The Rise of the Service Economy

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Web Appendix

In this online appendix we provide the proofs of the results in the paper, and present additional results that were informally discussed in the main body of the paper.

1 Proofs of Results in the Paper

The following two conditions on the parameters of the model are used in the proof of Proposition 2. Assumption A.1, given in the body of the paper, guarantees that for $A < A_1$ low-skilled individuals are used to produced manufacturing goods.

Assumption A.1: Assume the following joint restriction on the parameters ϕ, q, and the function $\theta(.)$,

$$\frac{\phi f^h \left(1 - \theta\left(f^h\right)\right)}{1 - f^h} < q,$$

where f^h solves $\phi\left[1 - \theta\left(f^h\right) - f^h \theta'\left(f^h\right)\right] = 1$.

We prove Proposition 2 for the case $A_1 < A_2$, which is guaranteed by Assumption A.2 below. Essentially, we assume that for $A = A_1$ the supply of high-skilled workers is more than sufficient to produce all the goods and services for which high-skilled workers have a strict comparative advantage.

(When Assumption A.2 does not hold, then $A_1 \geq A_2$, and an analogous proof follows.)

Assumption A.2: Assume the following joint restriction on the parameters q, λ_l, λ_h, ν, ϕ, and the function $\theta(.)$,

$$q \int_1^{\bar{z}_1} z^{\lambda_h} dz < A_1 \phi f^h \left(1 - \theta\left(f^h\right)\right),$$

where $\bar{z}_1 = \left[1 + \frac{1-v}{v}(1+q)\right]^{\frac{1}{\lambda_l - \lambda_h}}$ is the highest want for which $\underline{z} = \bar{z}$ is optimal when $w = \phi$, f^h solves $\phi\left[1 - \theta\left(f^h\right) - f^h \theta'\left(f^h\right)\right] = 1$, and

$$A_1 = \frac{(1+q)\int_0^1 z^{\lambda_l} dz + (1+q)\int_1^{\bar{z}_1} z^{\lambda_h} dz}{\left[\phi f^h \left(1 - \theta\left(f^h\right)\right) + 1 - f^h\right]},$$

which is the productivity level for which the budget constraint is satisfied for the threshold $\bar{z}_1$,

$$\int_0^{\bar{z}_1} z^{\lambda_l} dz + q \int_0^1 z^{\lambda_l} dz + q \int_1^{\bar{z}_1} z^{\lambda_h} dz = A_1 \left[\phi f^h \left(1 - \theta\left(f^h\right)\right) + 1 - f^h\right].$$

Proof of Proposition 2: Using the characterization in Proposition 1, the equilibrium allocations solve the following problem:

$$\max_{0 \leq f^h \leq 1, n \leq 1 - f^h, \underline{z} \leq \bar{z}} (1-v)\underline{z} + v\bar{z}$$

subject to

$$\begin{aligned} & q \int_0^{\underline{z}} \min\left\{z^{\lambda_l}, \frac{w}{\phi} z^{\lambda_h}\right\} dz + (1+q) \int_{\underline{z}}^{\bar{z}} \min\left\{z^{\lambda_l}, \frac{w}{\phi} z^{\lambda_h}\right\} dz \\ = \; & wAf^h \left(1 - \theta\left(f^h\right)\right) + A\left(1 - f^h - n\right), \end{aligned}$$

$$\int_0^{\underline{z}} \frac{z^{\lambda_l}}{A} dz = n,$$

and the high-skilled labor market clearing condition

$$q\int_{\hat{z}}^{\underline{z}} \min\left\{z^{\lambda_l}, z^{\lambda_h}\right\} dz + (1+q)\int_{\underline{z}}^{\bar{z}} \min\left\{z^{\lambda_l}, z^{\lambda_h}\right\} dz = \phi A f^h \left(1-\theta\left(f^h\right)\right).$$

Notice that, abusing notation, we use $\hat{z} \leq \left(\frac{w}{\phi}\right)^{\frac{1}{\lambda_l-\lambda_h}}$ to denote the least complex want that is produced with high-skilled labor.

Remark: If $w = \phi$, then firms are indifferent between using low or high-skilled workers to produce wants with complexity $z \leq 1$, i.e., the aggregate relative demand for high-skilled labor is infinitely elastic at the wage $w = \phi$. In this case, we adopt the convention that high-skilled workers are employed to produced the set of most complex wants.

We proceed by analyzing the cases of (i) low productivity, $A < A_1$, (ii) intermediate productivity, $A_1 \leq A < A_2$, and (iii) high productivity, $A \geq A_2$.

Low Productivity Case: $A < A_1$

Provided that assumptions A.1 and A.2 hold, an equilibrium for $A < A_1$ is given by the wage $w = \phi$, no market service $\underline{z} = \bar{z}$, and the values for the supply of high-skilled workers f^h and the upper bound of the wants that are satisfied $\bar{z}$ that solve optimality in f^h

$$\phi\left[1-\theta\left(f^h\right) - f^h\theta'\left(f^h\right)\right] = 1 \tag{1}$$

and the budget constraint

$$\begin{aligned} & \int_0^{\bar{z}} z^{\lambda_l} dz + q\int_0^{\min\{1,\bar{z}\}} z^{\lambda_l} dz + q\int_{\min\{1,\bar{z}\}}^{\bar{z}} z^{\lambda_h} dz \\ = \;& A\left[\phi f^h\left(1-\theta\left(f^h\right)\right) + 1 - f^h\right]. \end{aligned} \tag{2}$$

It is straightforward from these equations that $\frac{\partial f^h}{\partial A} = 0$ and $\frac{\partial \underline{z}}{\partial A}\frac{1}{\underline{z}} = \frac{\partial \bar{z}}{\partial A}\frac{1}{\underline{z}} > 0$.

Intermediate Productivity Case: $A_1 \leq A < A_2$

For $A_1 \leq A < A_2$, an equilibrium is given by the wage $w = \phi$ and allocations that solve equations (1),

$$(1+q)\,\frac{1-v}{v} = \frac{\underline{z}^{\lambda_l} - \underline{z}^{\lambda_h}}{\bar{z}^{\lambda_h}} \tag{3}$$

and

$$\begin{aligned} &\int_0^{\underline{z}} z^{\lambda_l} dz + q \int_0^1 z^{\lambda_l} dz + q \int_1^{\underline{z}} z^{\lambda_h} dz + (1+q) \int_{\underline{z}}^{\bar{z}} z^{\lambda_h} dz \\ = \;& A\left[\phi f^h \left(1 - \theta\left(f^h\right)\right) + 1 - f^h\right] \end{aligned} \tag{4}$$

The threshold A_2 is defined by the solutions of the following equation

$$q \int_1^{\underline{z}(A_2)} z^{\lambda_h} dz + (1+q) \int_{\underline{z}(A_2)}^{\bar{z}(A_2)} z^{\lambda_h} dz = A_2 \phi f^h \left(1 - \theta\left(f^h\right)\right)$$

where $\underline{z}\left(A_2\right)$ and $\bar{z}\left(A_2\right)$ are implicitly defined by equations (3) and (4) specialized to $A = A_2$, and f^h solves equation (1). Notice that Assumption A.2 guarantees $A_2 \geq A_1$.

By totally differentiating (3), we obtain

$$d \log \bar{z} = \frac{\frac{\lambda_l}{\lambda_h} \underline{z}^{\lambda_l} - \underline{z}^{\lambda_h}}{\underline{z}^{\lambda_l} - \underline{z}^{\lambda_h}} d \log \underline{z} > d \log \underline{z}, \tag{5}$$

where the last inequality follows from the assumption that $\lambda_l > \lambda_h$. Again, it is straightforward to see that $\frac{\partial f^h}{\partial A} = \frac{\partial w}{\partial A} = 0$.

High Productivity Case: $A \geq A_2$

For $A \geq A_2$, the equilibrium is characterized by optimality in f^h

$$w\left[1 - \theta\left(f^h\right) - f^h \theta'\left(f^h\right)\right] = 1, \tag{6}$$

marginal indifference between $\underline{z}$ and $\bar{z}$

$$(1+q)\,\frac{1-v}{v} = \frac{\underline{z}^{\lambda_l} - \frac{w}{\phi} \underline{z}^{\lambda_h}}{\frac{w}{\phi} \bar{z}^{\lambda_h}}, \tag{7}$$

and labor market clearing conditions for low- and high-skilled labor, respectively

$$q\int_0^{\hat{z}} z^{\lambda_l} dz + \int_0^{\underline{z}} z^{\lambda_l} dz = A\left(1-f\right), \tag{8}$$

and

$$q\int_{\hat{z}}^{\underline{z}} z^{\lambda_h} dz + (1+q)\int_{\underline{z}}^{\bar{z}} z^{\lambda_h} dz = \phi A f\left(1-\theta\left(f\right)\right). \tag{9}$$

Totally differentiating (7), we obtain the following relationship between the growth of $d\log\underline{z}$, $d\log\bar{z}$ and $d\log w$

$$\frac{\lambda_l \underline{z}^{\lambda_l} - \lambda_h \frac{w}{\phi}\underline{z}^{\lambda_h}}{\underline{z}^{\lambda_l} - \frac{w}{\phi}\underline{z}^{\lambda_h}} d\log\underline{z} - \lambda_h d\log\bar{z} - \frac{\underline{z}^{\lambda_l}}{\underline{z}^{\lambda_l} - \frac{w}{\phi}\underline{z}^{\lambda_h}} d\log w = 0. \tag{10}$$

Again totally differentiating equations (8) and (9) under the assumption that the supply of high-skilled workers does not change, we obtain the following upper bound on the equilibrium growth of the relative price of high-skilled labor

$$\begin{aligned} d\log w \quad \leq \quad & (\lambda_l - \lambda_h)\frac{\frac{(\lambda_h+1)(1+q)\bar{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1}-\underline{z}^{\lambda_h+1}-q\hat{z}^{\lambda_h+1}}}{\frac{(\lambda_h+1)q\hat{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1}-\underline{z}^{\lambda_h+1}-q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)q\hat{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1}+\underline{z}^{\lambda_l+1}}} d\log\bar{z} \\ & -(\lambda_l - \lambda_h)\frac{\frac{(\lambda_h+1)\underline{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1}-\underline{z}^{\lambda_h+1}-q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)\underline{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1}+\underline{z}^{\lambda_l+1}}}{\frac{(\lambda_h+1)q\hat{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1}-\underline{z}^{\lambda_h+1}-q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)q\hat{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1}+\underline{z}^{\lambda_l+1}}} d\log\underline{z}. \end{aligned}$$

Substituting this upper-bound for $d\log w$ into equation (10), we obtain the following lower-bound on the equilibrium behavior of $d\log\bar{z}$

$$d\log\bar{z} \geq \frac{\frac{\lambda_l \underline{z}^{\lambda_l} - \lambda_h \frac{w}{\phi}\underline{z}^{\lambda_h}}{\underline{z}^{\lambda_l} - \frac{w}{\phi}\underline{z}^{\lambda_h}} + (\lambda_l - \lambda_h)\frac{\underline{z}^{\lambda_l}}{\underline{z}^{\lambda_l} - \frac{w}{\phi}\underline{z}^{\lambda_h}} \frac{\frac{(\lambda_h+1)\underline{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)\underline{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1} + \underline{z}^{\lambda_l+1}}}{\frac{(\lambda_h+1)q\hat{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)q\hat{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1} + \underline{z}^{\lambda_l+1}}}}{\lambda_h + (\lambda_l - \lambda_h)\frac{\underline{z}^{\lambda_l}}{\underline{z}^{\lambda_l} - \frac{w}{\phi}\underline{z}^{\lambda_h}} \frac{\frac{(\lambda_h+1)(1+q)\bar{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}}}{\frac{(\lambda_h+1)q\hat{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)q\hat{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1} + \underline{z}^{\lambda_l+1}}}} d\log\underline{z},$$

after simplification

$$d\log\bar{z} \geq \frac{\lambda_h + (\lambda_l - \lambda_h)\frac{\underline{z}^{\lambda_l}}{\underline{z}^{\lambda_l} - \frac{w}{\phi}\underline{z}^{\lambda_h}} \frac{\frac{(\lambda_h+1)(1+q)\bar{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}} + \lambda_l - \lambda_h}{\frac{(\lambda_h+1)q\hat{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)q\hat{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1} + \underline{z}^{\lambda_l+1}}}}{\lambda_h + (\lambda_l - \lambda_h)\frac{\underline{z}^{\lambda_l}}{\underline{z}^{\lambda_l} - \frac{w}{\phi}\underline{z}^{\lambda_h}} \frac{\frac{(\lambda_h+1)(1+q)\bar{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}}}{\frac{(\lambda_h+1)q\hat{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)q\hat{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1} + \underline{z}^{\lambda_l+1}}}} d\log\underline{z} > d\log\underline{z}.$$

Thus, $d\log\bar{z} > d\log\underline{z}$ for $A \geq A_2$.

Finally, to show that the supply and the relative price of high-skilled labor increases for $A \geq A_2$, we totally differentiate equations (7), (8), and (9) under the assumption that f^h and w remain constant to obtain the following relationship between $d\log\hat{z}$ and $d\log\bar{z}$

$$\begin{aligned} d\log\hat{z} = {} & \frac{\frac{(\lambda_h+1)(1+q)\bar{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}}}{\frac{(\lambda_h+1)q\hat{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)q\hat{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1} + \underline{z}^{\lambda_l+1}}} d\log\bar{z} \\ & - \frac{\frac{(\lambda_h+1)\underline{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)\underline{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1} + \underline{z}^{\lambda_l+1}}}{\frac{(\lambda_h+1)q\hat{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)q\hat{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1} + \underline{z}^{\lambda_l+1}}} \frac{\lambda_h}{\frac{\lambda_l \underline{z}^{\lambda_l} - \lambda_h \frac{w}{\phi}\underline{z}^{\lambda_h}}{\underline{z}^{\lambda_l} - \frac{w}{\phi}\underline{z}^{\lambda_h}}} d\log\bar{z} \\ & > (\lambda_l - \lambda_h)\frac{\frac{\underline{z}^{\lambda_l} + \lambda_h \frac{w}{\phi}\underline{z}^{\lambda_h}}{\lambda_l \underline{z}^{\lambda_l} - \lambda_h \frac{w}{\phi}\underline{z}^{\lambda_h}}}{\frac{(\lambda_h+1)q\hat{z}^{\lambda_h+1}}{(1+q)\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_h+1} - q\hat{z}^{\lambda_h+1}} + \frac{(\lambda_l+1)q\hat{z}^{\lambda_l+1}}{q\hat{z}^{\lambda_l+1} + \underline{z}^{\lambda_l+1}}} d\log\bar{z} > 0, \end{aligned}$$

where the last inequality follows simple algebraic manipulations and the fact that $\lambda_l > \lambda_h$. This shows that the relative demand for high-skilled labor increases, and therefore the least complex want produced with high-skilled labor must increase, $d \log \hat{z} > 0$, to clear the low and high-skilled labor markets. In equilibrium this can only happen if $\frac{\partial w}{\partial A} > 0$, which, by equation (6), implies that $\frac{\partial f^h}{\partial A} > 0$.

Proof of Proposition 3: The value of the productivity A_0 consumption baskets of services relative to that of goods, evaluated at productivity A prices is given by the following expression

$$\frac{P_S(A, A_0)}{P_G(A, A_0)} = \frac{\int_{\underline{z}(A_0)}^{\bar{z}(A_0)} p_S(z; A) dz}{\int_0^{\underline{z}(A_0)} p_G(z; A) dz}.$$

Differentiating with respect to A

$$\frac{\partial}{\partial A}\left(\frac{P_S(A, A_0)}{P_G(A, A_0)}\right) = \frac{\partial w}{\partial A}\frac{1}{w}\frac{P_S(A, A_0)}{P_G(A, A_0)}\left[1 - \frac{\int_{\hat{z}(A_0)}^{\underline{z}(A_0)} p_G(z; A) dz}{\int_0^{\underline{z}(A_0)} p_G(z; A) dz}\right] > 0,$$

where the last inequality uses that $\frac{\partial w}{\partial A} > 0$ for $A \geq A_2$.

2 Characterization of the Behavior of the Economy as $A \to \infty$

The limiting behavior of the economy as productivity goes to infinity is summarized in the following proposition.

Proposition A.1: If $\lambda_l > \lambda_h \geq 0$, $0 \leq \lim_{A\to\infty} \frac{\partial \ln \underline{z}}{\partial \ln A} = 1/(\lambda_l + 1) < \lim_{A\to\infty} \frac{\partial \ln \bar{z}}{\partial \ln A} = 1/(\lambda_h + 1)$, $\lim_{A\to\infty} f^h = \bar{f}^h \in (0, 1)$, and $\lim_{A\to\infty} \frac{\partial \ln w}{\partial \ln A} = (\lambda_l - \lambda_h)/[(\lambda_l + 1)(1 + \lambda_h)]$.

Proof: We proceed by proving a series of lemmas, which taken together prove this Proposition.

Lemma A.2: $\lim_{A\to\infty} \frac{\partial \ln \underline{z}}{\partial \ln A} = 1/\left(\lambda_l + 1\right) < \lim_{A\to\infty} \frac{\partial \ln \bar{z}}{\partial \ln A} = 1/\left(\lambda_h + 1\right)$.
Proof: The fact that $\hat{z} \leq \underline{z}$ and $\lim_{A\to\infty} f^h\left(A\right) < 1$, together with the low-skilled labor market clearing condition, equation (8), imply that $\lim_{A\to\infty} \frac{\underline{z}(A)^{\lambda_l+1}}{A}$ converges to a strictly positive and finite limit, i.e., $\lim_{A\to\infty} \frac{\partial \ln \underline{z}}{\partial \ln A} = 1/\left(\lambda_l + 1\right)$. Similarly, the high-skilled labor market clearing condition, equation (9), implies that $\lim_{A\to\infty} \frac{\bar{z}(A)^{\lambda_h+1}}{A}$ converges to a strictly positive and finite limit, i.e., $\lim_{A\to\infty} \frac{\partial \ln \bar{z}}{\partial \ln A} = 1/\left(\lambda_h + 1\right)$.
Lemma A.3: $\lim_{A\to\infty} \frac{\partial \ln w}{\partial \ln A} = \left(\lambda_l - \lambda_h\right)/\left[\left(\lambda_l + 1\right)\left(1 + \lambda_h\right)\right]$.
Proof: This result follows from the previous lemma and equation (10).
Lemma A.4: $\lim_{A\to\infty} f^h = \bar{f}^h \in (0, 1)$.
Proof: This result follows from the previous lemma and equation (6).

3 Behavior of the Share of Services for the case $A_1 < A < A_2$

The ratio of expenditures in services relative to goods equals:

$$\begin{aligned}\frac{C_S}{C_G} &= \frac{(1+q)\int_{\underline{z}}^{\bar{z}} z^{\lambda_h} dz}{q\int_0^1 z^{\lambda_l} dz + q\int_1^{\underline{z}} z^{\lambda_h} dz} \\ &= \frac{\bar{z}^{\lambda_h+1} - \underline{z}^{\lambda_l+1}}{\underline{z}^{\lambda_h+1} - \frac{\lambda_l-\lambda_h}{\lambda_l+1}}.\end{aligned}$$

Differentiating with respect to A

$$
\begin{aligned}
\frac{\partial\left(\frac{C_S}{C_G}\right)}{\partial A} = & \frac{\lambda_h+1}{\left(\underline{z}^{\lambda_h+1}-\frac{\lambda_l-\lambda_h}{\lambda_l+1}\right)^2}\frac{1+q}{q} \\
& \left[\left(\bar{z}^{\lambda_h}\frac{\partial\bar{z}}{\partial A}-\underline{z}^{\lambda_h}\frac{\partial\underline{z}}{\partial A}\right)\left(\underline{z}^{\lambda_h+1}-\frac{\lambda_l-\lambda_h}{\lambda_l+1}\right)\right. \\
& \left.-\underline{z}^{\lambda_h}\frac{\partial\underline{z}}{\partial A}\left(\bar{z}^{\lambda_h+1}-\underline{z}^{\lambda_l+1}\right)\right] \\
= & \frac{\lambda_h+1}{\left(\underline{z}^{\lambda_h+1}-\frac{\lambda_l-\lambda_h}{\lambda_l+1}\right)^2}\frac{1+q}{q} \\
& \left[\bar{z}^{\lambda_h}\frac{\partial\bar{z}}{\partial A}\left(\underline{z}^{\lambda_h+1}-\frac{\lambda_l-\lambda_h}{\lambda_l+1}\right)-\underline{z}^{\lambda_h}\frac{\partial\underline{z}}{\partial A}\left(\bar{z}^{\lambda_h+1}-\frac{\lambda_l-\lambda_h}{\lambda_l+1}\right)\right].
\end{aligned}
$$

Using the inequality in (5),

$$
\begin{aligned}
\frac{\partial\left(\frac{C_S}{C_G}\right)}{\partial A} > & \frac{\lambda_h+1}{\left(\underline{z}^{\lambda_h+1}-\frac{\lambda_l-\lambda_h}{\lambda_l+1}\right)^2}\frac{1+q}{q}\frac{1}{\underline{z}}\frac{\partial\underline{z}}{\partial A} \\
& \left[\bar{z}^{\lambda_h+1}\frac{\frac{\lambda_l}{\lambda_h}\underline{z}^{\lambda_l}-\underline{z}^{\lambda_h}}{\underline{z}^{\lambda_l}-\underline{z}^{\lambda_h}}\left(\underline{z}^{\lambda_h+1}-\frac{\lambda_l-\lambda_h}{\lambda_l+1}\right)-\underline{z}^{\lambda_h+1}\left(\bar{z}^{\lambda_h+1}-\frac{\lambda_l-\lambda_h}{\lambda_l+1}\right)\right] \\
= & \frac{\lambda_h+1}{\left(\underline{z}^{\lambda_h+1}-\frac{\lambda_l-\lambda_h}{\lambda_l+1}\right)^2}\frac{1+q}{q}\frac{1}{\underline{z}}\frac{\partial\underline{z}}{\partial A} \\
& \left[\frac{\lambda_l-\lambda_h}{\lambda_h}\frac{\bar{z}^{\lambda_h+1}\underline{z}^{\lambda_l}}{\underline{z}^{\lambda_l}-\underline{z}^{\lambda_h}}\left(\underline{z}^{\lambda_h+1}-\frac{\lambda_l}{\lambda_l+1}\right)+\frac{\lambda_l-\lambda_h}{\lambda_l+1}\left(\bar{z}^{\lambda_h+1}\frac{\underline{z}^{\lambda_h}}{\underline{z}^{\lambda_l}-\underline{z}^{\lambda_h}}+\underline{z}^{\lambda_h+1}\right)\right] \\
> & 0,
\end{aligned}
$$

where the last inequality follows from $\lambda_l > \lambda_h$ and $\underline{z} \leq 1$.

This shows that the real share of services in consumption expenditure increases with productivity when $A_1 < A < A_2$. It is straightforward to see that the nominal share of services in consumption expenditure increases, as the relative wage w is independent of productivity when $A < A_2$.

4 Behavior of the Share of Services for the case $\lambda_h = 0$

Given the previous analysis, we only need to consider the case $A > A_2$. When $\lambda_h = 0$ and $A > A_2$, the wage rate equals $w = \phi \hat{z}^{\lambda_l}$, and the first order conditions simplify to

$$(1+q)\frac{1-v}{v} = \left(\frac{\underline{z}}{\hat{z}}\right)^{\lambda_l} - 1 \tag{11}$$

and

$$\phi \hat{z}^{\lambda_l}\left[1-\theta\left(f^h\right) - f^h\theta'\left(f^h\right)\right] = 1. \tag{12}$$

Using these equations and the counterparts to equations (8) and (9) we can derive the following bound on the equilibrium behavior of $(1/\bar{z})(\partial\bar{z}/\partial A)$ relative to $(1/\underline{z})(\partial\underline{z}/\partial A)$ for the case when the relative wage increases, i.e, when $A > A_2$:

Lemma A.5: If $\lambda_h = 0$ and $A \geq A_2$ then

$$\frac{1}{\bar{z}}\frac{\partial\bar{z}}{\partial A} > \left(1+\lambda_l\frac{\bar{z}-\underline{z}}{\bar{z}}\right)\frac{1}{\underline{z}}\frac{\partial\underline{z}}{\partial A}. \tag{13}$$

Proof: Applying the Implicit Function Theorem on the equations (11), (12), (8) and (9) we obtain the following system of equations on the derivative of the thresholds, $\hat{z}$, $\underline{z}$, and $\bar{z}$, and the supply of skills f^h, with respect to productivity A

$$\frac{1}{\hat{z}}\frac{\partial\hat{z}}{\partial A} = \frac{1}{\underline{z}}\frac{\partial\underline{z}}{\partial A}, \tag{14}$$

$$\frac{\partial f^h}{\partial A} = \lambda_l\frac{1-\theta\left(f^h\right)-f^h\theta'\left(f^h\right)}{2\theta'\left(f^h\right)+f^h\theta''\left(f^h\right)}\frac{1}{\underline{z}}\frac{\partial\underline{z}}{\partial A},$$

$$\left[q\hat{z}^{\lambda_l+1}+\underline{z}^{\lambda_l+1}\right]\frac{1}{\underline{z}}\frac{\partial\underline{z}}{\partial A}=\left(1-f^h\right),$$

$$(1+q)\frac{\partial\bar{z}}{\partial A}-q\frac{\partial\hat{z}}{\partial A}-\frac{\partial\underline{z}}{\partial A}=f^h\left(1-\theta\left(f^h\right)\right)+A\left[1-\theta\left(f^h\right)-f^h\theta'\left(f^h\right)\right]\frac{\partial f^h}{\partial A},$$

This system of equations can be solved to obtain the following relationship between $(1/\bar{z})(\partial\bar{z}/\partial A)$ and $(1/\underline{z})(\partial\underline{z}/\partial A)$,

$$\begin{aligned}\frac{1}{\bar{z}}\frac{\partial\bar{z}}{\partial A} &= \left\{\frac{(1+q)\,\bar{z}-q\hat{z}-\underline{z}}{(1+q)\,\bar{z}}\left[1+\lambda_l+\frac{\lambda_l}{1-f}\frac{1-\theta(f)-f\theta'(f)}{2\theta'(f)+f\theta''(f)}\right]\right.\\ &\quad\left.+\frac{\lambda_l A}{(1+q)\,\bar{z}}\frac{\left[1-\theta(f)-f\theta'(f)\right]^2}{2\theta'(f)+f\theta''(f)}+\frac{(\hat{z}q+\underline{z})}{(1+q)\,\bar{z}}\right\}\frac{1}{\underline{z}}\frac{d\underline{z}}{\partial A}.\end{aligned}$$

Using that the labor supply increases when $A>A_2$

$$\begin{aligned}\frac{1}{\bar{z}}\frac{\partial\bar{z}}{\partial A} &\geq \left[\frac{(1+q)\,\bar{z}-q\hat{z}-\underline{z}}{(1+q)\,\bar{z}}(1+\lambda_l)+\frac{(\hat{z}q+\underline{z})}{(1+q)\,\bar{z}}\right]\frac{1}{\underline{z}}\frac{d\underline{z}}{\partial A}\\ &= \left[1+\lambda_l\frac{((1+q)\,\bar{z}-q\hat{z}-\underline{z})}{(1+q)\,\bar{z}}\right]\frac{1}{\underline{z}}\frac{d\underline{z}}{\partial A}.\end{aligned}$$

Finally, using that $\underline{z}>\hat{z}$ we obtain

$$\frac{1}{\bar{z}}\frac{\partial\bar{z}}{\partial A}>\left[1+\lambda_l\frac{\bar{z}-\underline{z}}{\bar{z}}\right]\frac{1}{\underline{z}}\frac{d\underline{z}}{\partial A},$$

which completes the proof of the desired result.

Using this lemma, we can show that $(1/C_S)(\partial C_S/\partial A)>(1/C_S)(\partial C_S/\partial A)$, which implies that $\partial(C_S/C_G)/\partial A > 0$:

$$\begin{aligned}
\frac{1}{C_S}\frac{\partial C_S}{\partial A} &= \frac{\partial\bar{z}/\partial A - \partial\underline{z}/\partial A}{\bar{z}-\underline{z}} \\
&> \frac{\left(1+\lambda_l\frac{\bar{z}-\underline{z}}{\bar{z}}\right)\bar{z}-\underline{z}}{\bar{z}-\underline{z}}\frac{1}{\underline{z}}\frac{\partial\underline{z}}{\partial A} \\
&= (\lambda_l+1)\frac{1}{\underline{z}}\frac{\partial\underline{z}}{\partial A} \\
&> \left(\lambda_l\frac{\frac{\hat{z}^{\lambda_l+1}}{\lambda_l+1}}{\frac{\hat{z}^{\lambda_l+1}}{\lambda_l+1}+\frac{w}{\phi}\left(\underline{z}-\hat{z}\right)}+1\right)\frac{1}{\underline{z}}\frac{\partial\underline{z}}{\partial A} \\
&= \frac{(\lambda_l+1)\frac{\hat{z}^{\lambda_l+1}}{\lambda_l+1}+\frac{w}{\phi}\left(\underline{z}-\hat{z}\right)}{\frac{\hat{z}^{\lambda_l+1}}{\lambda_l+1}+\frac{w}{\phi}\left(\underline{z}-\hat{z}\right)}\frac{1}{\underline{z}}\frac{\partial\underline{z}}{\partial A} \\
&= \frac{(\lambda_l+1)\frac{\hat{z}^{\lambda_l}}{\lambda_l+1}\frac{\partial\hat{z}}{\partial A}+\frac{w}{\phi}\left(\frac{\partial\underline{z}}{\partial A}-\frac{\partial\hat{z}}{\partial A}\right)}{\frac{\hat{z}^{\lambda_l+1}}{\lambda_l+1}+\frac{w}{\phi}\left(\underline{z}-\hat{z}\right)} \\
&= \frac{1}{C_M}\frac{\partial C_M}{\partial A},
\end{aligned}$$

where the next to last equality follows from equation (14).