Web Appendix for *Maturity, Indebtedness and Default Risk* by Satyajit Chatterjee and Burcu Eyigungor.

Appendix A: Proofs of Propositions 1-4

The proofs of these propositions requires some results that are collected in the following preliminary proposition and lemma.

Proposition 0 (Existence, Continuity and Monotonocity of Value Functions): Given any $q(y, b') \geq 0$, there exists a unique, bounded function $W(y, m, b)$ continuous in m that solves the functional equation (11). Furthermore, $X(y, m)$ is strictly increasing and continuous in m; in the region where repayment is feasible, $V(y, m, b)$ is strictly increasing in b and m and continuous in m; and $Z(y, b') = E_{(y',m')|y)}W(y', m', b')$ is strictly increasing in b', provided there is positive probability of repayment for every debt level.

Proof: Let $\mathcal{W}$ be the set of all continuous (in m) functions on $Y \times M \times B$ that take values in the bounded interval $[u(0)/(1-\beta), U/(1-\beta)]$. Equip $\mathcal{W}$ with sup norm (uniform) metric $||\cdot||_\infty$. Then $(\mathcal{W}, ||\cdot||_\infty)$ is a complete metric space.

For $W \in \mathcal{W}$, let $X(y, m; W)$ be the solution to (3). The solution exists because (3) defines a contraction mapping in X with modulus $\beta(1-\xi)$. By standard contraction mapping arguments, $X(y, m)$ is continuous in m because $c = y - \phi(y) + m$ is continuous in m and u is continuous in c.

For $W \in \mathcal{W}$, let $V(y, m, b; W, q)$ be the solution to (4). We index this solution by q because the function $q(y, b')$ appears as a parameter in (4). Here, however, we need to address the fact that V may not be well-defined because there may not be any feasible b' for some (y, m, b) and q. To extend the definition of V over the entire domain, we will assume that the utility from a choice of b' under repayment, denoted $V_{b'}(y, m, b; W, q)$, is given by $u(\max\{0, y+m+[\lambda+(1-\lambda)z]b+q(y, b')[b'-(1-\lambda)b]\}) + \beta E_{(y',m'|y)}W(y', m', b')$. Thus, for an infeasible choice of b', current consumption is set to 0. Then, $V(y, m, b; W, q) = \max_{b' \in B} V_{b'}(y, m, b; W, q)$. Since B is a finite set, $V(y, m, b; W, q)$ exists for all (y, m, b) and q. Also $V_{b'}$ is continuous in m for every b' since $\max\{0, y+m+[\lambda+(1-\lambda)z]b+q(y,b')[b'-(1-\lambda)b]\}$ is continuous in m and u is continuous in c. Therefore, $V(y,m,b;W,q)$ is continuous in m since the maximum of a finite set of continuous functions is also continuous. Furthermore, both $X(y,m;W)$ and $V(y,m,b;W,q) \in [u(0)/(1-\beta), U/(1-\beta)]$ for all y, m and b.

Next, define the operator

$$T(W)(y,m,b;q) = \max\{V(y,m,b;W,q), X(y,-\bar{m};W)\} \tag{A1}$$

on the space of functions $\mathcal{W}$. Then, (i) $T(\mathcal{W})(y,m,b;q) \subset \mathcal{W}$ which is obvious; (ii) If $W \geq \hat{W}$ then $T(W) \geq T(\hat{W})$, which follows because $V(y,m,b,W,q)$ is clearly increasing in W and standard contraction mapping arguments can establish that $X(y,m;W)$ is increasing in W; and (iii) $T(W+k) \leq T(W) + \eta k$, where $\eta = \max\{\beta\xi/[(1-\beta)+\beta\xi], \beta\} < 1$. To see (iii), note that $V(y,m,b;W+k,q) = V(y,m,b;W,q) + \beta k$ and $X(y,m;W+k) = X(y,m;W) + (\beta\xi/[(1-\beta)+\beta\xi])k$. Therefore,

$$\begin{aligned} &T(W+k)(y,m,b;q) = \\ &\max\left\{V(y,m,b;W,q) + \beta k,\ X(y,-\bar{m};W) + \frac{\beta\xi}{[(1-\beta)+\beta\xi}k\right\} \end{aligned}$$

and (iii) follows. Therefore, T is a contraction mapping with modulus η and the existence of a unique solution to (A1) in $\mathcal{W}$, denoted $W_q^*(y,m,b)$, follows from the Contraction Mapping Theorem.

The strict monotonicity of $X(y,m)$ with respect to m follows from the endowment being strictly increasing in m, u being strictly increasing in c, and the fact that m does not affect the probability distribution of (y', m').

For the strict monotonicity of V with respect to m, observe that if $m^0 < m^1$ then every b' that is feasible under (y, m^0, b) is also feasible under (y, m^1, b) and yields strictly higher consumption. In the region where repayment is feasible, there must be at least one b' that is feasible. Then, since m does not affect the probability distribution of (y', m'), strict monotonicity of u implies $V(y, m^0, b) < V(y, m^1, b)$. For strict monotonicity with regard to b, observe that for $b^0 < b^1$ we have $[\lambda + (1-\lambda)z]b^0 + q(y, b')[1-\lambda]b^0 < [\lambda + (1-\lambda)z]b^1 + q(y, b')[1-\lambda]b^1$ for every feasible $b' \in B$ and every $y \in Y$. This follows because $[\lambda + (1-\lambda)z] > 0$ and $q(y, b') \geq 0$. Hence, every b' that is feasible under (y, m, b^0) is also feasible under (y, m, b^1) and affords strictly greater consumption. Again, in the region where repayment is feasible, there must be at least one feasible b'. Therefore, by the strict monotonicity of u, $V(y, m, b^0) < V(y, m, b^1)$.

From the strict monotonicity of V with respect to b, it follows that for $b'^1 > b'^0$, $W(y', m', b'^1) \geq W(y', m', b'^0)$. Hence, $Z(y, b'^1) \geq Z(y, b'^0)$. To show the inequality is strict, we will assume b_I (the smallest $b \in B$) is bounded below as

$$b_I > -[\phi(y_{\max}) + 2\bar{m}]/[\lambda + (1-\lambda)z], \tag{A2}$$

where $y_{\max}$ is the largest $y \in Y$. Then, observe that (A2) implies that $u(y_{\max} + [\lambda + (1-\lambda)z]b + \bar{m}) + \beta Z(y, 0) > u(y_{\max} - \phi(y_{\max}) - \bar{m}) + \beta Z(y, 0)$ for all $b \in B$. Also, observe that $Z(y, 0) > E_{(y', m')|y}[\xi W(y', m', 0) + (1-\xi)X(y', m')]$, since $X(y', m') < W(y', m', 0)$ for all $(y', m') \in Y \times M$. Thus, for every debt level there is a range of m values for which repayment without new borrowing is better than default if y is at its highest value. Therefore, for every $b' \in B$, there is a range of m' for which $V(y_{\max}, m', b') > X(y_{\max}, -\bar{m})$. By the strict monotonicity of V with respect to b, every m' for which $V(m', b'^0, y_{\max}) > X(y_{\max}, -\bar{m})$ it is also true that $V(m', b'^1, y_{\max}) > X(y_{\max}, -\bar{m})$. Thus, there is a range of m values for which $W(y_{\max}, m', b'^1) > W(y_{\max}, m', b'^0)$. Since $F(y, y_{\max}) > 0$ for all y, it follows that $Z(y', b'^1) > Z(y', b'^0)$. ■

Since we extended the domain of the definition of V to infeasible choices, we need to verify that this extension does not result in the sovereign actually choosing infeasible b'. In the following Lemma we establish that if $u(0)$ is set to a sufficiently low number, then it is never optimal to choose infeasible actions.

Lemma 0: If $u(0)+\beta U/(1-\beta) < u(y_{\min}-\phi(y_{\min})-\bar{m})/(1-\beta)$, where $y_{\min}$ is the smallest value in Y, then optimal consumption under repayment, $c(y,m,b)$, is uniformly bounded below by some strictly positive number $\bar{c}$.

Proof: By continuity of u there exists $\bar{c}>0$ such that $u(\bar{c})+\beta U/(1-\beta) < u(y_{\min}-\phi(y_{\min})-\bar{m})/(1-\beta)$. Since the sovereign can choose to consume its endowment each period, and it can always consume at least $y_{\min}-\phi(y_{\min})-\bar{m}$ in every period, its lifetime utility in any period is bounded below by $u(y_{\min}-\phi(y_{\min})-\bar{m})/(1-\beta)$. On the other hand, the highest utility from selecting any action that leads to consumption $\bar{c}$ or less is $u(\bar{c})+\beta U/(1-\beta)$. By assumption the former dominates the latter. Thus it can never be optimal to choose to consume $\bar{c}$ or less. In particular, it can never be optimal to choose an action that leads to 0 consumption. ■

Proof of Proposition 1: Suppose, to get a contradiction, that for some (y,m) we have $d(y,m,b^0;q) < d(y,m,b^1;q)$. Then $d(y,m,b^0;q)=0$ and $d(y,m,b^1;q)=1$. The former implies that $V(y,m,b^0)\geq X(y,-\bar{m})$ and the latter implies $X(y,-\bar{m}) > V(y,m,b^1)$. But these inequalities imply $V(y,m,b^0) > V(y,m,b^1)$, which contradicts Proposition 0. Hence, $d(y,m,b^0)\geq d(y,m,b^1)$.

Proof of Proposition 2: Fix m and y. Denote $a(y,m,b^0)$ by b'^0 and the associated consumption level by c^0. Let $\hat{b}'$ be some other feasible choice greater than b'^0 and let $\hat{c}$ be the associated consumption level. Then, by optimality and the tie-breaking rule that if the sovereign is indifferent between two b's it always chooses the higher one, we have

$$u(c^0)+\beta Z(y,b'^0) > u(\hat{c})+\beta Z(y,\hat{b}'). \tag{A3}$$

Since $Z(y,\hat{b}') > Z(y,b'^0)$ (Proposition 0), (A3) implies $c^0>\hat{c}$. Let $\Delta(b^0)=c^0-\hat{c}>0$. Thus, $\Delta(b^0)$ is the loss in current consumption from choosing $\hat{b}'$ over b'^0 when the beginning-of-period debt is b^0. From the budget constraint we have that $\Delta(b^0)+q(y,b'^0)b'^0-q(y,\hat{b}')\hat{b}' = [1-\lambda](-b^0)[q(y,\hat{b}')-q(y,b'^0)]$. Holding fixed $\hat{b}'$

and ${b'}^0$, let $\Delta(b^1)$ be the value of Δ that solves $\Delta(b^1)+q(y,{b'}^0){b'}^0-q(y,\hat{b}')\hat{b}' = [1-\lambda](-b^1)[q(y,\hat{b}')-q(y,{b'}^0)]$. Then $\Delta(b^1)$ is the change in current consumption from choosing $\hat{b}'$ over ${b'}^0$ when the beginning-of-period debt is b^1. Since, by assumption, $[q(y,\hat{b}') - q(y,{b'}^0)] \geq 0$, $b^1 < b^0$ implies $\Delta(b^1) \geq \Delta(b^0)$. Thus the loss in current consumption from choosing $\hat{b}'$ over ${b'}^0$ is at least as large when the beginning-of-period debt is b^1 compared with b^0. Next, note that since $[\lambda + (1-\lambda)z] > 0$ and $q(y,b') \geq 0$, $b^1 < b^0$ implies $[\lambda + (1-\lambda)z]b^1 + (1-\lambda)q(y,b'^0)b^1 < [\lambda + (1-\lambda)z]b^0 + (1-\lambda)q(y,b'^0)b^0$. Therefore, from the budget constraint it follows that if the beginning-of-period debt is b^1, choosing ${b'}^0$ implies consumption $\tilde{c}$ strictly less than c^0. To complete the proof, observe that the strict concavity of u implies $u(\tilde{c})-u(\tilde{c}-\Delta(b^1)) > u(c^0)-u(c^0-\Delta(b^0)) = u(c^0)-u(\hat{c})$. Therefore, (A3) implies that $u(\tilde{c})+\beta Z(y,{b'}^0) > u(\tilde{c}-\Delta(b^1))+\beta Z(y,\hat{b}')$. Since $\hat{b}'$ is any feasible b' greater than ${b'}^0$, the optimal choice of b' under repayment when beginning-of-period debt is b^1 cannot be greater than ${b'}^0$. Therefore, $a(y,m,b^1) \leq a(y,m,b^0)$.

The proof of Proposition 3 requires that the operator H be continuous. The next three Lemmas prove some results needed to establish this point. Lemma 4 proves the necessary continuity result.

Lemma 1: $W_q^*(y,m,b)$, $V(y,m,b;W_q^*,q)$, $X(y,m;W_q^*)$ and $Z_q^*(y,b')$ are all continuous in q.

Proof: To prove that $W^*(y,m,b;q)$ is continuous in q, it is sufficient to prove that the contraction operator $T(W)(y,m,b;q)$ is continuous in q (see Theorem 4.3.6 in Hutson and Pym (1980), pp. 117-118). In order to establish this, we need to prove only that $V(y,m,b;W,q)$ is continuous in q. Fix (y,m,b) and W. Observe that $V_{b'}(y,m,b;W,q)$ is continuous in q because $\max\{0, y+m+[\lambda + (1-\lambda)z]b + q(y,b')[b' - (1-\lambda)b]\}$ is continuous in q and u is continuous in c. Thus, $V(y,m,b;W,q)$, being the maximum of a finite set of continuous functions, is also continuous in q. Hence $W_q^*(y,m,b)$ is continuous in q. The continuity of $V(y,m,b;W_q^*,q)$ with respect to q follows from the same logic as before: $V_{b'}(y,m,b;W_q^*,q)$ is continuous in q for each $b'$and hence the maximum over b'must also be continuous in q; the continuity of $Z_q^*(y,b')$ with respect to q follows directly from its definition; and the continuity of $X(y,m;W_q^*)$ with respect q follows from noting that the contraction operator defining $X(y,m;W)$ depends on W via the quantity $Z(y,0)$ and that the operator is continuous in $Z(y,0)$. Since $Z_q^*(0,y')$ is continuous in q, it follows from another application of Theorem 4.3.6 of Hutson and Pym that $X(y,m;W_q^*)$ is continuous in q. ■

Lemma 2 establishes that the sovereign can be indifferent between default and repayment at exactly one value of m and it can be indifferent between any two borrowing levels at exactly one value of m. These results are needed for Lemma 3, which establishes almost sure convergence of decision rules with respect to prices q.

Lemma 2: (i) For any given b'^0, there can be at most one value of m for which choosing b'^0 gives the same lifetime utility as defaulting and (ii) for any given $b'^0 < b'^1$ there can be at most one value of m for which choosing the two debt levels gives the same lifetime utility.

Proof: (i) Fix y and b. (i) Suppose that there is an $\hat{m}$ such that $V_{b'^0}(y,\hat{m},b) = X(y,-\bar{m})$. Since the l.h.s is strictly increasing in m, there cannot be any other $m \neq \hat{m}$ for which the same equality holds. (ii) Suppose there is an $\hat{m}$ for which $u(c^0(\hat{m})) + \beta Z(y,b'^0) = u(c^1(\hat{m})) + Z(y,b'^1)$, where $c^0(\hat{m})$ and $c^1(\hat{m})$ are the levels of consumption when b'^0 and b'^1 are chosen, respectively. Since $Z(y,b'^1) > Z(y,b'^0)$ (Proposition 0), it follows that $c^0(\hat{m}) > c^1(\hat{m})$. Suppose, to get a contradiction, there is another $\tilde{m} > \hat{m}$ such that $u(c^0(\tilde{m})) + \beta Z(y,b'^0) = u(c^1(\tilde{m})) + Z(y,b'^1)$. Then, $u(c^0(\hat{m})) - u(c^0(\tilde{m})) = u(c^1(\hat{m})) - u(c^1(\tilde{m}))$ and (from the budget constraint) $c^i(\tilde{m}) = c^i(\hat{m}) + [\tilde{m} - \hat{m}]$ for $i = 0,1$. Thus, we must have $u(c^0(\hat{m})) - u(c^0(\hat{m}) + [\tilde{m} - \hat{m}]) = u(c^1(\hat{m})) - u(c^1(\hat{m}) + [\tilde{m} - \hat{m}])$. But, since $c^0(\hat{m}) > c^1(\hat{m})$, the preceding equality violates the strict concavity of u. Hence there can only be at most one m for which $u(c^0(m)) + \beta Z(y,b'^0) = u(c^1(m)) + \beta Z(y,b'^1)$. ■

Corollary to Lemma 2: The thresholds $\{-\bar{m} < m^{K-1} < m^{K-2} < \ldots < m^1 <$

$\bar{m}\}$ and the corresponding debt choices $\{b'^K < b'^{K-1} < \ldots < b'^1\}$ are unique.

Proof: Suppose to the contrary that there are two distinct pairs $\{m^{k-1}, b'^k\}$ and $\{\hat{m}^{k-1}, \hat{b}'^k\}$. Without loss of generality, assume that these lists deviate from each other for $k = 1$. That is, according to the first list the sovereign is indifferent between choosing 0 and b'^1 at m^1 and according to the second list it is indifferent between choosing 0 and $\hat{b}'^1$ at $\hat{m}^1$. Suppose also that $\hat{b}'^1 > b'^1$. If $\hat{m}^1 \neq m^1$, then there are two distinct values of m for which $\hat{b}'^1$ and b'^1 give the same utility. This contradicts Lemma 2(ii). And if $\hat{m}^1 = m^1$ then b'^1 is inconsistent with our assumption that, all else the same, among two b' choices that give the same utility, the sovereign chooses the larger one. ■

Lemma 3: Let $q^n(y, b')$ be a sequence converging to $\bar{q}(y, b')$. Let $d(y, m, b; q^n)$, $a(y, m, b; q^n)$ and $d(y, m, b; \bar{q})$, $a(y, m, b; \bar{q})$ be the corresponding optimal decision rules. Then, $d(y, m, b; q^n)$ converges pointwise to $d(y, m, b; \bar{q})$ and $a(y, m, b; q^n)$ converges pointwise to $a(y, m, b; \bar{q})$ except, possibly, at a finite number of points.

Proof: (Convergence of $a(y, m, b; q^n)$). Let $q^n \to \bar{q}$. Fix y and b. For a given m, let $b'^0 = a(y, m, b; \bar{q})$. Let $V_{b'}(y, m, b; W^*_{\bar{q}}, \bar{q})$ denote the lifetime utility if the sovereign chooses to borrow b' in the current period but follows the optimal plan in all future periods. Two cases are possible: (i) $V(y, m, b; W^*_{\bar{q}}, \bar{q}) > V_{b'}(y, m, b; W^*_{\bar{q}}, \bar{q})$ for all $b' \neq b'^0$ and (ii) $V(y, m, b; W^*_{\bar{q}}, \bar{q}) = V_{b'}(y, m, b; W^*_{\bar{q}}, \bar{q})$ for some $b' \neq b'^0$. Consider case (i). Let $V(y, m, b; W^*_{\bar{q}}, \bar{q}) - V_{b'}(y, m, b; W^*_{\bar{q}}, \bar{q}) = \Delta$. Since $V(y, m, b; W^*_q, q)$ is continuous in q there exists N_1 such that for all $n \geq N_1$ $V(y, m, b; W^*_{q^n}, q^n) > V(y, m, b; W^*_{\bar{q}}, \bar{q}) - \Delta/2$. Next, note that

$$V_{b'}(y, m, b; W^*_{q^n}, q^n) = u(y + m + [\lambda + (1-\lambda)z]b - q^n(y, b')[b' - (1-\lambda)b]) + \beta Z^*_{q^n}(y, b').$$

Since $Z^*_q(y, b')$ is continuous in q it follows that there exists N_2 such that for all $n \geq N_2$ $V_{b'}(y, m, b; W^*_{q^n}, q^n) < V_{b'}(y, m, b; W^*_{\bar{q}}, \bar{q}) + \Delta/2$. Then $V(y, m, b; W^*_{q^n}, q^n) - V_{b'}(y, m, b; W^*_{q^n}, q^n) > V(y, m, b; W^*_{\bar{q}}, \bar{q}) - \Delta/2 - V_{b'}(y, m, b; W^*_{\bar{q}}, \bar{q}) - \Delta/2 = 0$ for all $n \geq \max\{N_1, N_2\}$. Hence $a(y, m, b; q^n) = b'^0$ for all $n > \max\{N_1, N_2\}$.

Now consider case (ii). In this case, convergence may fail because $a(y,m,b;q^n)$ may converge to b' rather than b'^0. However, by Lemma 1 there can be only a finite number of m values for which case (ii) can hold. Therefore, $a(y,m,b;q^n)$ converges pointwise to $a(y,m,b;\bar{q})$ except, possibly, for a finite number of m.

(Convergence of $d(y,m,b;q^n)$). Let $q^n \to \bar{q}$. Fix y and b. Again, two cases are possible. (i) $X(y,-\bar{m};W^*_{\bar{q}}) \neq V(y,m,b;W^*_{\bar{q}},\bar{q})$ and (ii)$X(y,-\bar{m};W^*_{\bar{q}}) = V(y,m,b;W^*_{\bar{q}},\bar{q})$. Consider case (i). For concreteness, suppose $X(y,-\bar{m};W^*_{\bar{q}}) - V(y,m,b;W^*_{\bar{q}},\bar{q}) = \Delta > 0$. By continuity of $V(y,m,b;W^*_q,q)$ and $X(y,m;W^*_q)$ with respect to q there exists N_1 such that for all $n \geq N_1$, $V(y,m,b;W^*_{q^n},q^n) < V(y,m,b;W^*_{\bar{q}},\bar{q}) + \Delta/2$. And, by the continuity of $X(y,-\bar{m};W^*_q)$ with respect to q, there exists N_2, such that for $n \geq N_2$, $X(y,-\bar{m};W^*_{q^n}) > X(y,-\bar{m};W^*_{\bar{q}}) - \Delta/2$. Then, for all $n \geq \max\{N_1,N_2\}$, $X(y,-\bar{m};W^*_{q^n}) - V(y,m,b;W^*_{q^n},q^n) > X(y,-\bar{m};W^*_{\bar{q}}) - V(y,m,b;W^*_{\bar{q}},\bar{q}) - \Delta = 0$. Hence $d(y,m,b;q^n) = d(y,m,b;\bar{q}) = 1$ for all $n \geq \max\{N_1,N_2\}$. If $\Delta < 0$, we can use a similar argument to show that there exists some N such that for all $n \geq N$, $V(y,m,b;W^*_{q^n},q^n) > X(y,-\bar{m};W^*_{q^n})$. Hence, for all such n, $d(y,m,b;q^n) = d(y,m,b;\bar{q}) = 0$. Now consider case (ii). Again, convergence may fail in this case because $d(y,m,b;q^n)$ may converge to 1 or 0 while $d(y,m,b;\bar{q})$ is 0 or 1. However, by Lemma 2(i), there can only be one value of m for which this can happen. Therefore, $d(y,m,b;q^n)$ converges pointwise to $d(y,m,b;\bar{q})$ except, possibly, for one value of m. ■

Lemma 4: H is continuous in $q(y,b')$.

Proof: Let $\{q^n\} \subset Q$ converge to $\bar{q} \in Q$. Let $\{d(y,m,b;q^n),\ a(y,m,b;q^n)\}$ and $\{d(y,m,b;\bar{q}),\ a(y,m,b;\bar{q})\}$ be the corresponding default and debt decision rules. Then

$$(Hq^n)(y,b') = \\ E_{(y'\,m')|y}\left[[1-d(y',m',b';q^n)]\frac{\lambda + [1-\lambda][z + q^n(y',a(y',m',b';q^n))]}{1+r_f}\right].$$

Or,

$$(Hq^n)(y,b') = \sum_{y'} F(y,y') \times$$
$$\left[\frac{\int_M [1-d(y',m',b';q^n)]\,[\lambda+[1-\lambda][z+q^n(y',a(y',m',b';q^n))]]\,dG(m')}{1+r_f}\right].$$

Fix y' and b'. By Lemma 3, $\lim_n [1-d(y',m',b';q^n)] = [1-d(y',m',b';\bar{q})]$ for all but a finite number of points (possibly) of m'. Since individual points of m have probability zero, $[1-d(y',m',b';q^n)]$ converges almost surely to $[1-d(y',m',b';\bar{q})]$ with respect to the measure induced by $G(m)$. Also, by Lemma 3, $\lim_n a(y',m',b';q^n) = a(y',m',b';\bar{q})$ for all but a finite number of points (possibly) of m'. If convergence holds then, since $a(\cdot;q^n)$ takes values in a finite set B, there must exist N such that for all $n > N$ $a(y',m',b';q^n) = a(y',m',b';\bar{q})$. Therefore, for $n > N$, $q^n(y',a(y',m',b';q^n)) = q^n(y',a(y',m',b';\bar{q}))$. Since $q^n \to \bar{q}$, it follows that $\lim_n q^n(y',a(y',m',b';\bar{q})) = \bar{q}(y',a(y',m',b';\bar{q}))$. Thus, viewed as a function of m', $q^n(y',a(y',m',b';q^n))$ converges almost surely to $\bar{q}(y',a(y',m',b';\bar{q}))$. Therefore, we have that

$$\lim_n [1-d(y',m',b';q^n)]\left[\lambda+[1-\lambda][z+q^n(y',a(y',m',b';q^n))]\right] =$$
$$[1-d(y',m',b';\bar{q})]\left[\lambda+[1-\lambda][z+\bar{q}(y',a(y',m',b';\bar{q}))]\right]$$

except, possibly, at a finite number of points.

Now observe that each function in the sequence is non-negative and bounded above by $\lambda+(1-\lambda)[z+\bar{q}]$. Thus, by the Lebesgue Dominated Convergence Theorem, we have that

$$\lim_n \int_M [1-d(y',m',b';q^n)]\left[\lambda+[1-\lambda][z+q^n(y',a(y',m',b';q^n))]\right] dG(m') =$$
$$\int_M [1-d(y',m',b';\bar{q})]\left[\lambda+[1-\lambda][z+\bar{q}(y',a(y',m',b';\bar{q}))]\right] dG(m').$$

Putting these results together, we get

$$\begin{aligned} & \lim_n (Hq^n)(y,b') = \sum_{y'} F(y',y) \times \\ & \left[\frac{\int_M [1-d(y',m',b';q^n)]\,[\lambda + [1-\lambda][z + q^n(y', a(y',m',b';q^n))]]\, dG(m')}{1+r_f} \right]. \\ = \quad & (H\bar{q})(y,b'). \end{aligned}$$

Thus H is continuous in $q(y,b')$. ■

Proof of Proposition 3: Let $\bar{q} = [\lambda + [1-\lambda]z]/[\lambda + r_f]$. Then $\bar{q}$ is the present discounted value of a bond with coupon payment z and probability of maturity λ on which there is no risk of default. Let S be the set of all nonnegative functions $q(y,b')$ defined on $Y \times B$ and let $Q \subset S$ be the subset of functions that are increasing in b' and bounded above by $\bar{q}$. Define the $(Hq)(y,b') : Q \to S$ as

$$E_{(y'\,m')|y} \left[[1-d(y',m',b';q)] \frac{\lambda + [1-\lambda][z + q(y', a(y',m',b';q))]}{1+r_f} \right],$$

where $d(y,m,b;q)$ and $a(y,m,b;q)$ are the default and debt decision rule, given q. Then H has the following properties:

(a) $(Hq)(y,b') \in Q$. (i) Nonnegativity is obvious. (ii) To show that $(Hq)(y,b') \leq \bar{q}$, observe that $\bar{q}$ satisfies the equation $\bar{q} = [\lambda + (1-\lambda)[z+\bar{q}]]/(1+r_f)$. Then, since $1 - d(y',m',b') \leq 1$ and $q(y', a(y',m',b';q)) \leq \bar{q}$ for every (y',m',b'), it follows that

$$\left[[1-d(y',m',b';q)] \frac{\lambda + [1-\lambda][z + q(y', a(y',m',b';q))]}{1+r_f} \right] \leq \bar{q} \text{ for every } y',m',b'.$$

Hence $(Hq)(y,b') \leq \bar{q}$. (iii) Finally, we will show that $(Hq)(y,b')$ is increasing in b'. Fix y' and m'. Since $q(y,b') \in Q$, $q(y,b')$ is increasing in b' and, by Proposition 2, $a(y',m',b';q)$ is increasing in b'. Thus, $q(y', a(y',m',b';q))$ is increasing in b'. And, by Proposition 2 again, $[1-d(y',m',b';q)]$ is increasing in b'. Hence $(Hq)(y,b')$ is increasing in b'. (b) By Lemma 4, $(Hq)(y,b')$ is continuous in $q(y,b')$.

To complete the proof of the proposition, note that Q is a compact and convex set. Since (Hq) is continuous in $q(y, b')$, by Brouwer's Fixed Point Theorem there exists $q^* \in Q$ such that $q^*(y, b') = (Hq^*)(y, b')$. This establishes the existence of an equilibrium price function that is increasing in b'.

Proposition 4: $a(y, m, b)$ is increasing in m and $d(y, m, b)$ is decreasing in m.

Proof: To prove $a(y, m, b)$ is increasing in m, fix y and b and let $m^1 > m^0$. Assume also that repayment is feasible for both m^1 and m^0. Denote $a(y, m^1, b)$ by b'^1 and the associated consumption by c^1. Let $\hat{b}' > b'^1$ be some other feasible choice of b' greater than $\hat{b}'^1$ and denote the associated consumption by $\hat{c}$. Then, by optimality $u(c^1) + \beta Z(y, b'^1) > u(\hat{c}) + \beta Z(y, \hat{b}')$. Since $Z(y, \hat{b}') > Z(y, b'^1)$ (Proposition 0), the above inequality implies $c^1 > \hat{c}$. Let $\Delta = c^1 - \hat{c}$ denote the loss in current consumption from choosing $\hat{b}'$ over b'^1 when the transitory shock takes the value m^1. Now observe that the loss in current consumption from choosing $\hat{b}'$ over b'^1 when the transitory shock takes the value m^0 is also Δ. However, the level of consumption when the transitory shock takes the value m^0 and the sovereign chooses b'^1, denoted $\tilde{c}$, is strictly less than c^1. From the strict concavity of u, it follows that $u(\tilde{c}) - u(\tilde{c} - \Delta) > u(c^1) - u(c^1 - \Delta)$. Therefore, $u(\tilde{c}) + \beta Z(y, b'^1) > u(\tilde{c} - \Delta) + \beta Z(y, \hat{b}')$. Since $\hat{b}'$ was any b' greater than b'^1, $a(y, m^0, b)$ cannot exceed b'^1. Thus, $a(y, m^0, b) \leq a(y, m^1, b)$. To see that $d(y, m, b)$ is decreasing in m, note that $V(y, m, b)$ is strictly increasing in m (Proposition 0) and the utility from default, $X(y, -\bar{m})$, is independent of m.

Appendix B: Logic of the Computation Algorithm

In this appendix, we describe the logic of our solution algorithm. The first part gives the logic of the algorithm for calculating the optimal debt choice as a function of m. The second part, taking the algorithm in the first part as given, provides the logic for the solution algorithm.

B1. Method For Recovering $a(y,m,b;q)$ Given (y,b) and $q(y,b')$

Proposition 5 implies that given (y,b) and q there exists $\{-\bar{m} < m^{K-1} < m^{K-2} < \ldots < m^1 < \bar{m}\}$ and $\{b'^K < b'^{K-1} < \ldots < b'^1\}$ such that b'^K is chosen for $m \in [-\bar{m}, m^{K-1})$, b'^{K-1} is chosen for $m \in [m^{K-1}, m^{K-2})$, $\ldots$, b'^1 is chosen for $m \in (m^1, \bar{m}]$ ($K = 1$ means that b'^1 is chosen for all $m \in M$).

Since b'^k need not be adjacent to b'^{k+1} on the grid, the algorithm has to find both $\{m^k\}$ *and* $\{b'^k\}$. The decision rule is constructed recursively. The choice problem is initially solved for a choice set containing only one b'. The choice set is then expanded in steps until the entire set B is available, with the solution from each step being used to construct the solution for the next step.

Suppose we have located $\{(m^{h-1}, b'^h), (m^{h-2}, b'^{h-1}), \ldots (\bar{m}, b'^1)\}$ such that if the sovereign is permitted to choose *only from the set* $b' \geq b'^h$, the sovereign would choose b'^h for $m \in [-\bar{m}, m^{h-1})$, b'^{h-1} for $m \in [m^{h-1}, m^{h-2})$, $\ldots$, $b'^1 \in (m^1, \bar{m}]$. The next step is to compare the utility from choosing b'^h with the utility from choosing the next lower b' (i.e., next higher debt level) on the grid, denoted b'^-. Two cases are possible.

1) $-q(y,b'^-)[b'^- - (1-\lambda)b] \leq -q(y,b'^h)[b'^h - (1-\lambda)b]$. Then, the lifetime utility from b'^h is at least as high as the lifetime utility from b'^- for all $m \in M$. So we drop b'^- from further consideration and move to comparing b'^h to the next lower b' on the grid.

2) $-q(y,b'^-)[b'^- - (1-\lambda)b] > -q(y,b'^h)[b'^h - (1-\lambda)b]$. Then $\Delta(m) = u(\ldots m - q(y,b'^-)[b'^- - (1-\lambda)b]\ldots) - u(\ldots m - q(y,b'^h)[b'^h - (1-\lambda)b]\ldots) > 0$ for all m, where $u(\ldots m - q(y,b')[b' - (1-\lambda)b]\ldots)$ is the current utility from choosing b' (we have suppressed terms that do not depend on m and b'). Furthermore, from the strict concavity of u, $\Delta(m)$ is decreasing in m. Three subcases are possible.

   a) $\Delta(-\bar{m}) + \beta\{Z(y,b'^-) - Z(y,b'^h)\} \leq 0$. Then b'^h is at least as good as b'^- for all m and we can drop b'^- from further consideration.

b) $\Delta(-\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} > 0$ and $\Delta(\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} \leq 0$. Then there must exist a unique $\tilde{m} \in (-\bar{m}, \bar{m}]$ such that $\Delta(\tilde{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} = 0$. If $\tilde{m} < m^{h}$, we prepend $(\tilde{m}, b'^{-})$ to the list of pairs and proceed to compare the utility between b'^{-} with the next lower b' on the grid. If $\tilde{m} \geq m^{h}$, we drop b'^{h} from further consideration and proceed backwards to compare b'^{-} with b'^{h-1}. The reason is that $\tilde{m} \geq m^{h}$ implies that b'^{-} is preferred to b'^{h} for any $m < \tilde{m}$ and at the same time b'^{h-1} is preferred to b'^{h} for any $m \geq m^{h}$. Thus, b'^{h} is dominated by the choices of b'^{h-1} and b'^{-} and can be dropped from further consideration. When this is the case, b'^{-} needs to be compared to b'^{h-1}. The process is continued by finding a new $\widetilde{m}$ between the choices of b'^{-} and b'^{h-1}. If $\tilde{m} < m^{h-1}$, we add $(\tilde{m}, b'^{-})$ to the list of pairs $\{(m^{h-2}, b'^{h-1}), \ldots (\bar{m}, b'^{1})\}$ and proceed to compare the utility between b'^{-} with the next lower level of assets. If $\tilde{m} \geq m^{h-1}$, we drop b'^{h-1} from further consideration and continue to go backwards through the list. This process will either end in finding m^{h-j} such that $\tilde{m} < m^{h-j}$ or in the exhaustion of all pairs in the list $\{m^{k}, b'^{k}\}$. If the latter, we conclude that b'^{-} dominates any $b' > b'^{-}$ for all m (i.e., the list becomes a singleton $\{(\bar{m}, b'^{-})\}$ and proceed to compare b'^{-} with the next lower b' on the grid.

c) $\Delta(-\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} > 0$ and $\Delta(\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} > 0$. Then b'^{-} dominates b'^{h} for all m and we can drop b'^{h} from further consideration. We then move to compare b'^{-} with b'^{h-1}.

3) To implement this algorithm we start off with the choice set being $\{0\}$.The solution for this stage is the list $\{(\bar{m}, 0)\}$ (meaning that no borrowing is optimal for all m). We then proceed to compare 0 with the next lower b' on the grid. The algorithm is applied until every element of B has been compared.

B2. Method for Computing the Solution

We discretize the state space into N_y grids for persistent output shock and N_b grids for bonds. We enter the k-th iteration with guesses for $q^k(y, b')$ and $Z^k(y, b')$, where $Z^k(y, b') = E_{(y',m'|y)} W^k(y', m', b')$. All calculations below are for some specific (y, b) and k.

1) Given these guesses, we find what the sovereign would do if it repayed. This entails finding the decision rule for debt. The algorithm to accomplish this was outlined above and gives $\{(m^{K-1}, b'^K), (m^{K-2}, b'^{K-1}), \ldots, (\bar{m}, b'^1)\}$.

2) In the second step, we find default thresholds. For each interval from step 1, we compare the lifetime utility from choosing the indicated quantity of debt with the lifetime utility derived from default. Suppose that for $m \in (m^i, m^{i-1}]$ the sovereign chooses b'^i. Define

$$\Delta(m) = u\left(y + m - q^k(y, b'^i)[b'^i - [1-\lambda]b]\right) + \beta Z^k\left(y, b'^i\right) - X(y, -\bar{m}).$$

Evidently, $\Delta(m)$ is increasing in m. If $\Delta(m^i) \cdot \Delta(m^{i-1}) < 0$, there exists an $\tilde{m}$ such that default is optimal for $(m^i, \tilde{m})$ and b'^i is optimal for $[\tilde{m}, m^{i-1}]$. If $\Delta(m^i) \cdot \Delta(m^{i-1}) \geq 0$, then either default is optimal over the entire interval or b'^i is optimal over the interval. At the end of this stage, we have a maximum of $2(N_b - 1)$ intervals. Within each interval we know whether default or repayment is chosen and if repayment is chosen, the corresponding debt choice. Although the maximum number of intervals can be very large, in practice the number of intervals is usually less than 20.

3) Finally, with these intervals in hand we compute the functions $Z^{new}(y, b')$ and $q^{new}(y, b')$. We check if

$$\left| Z^{new}\left(y, b'\right) - Z^k\left(y, b'\right) \right| < \varepsilon_1 \text{ and } \left| q^{new}(y, b') - q^k(y, b') \right| < \varepsilon_2$$

where ε_1 and ε_2 are very small numbers. If these conditions hold, we end the program. If one of them does not hold, we update

$$q^{k+1}(y,b') = (1-\zeta)\cdot q^{new}(y,b') + \zeta\cdot q^{k}(y,b')$$
$$Z^{k+1}\left(y,b'\right) = (1-\nu)\cdot Z^{new}\left(y,b'\right) + \nu\cdot Z^{k}\left(y,b'\right),$$

where $\zeta,\nu\in[0,1)$ and continue with step 1 (ν can be set to 0 without any impairment in performance).

4) To compute $Z^{new}\left(y,b'\right)$ and $q^{new}(y,b')$ we need to integrate W^k and q^k with respect to m. To integrate, we divide M into 11 equally spaced intervals and assume that within each interval m is uniformly distributed. Consider an interval (m_1,m_2) and suppose that it contains one threshold, say $\hat{m}\in[m_1,m_2]$, where the optimal decision changes from a debt of b' to a debt of $\hat{b}'$. Denote $[\lambda+z(1-\lambda)]b+q(y,b')[b'-(1-\lambda)b])$ by $s(b',b)$. Then,

$$\int_{m_1}^{m_2} W(y,m,b)dG(m) \simeq \int_{m_1}^{m_2} dG(m)\times$$
$$\left[\left(\frac{\hat{m}-m_1}{m_2-m_1}\right)\cdot\left(u(y+m_{12}+s(b',b))+\beta Z(y,b')\right)+\right.$$
$$\left.\left(\frac{m_2-\hat{m}}{m_2-m_1}\right)\cdot\left(u(y+m_{12}+s(\hat{b}',b))+\beta Z(y,\hat{b}')\right)\right].$$

In other words, over each interval, we replace m by the midpoint of the interval but recognize that the choice of debt may switch as m varies over the interval. The overall variation in m is small and, with 11 intervals, the variation within each interval is smaller still. Thus, the differences between m and m_{12} are of little consequence for the evaluation of utility, given the choice of debt. Having obtained $\int_m W(y,m,b)dG(m)$ in this way for each y and b, we obtain $Z(y,b')$ as $\sum_{y'}[\int_m W(y',m',b')dG(m)]F(y,y')$.

The procedure for integrating the price function is similar:

$$\int_{m_1}^{m_2} q(y', a(y', m', b'))dG(m') \simeq$$
$$\int_{m_1}^{m_2} dG(m') \times \left[\left(\frac{\hat{m} - m_1}{m_2 - m_1}\right) \cdot q(y', b') + \left(\frac{m_2 - \hat{m}}{m_2 - m_1}\right) \cdot (q(y, \hat{b}')\right].$$

Appendix C: Performance of the Computational Algorithm

As explained in the computation section, the reason for adding the m shock (and calculating thresholds to solve the decision problem) is to ensure that (6) has a solution and that the iteration (7) converges. In this appendix we show that alternative methods that do not use "randomization" have significantly worse convergence performance. We make these comparisons by fixing all parameter values at baseline values and iterate each solution method 3000 times and report the maximum absolute error in the final 100 iterations as well as the relative value of this maximum error. The error for iteration k is defined as the largest absolute change in the price matrix from iteration $k-1$ to k. For purely discrete models, we also report the jumps in asset choice (i.e., the maximum number of grid points skipped) from one iteration to the next, for the final 100 iterations. All computations were implemented via parallelized (MPI) Fortran 90/95 running on a 16-node cluster.

C1. Omitting M and refining Y

Table C1 compares the baseline method (Method I) with three other methods. Method II is the model without M, method III is the model without M in which the Y grid is doubled, and method IV is the baseline model but the M is discretized and thresholds are not computed.

With the baseline method, we get convergence for very tight convergence criteria. In contrast, for Method II, where we omit M, even after 3000 iterations the price matrix is far from convergence; the error can be as much as 20 per-

TABLE C1—OMITTING M AND REFINING Y

	Baseline	II	III	IV
Number of Y Grids	200	200	400	200
Number of B Grids	350	350	350	350
$\Delta\lvert q^k - q^{k-1}\rvert$	9.47×10^{-14}	2.33×10^{-2}	1.14×10^{-2}	3.34×10^{-3}
$\Delta\left\lvert\frac{(q^k - q^{k-1})}{0.001+q^k}\right\rvert$	4.85×10^{-13}	2.11×10^{-1}	1.01×10^{-1}	2.61×10^{-2}
$\Delta\lvert V^k - V^{k-1}\rvert$	1.78×10^{-14}	2.65×10^{-4}	1.48×10^{-4}	2.40×10^{-5}
$\Delta\left\lvert\frac{(V^k - V^{k-1})}{0.001+V^k}\right\rvert$	8.84×10^{-13}	1.31×10^{-5}	7.27×10^{-6}	1.04×10^{-6}
Jump in b'	NA	15	14	14

Method II: M is omitted; Method III: M is omitted but Y grid is refined; Method IV: Baseline but M is discrete and thresholds are not computed

cent. The maximum change in debt choice is 15 grid points; these jumps occur because nonconvexities lead to multiple local maxima and the solution meanders between these local maxima (as discussed in the text in relation to Figure 2). In Method III, we double the grid on Y to 400. There is not much improvement in the results. Finally, in the last column, M is discretized and thresholds are not used. Convergence is somewhat better but still nowhere close to Method I.[43] For models II- IV, the convergence performance of value functions is considerably better than the convergence performance of the price function. This is because the jumps occur between actions that give roughly the same utility and, therefore, do not affect value functions as much (the same is true of models V and VI discussed below).

C2. Omitting M and Refining B

Table C2 shows that the poor performance of the baseline model without M (Model II above) cannot be rectified by refining the B, or asset, dimension.

The column labeled Method V shows the case where we omit the M shock

[43] Also this method takes longer to run relative to the baseline method because the discounted utility of the country is calculated for *all* current states (m, y, b) and for all choices of b'. In the baseline method, given current states (y, b), we find the thresholds for which there is a switch between different choices of assets. As those switches do not happen very frequently, the utility level given the choice of b' is computed much less frequently.

Table C2—Omitting M and Refining B

	Method II	Method V	Method VI
Number of Y Grids	200	200	200
Number of B Grids	350	700	continuous
$\Delta\lvert q^t - q^{t-1}\rvert$	2.23×10^{-2}	2.47×10^{-2}	2.22×10^{-2}
$\Delta\left\lvert \frac{(q^t - q^{t-1})}{0.001+q^t} \right\rvert$	2.11×10^{-1}	2.13×10^{-1}	2.10×10^{-1}
$\Delta\lvert V^t - V^{t-1}\rvert$	2.65×10^{-4}	1.82×10^{-4}	1.84×10^{-4}
$\Delta\left\lvert \frac{(V^t - V^{t-1})}{(0.001+V^t)} \right\rvert$	1.31×10^{-5}	9.05×10^{-5}	9.10×10^{-6}
Jump in b'	15	31	-

and increase the number of grids for the asset level. Evidently, increasing the grids for B makes no difference to convergence. In Method VI we continue to omit the M shock but treat B as a continuous variable in the sense that we allow for asset choices off the grid. In particular, if income is y and beginning-of-period debt is b, then for a debt level b' between two adjacent grids b_j and b_{j-1}, $c = y + [\lambda + (1-\lambda)z]b + wq(b_j, y)[b_j - (1-\lambda)b] + (1-w)q(b_{j+1}, y)[b_{j+1} - (1-\lambda)b]$, and $E_{y'|y}V(y', b') = wE_{y'|y}V(y', b_j) + (1-w)E_{y'|y}V(y', b_{j+1})$ where w is $(b_{j+1} - b')/(b_{j+1} - b_j)$. Since there is more than one local maxima in our problem, we first find the b' that maximizes utility confining our choice to the initial discrete grids and then do a refined search to locate the best choice of b' around that grid (this is the procedure followed in Hatchondo, Martinez and Sapriza (2010)). Treating B continuous in this fashion also does not improve convergence. The lotteries between adjacent grid points do not help because the problematic cycles are between grids that are far apart.

C3. Sensitivity Analysis

It is known that model statistics in the Eaton-Gersovitz model can be sensitive to the choice of grid sizes (Hatchondo, Martinez and Sapriza 2010). To check for this, we doubled the grid sizes on Y and B, separately. Table C3 reports the results for the baseline model as well as for the one-period debt model. The statistics for the baseline model are virtually unaffected. For the one-period model, mean spreads decline somewhat with an increase in grid size but other statistics are unaffected.

Table C3—Sensitivity to Grid Sizes

	Models					
Moment	Baseline	I	II	1-Period Debt	A	B
Avg. $(r - r_f)$	0.0815	0.0814	0.0815	0.0815	0.0806	0.0809
$\sigma(r - r_f)$	0.0443	0.0443	0.0443	0.0443	0.0434	0.0438
Avg. b/y	−0.7	−0.7	−0.7	−0.7	−0.7	−0.7
$\sigma(c)/\sigma(y)$	1.11	1.11	1.11	1.59	1.59	1.59
$\sigma(NX/y)/\sigma(y)$	0.20	0.21	0.20	1.06	1.05	1.06
corr (c, y)	0.99	0.99	0.99	0.73	0.73	0.73
corr (NX, y)	−0.44	−0.44	−0.44	−0.16	−0.16	−0.16
corr $(r - r_f, y)$	−0.65	−0.65	−0.65	−0.55	−0.55	−0.55
Debt Service	0.055	0.055	0.055	0.699	0.699	0.701
Def Freq	0.068	0.068	0.068	0.073	0.072	0.073

Baseline: $N_y = 200$, $N_b = 350$;1-period debt model: $N_y = 200$,$N_b = 450$; Model I: Baseline with $N_y = 400$, $N_b = 350$; Model II: Baseline with $N_y = 200$, $N_b = 700$; Model A: 1-period debt model with $N_y = 400$,$N_b = 450$; Model B: 1-period debt model with $N_y = 200$,$N_b = 900$

Table C4 reports the results of three additional sensitivity analysis. In the first exercise, denoted Model I, the full average debt level of 1.0 is targeted. The model can successfully match all targets. There are some differences in the results. There are increases in the volatility of consumption and NX, and a measurable increase in the debt service. These increases are what we would expect for a higher average debt burden. The correlation patterns remain the same. Overall, model performance is somewhat inferior to the baseline model.

In Model II, we address one potential concern regarding the assumption about m in the period of default. Recall that we assumed that in the period of default, the value of m resets to $-\bar{m}$. This means that there is an additional source of punishment for default, and one may wish to know if this plays any role in the results. In Model II, we assume that in the period of default the value of m resets to 0 instead of $-\bar{m}$ – which might be viewed as a more neutral assumption. As

Table C4—Additional Sensitivity Analyses

Moment	Baseline	Model I	Model II	Model III
Avg. $(r - r_f)$	0.0815	0.0815	0.0815	0.0815
$\sigma(r - r_f)$	0.0443	0.0443	0.0443	0.0444
Avg. b/y	-0.7	-0.9996	-0.7	-0.7
$\sigma(c)/\sigma(y)$	1.11	1.15	1.11	1.11
$\sigma(NX/y)/\sigma(y)$	0.20	0.28	0.21	0.20
$corr(c, y)$	0.99	0.96	0.99	0.99
$corr(NX/y, y)$	-0.44	-0.44	-0.44	-0.44
$corr(r - r_f, y)$	-0.65	-0.62	-0.65	-0.65
Debt Service	0.055	0.078	0.055	0.055
Def Freq	0.068	0.067	0.068	0.068

Model I: Average b/y target = 1.0; Model II: Same targets as baseline but in period of default, m resets to 0; Model III: Same targets as baseline with $\rho = 0.948081$, $\sigma_\epsilon = 0.027203$, $\sigma_m = 0.002$.

is evident, there is virtually no difference in results between the baseline model and this one.

In the third sensitivity analysis, we examine if the results change with a lower standard deviation for m. We re-estimated the endowment process under the assumption that $\sigma_m = 0.002$, which is the lowest σ_m value for which we get convergence for our baseline model. The implied estimates of ρ and σ_ϵ are as reported in the table. As one would expect, ρ is somewhat lower, and σ_ϵ somewhat higher than in the baseline model. However, these changes in the endowment process have virtually no effect on model statistics.

Appendix D: The Trade-off Between σ_m and ζ in Achieving Convergence

This section of confirms that there is a trade-off between the variability of m and the relaxation parameter ζ with regard to convergence within $100,000$ iterations. We consider the model where all parameter values are as in the baseline model but the number of grids on Y, M, and B are 25, 50 and 100, respectively. With fewer grids, the computations take less time and we can demonstrate that convergence can be achieved for *very* small values of σ_m, provided the value of ζ is increased correspondingly.

Table D1—(σ_m, ζ) Pairs For Which Convergence Is Achieved

σ_m	ζ
0.001	0.98
0.0005	0.98
0.0001	0.98
0.00005	0.995
0.00001	0.998

Grids for $Y = 25$, grids for $M = 50$, and grids for $B = 100$.