk averse consumers. *This implication agrees with the spirit of Rogerson's Theorem 3, which deals with pure income uncertainty.*[2] *I should point out that the next result can be extended to pure income uncertainty when consumers are risk averse (the analogues of (iii-b) or (iii-c) for income uncertainty are compatible with risk aversion).*

I now consider the polar opposite case, independent private values. I must now also restrict how the shape of the demand varies with the type, and I assume that each consumer is "informationally small" in the sense that each consumer's belief about the equilibrium price distribution does not depend on that consumer's type realization (an assumption that is inconsistent with a finite number of consumers, just as the price-taking assumption is).

Proposition OA2 (Ex Ante Optimality: Independent Private Values)**.** *Let all the hypotheses of Theorem 1 in Schlee (forthcoming) hold, suppose that α^* maximizes aggregate expected surplus on $\mathcal{A}$, and suppose that*

(i) Consumer types are independently distributed; and each consumer i's belief about the equilibrium price distribution does not depend on i's type realization, $\tau_i(\omega)$;

(ii) $D(p, m_i, \cdot)$ is increasing on Θ for every $p \in P$;

(iii) At least ***one*** *of the following holds for every $i \le I_C$ and $\alpha \in \mathcal{A}$.*

(iii-a) $V_m(p, m_i, \cdot)$ is increasing on Θ; the demand D satisfies increasing differences in p and θ on $P \times \Theta$; and $\int_\Omega \widetilde{p}(\alpha,\omega)dF(\omega) \ge \int_\Omega \widetilde{p}(\alpha^*,\omega)dF(\omega)$;

(iii-b) $V_m(p, m_i, \cdot)$ is decreasing on Θ; the demand D satisfies increasing differences in p and $-\theta$ on $P \times \Theta$; and $\int_\Omega \widetilde{p}(\alpha,\omega)dF(\omega) \le \int_\Omega \widetilde{p}(\alpha^*,\omega)dF(\omega)$; or

(iii-c) Demand is additively separable in p and θ and $\int_\Omega \widetilde{p}(\alpha,\omega)dF(\omega) = \int_\Omega \widetilde{p}(\alpha^*,\omega)dF(\omega)$.

Then α^* is ***ex ante*** *Pareto Optimal.*

Proof: Again I establish an extension of Lemma 1 to this case of *ex ante* welfare: for each consumer $i \le I_C$, if i's expected surplus at α^* is not less than i's expected surplus at α, then i weakly prefers α^* to α; and if the inequality is strict, then the preference is strict.

Fix a consumer $i \le I_C$. Again, it is almost immediate that if either one of the sets Ω_- or Ω_+ from the proof of Lemma 1 has zero measure, then the conclusions in the last paragraph hold. For the rest of the proof suppose that Ω_+ and Ω_- have positive measure and let r be a crossing price for α and α^*. Suppose that *(i)* and *(ii)* hold and consider the inequality in equation (A2) in Schlee

[2] Rogerson's Theorem 3 shows that $V_{pm} = 0$ and $V_{mm} = 0$—his sufficient and necessary conditions for *ex ante* expected consumer's surplus to represent *ex ante* preferences over *all* policies which affect the price of good 1 and income—implies that $D_m = 0$ (so if $V_{pm} = 0 = V_{mm}$ for all consumers, then income changes can't be a source of demand changes).

(forthcoming). By *(i)* $F(\omega)$ can replace $F(\omega \,|\, i, \theta)$ in that inequality. Multiply the inequality by $\lambda_i(\theta)$—the probability that consumer i is type θ—and sum over types to find that

$$\sum_{\theta \in \Theta} \int \Big(V(\widetilde{p}(\alpha^*, \omega), m_i, \theta) - V(\widetilde{p}(\alpha, \omega), m_i, \theta) \Big) dF(\omega) \lambda_i(\theta)$$
$$\geq \sum_{\theta \in \Theta} \left[V_m(r, m_i, \theta) \times \int_\Omega \Big(\int_{\widetilde{p}(\alpha^*, \omega)}^{\widetilde{p}(\alpha, \omega)} D(p, m_i, \theta) \ dp \Big) dF(\omega) \right] \lambda_i(\theta) \tag{4}$$

The left side of (4) is the difference in i's *ex ante* expected utility between α^* and α. By Assumption *(iii-a)*, V_m is increasing in θ. The other term in brackets is the change in expected surplus for a type-θ consumer. If that term is increasing in θ, then the right side of (4) would be at least as large as

$$\sum_{\theta \in \Theta} \left[V_m(r, m_i, \theta) \lambda_i(\theta) \right] \times \sum_{\theta \in \Theta} \int_\Omega \Big(\int_{\widetilde{p}(\alpha^*, \omega)}^{\widetilde{p}(\alpha, \omega)} D(p, m_i, \theta) \ dp \Big) dF(\omega) \lambda_i(\theta).$$

which equals

$$\sum_{\theta \in \Theta} \left[V_m(r, m_i, \theta) \lambda_i(\theta) \right] \times \sum_{\theta \in \Theta} \big(E[S(\alpha^*, \omega, m_i, \theta)] - E[S(\alpha, \omega, m_i, \theta)] \big) \lambda_i(\theta),$$

from which the conclusion follows. To confirm that the change in expected surplus is indeed increasing in θ under *(ii)* and *(iii-a)*, let θ'' and θ' be any points in Θ with $\theta'' > \theta'$ and consider the difference in expected surplus

$$\Delta = \int_\Omega \Big(\int_{\widetilde{p}(\alpha^*, \omega)}^{\widetilde{p}(\alpha, \omega)} [D(p, m_i, \theta'') - D(p, m_i, \theta')] \ dp \Big) dF(\omega). \tag{5}$$

It suffices to show that $\Delta \geq 0$. If r be a crossing price for α^* and α, then for any $p^- \leq r \leq p^+$ we have

$$D(p^-, m_i, \theta'') - D(p^-, m_i, \theta') \leq D(r, m_i, \theta'') - D(r, m_i, \theta') \leq D(p^+, m_i, \theta'') - D(p^+, m_i, \theta'),$$

which implies that

$$\Delta \geq [D(r, m_i, \theta'') - D(r, m_i, \theta')] \times \int_\Omega \widetilde{p}(\alpha, \omega) - \widetilde{p}(\alpha^*, \omega) dF(\omega).$$

By *(ii)* the bracketed term is nonnegative and by *(iii-a)* expected price difference is nonnegative, so $\Delta \geq 0$ and the change in expected surplus is increasing in the preference type. This completes the proof for the case of *(iii-a)*. The proof in the case of *(iii-b)* is similar. For *(iii-c)*, note that if demand is additively separable in p and θ, then Δ in (5) is 0 if the mean prices are the same for policies α and α^*. ■

Except for the important special case of *(iii-c)*, both Propositions OA1 and OA2 assume monotonicity of V_m in the preference type θ (increasing or decreasing). To pin down each possibility from preferences, suppose that the indirect utility function is C^2 and that the demand is C^1. Write the direct utility function as $u(x, y, \theta)$, where x is consumption of good 1 and y is spending on all other goods, and assume that it is strongly quasiconcave. Routine calculation reveals that (subscripts denote partial derivative)

$$V_{\theta m} = D_m D_\theta H + u_{y\theta} \tag{6}$$

where $H = -u_{xx} + 2u_{xy}p - u_{yy}p^2 > 0$ and p is the price of good 1. So if demand is increasing in both income and the preference type and $u_{y\theta} \geq 0$, then V_m is increasing in θ. These conditions are satisfied, for example, if u is additively separable and strongly concave, and θ enters only the good-1 component function—$u(x, y, \theta) = f(x, \theta) + g(y)$. On the other hand, if demand does not depend on income (quasilinear preferences) but the consumer is risk averse over money lotteries, then V_m is decreasing in θ (since $u_{y\theta} \leq 0$ in that case).

The case of high consumer risk aversion is of special interest. In particular suppose that the indirect utility is of the form $V(p, m, \theta) = \phi(\widetilde{V}(p, m, \theta))$, where ϕ is strictly increasing and strictly concave with $\phi' > 0$. Differentiate with respect to m and θ and rearrange to find that

$$V_{m\theta} = \phi' \left[\frac{\phi''}{\phi'} \widetilde{V}_m \widetilde{V}_\theta - \widetilde{V}_{m\theta} \right].$$

The equation immediately implies that if Arrow-Pratt risk aversion is uniformly high enough and $V_\theta < 0$, then V_m is *increasing* in θ; if risk aversion is high enough and $V_\theta > 0$, then V_m is *decreasing* in θ. Since demand is increasing in the preference type, the conclusion can be rephrased: if the consumer is risk averse enough, then V_m is increasing in θ if demand and utility move in *opposite* directions in response to changes in θ; and V_m is decreasing in θ if demand and utility move in *same* direction in response to changes in θ.

Examples abound for each possibility. Suppose the uncertain state of the world affects the weather. Demand for heating and cooling is highest when the weather is extreme; and since most prefer moderate to extreme weather, a consumer's utility and demand for climate control and move in opposite directions. But demand for many outdoor recreation activities is highest when weather is moderate. If the uncertainty is about the quality of a *substitute* for the good, then utility and the demand for the good move in opposition directions—for example car gas mileage and the demand for gasoline; and possibly health care quality and the demand for hospital days (Feldstein (1977)). If the uncertainty is about the quality of the good itself—with 'higher' quality defined as something everyone prefers—then utility is by construction increasing in quality, but the demand for the good could be either increasing or decreasing in quality: if the durability of a firm's product increases, then its demand might increase if it has competitors, but might decrease if the firm is a (protected) monopolist.

Propositions OA1 and OA2 contain a long list of assumptions, so it might not be obvious how to apply them. Suppose first that the demand uncertainty is common values, as in Proposition OA1. If one adds the assumption that V_m is increasing in the preference type θ, then the results in Lewis and Sappington (1988), Deneckere, Marvel, and Peck (1997), and Rey and Tirole (1986) that were shown to be *interim* Pareto Optimal in Section 4 are also *ex ante* optimal for common-values demand uncertainty.

For private values, notice that condition *(iii-c)* includes the familiar case of linear demand with intercept uncertainty: $D(p, m, \theta) = g(\theta, m) - p$. Rey and Tirole (1986) impose this form in their main theorem on vertical restraints and welfare (Proposition 7). If retailers are risk neutral or infinitely risk averse, then the expected retail price is the same across the three policies for pure demand uncertainty and condition *(iii-c)* holds, so by Proposition 4, RPM is *ex ante* Pareto Optimal for private-values demand uncertainty. If retailers have intermediate risk aversion, then the expected retail price for exclusive territories can be higher than for RPM and competition, so that conclusion extends if we add the assumption that V_m is increasing in θ. Deneckere, Marvel, and Peck's (1997) Theorem A.2 imposes linear demand. They prove that the mean price is the same for flexible pricing and minimum RPM, so again condition *(iii-c)* holds. Deneckere, Marvel, and Peck (1997) also consider nonlinear demand. Their maintained assumptions on the curvature on the inverse demand are in general consistent with D satisfying increasing differences in either p and θ or p and *minus* θ but in some important special cases, the second must hold. They also point out that expected price under minimum RPM can be either higher or lower than under flexible prices, but for important classes of demands and costs, the mean price for flexible prices is no less than the mean price under minimum RPM. Thus assumption (*iii-b*) of Proposition 4 fits best for nonlinear demand when the demand uncertainty is iid in their model.

Finally consider Lewis and Sappington (1988) and private values. Although linear demand is their leading example of a functional form which fits their assumptions, Proposition OA2 requires that the mean price from the policy which maximizes aggregate expected surplus be higher or lower than the mean price from every other policy. That part of the sufficient condition fails in Lewis and Sappington (1988): the best hope for extending their results beyond quasilinear utility is for *interim* optimality, or common-values uncertainty for *ex ante* optimality.

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