Aggregation and the PPP Puzzle in a Sticky-Price Model

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Online Appendix – not for publication

A Proofs of propositions, corollaries, and lemmas

Proposition 1 *Under the assumptions of Section II, sectoral real exchange rates follow $AR(2)$ processes:*

$$(1-\rho_z L)(1-\lambda_k L)\, q_{k,t} = \varphi_k u_t,$$

where $\lambda_k \equiv 1-\alpha_k$ is the per-period probability of no price adjustment for a firm in sector k, $u_t \equiv \sigma_{\varepsilon_z}\left(\varepsilon_{z,t}-\varepsilon^*_{z,t}\right)$ is a white noise process, $\varphi_k \equiv \lambda_k - (1-\lambda_k)\frac{\rho_z\beta\lambda_k}{1-\rho_z\beta\lambda_k}$, and L is the lag operator.

Proof. From the optimal-price equations:

$$\begin{aligned}
x_{H,k,t} &= (1-\beta(1-\alpha_k))\, E_t \sum\nolimits_{s=0}^{\infty} \beta^s (1-\alpha_k)^s \left[c_{t+s}+p_{t+s}\right] \\
&= (1-\beta(1-\alpha_k))\, E_t \sum\nolimits_{s=0}^{\infty} \beta^s (1-\alpha_k)^s z_{t+s} \\
&= z_t + \frac{\rho_z\beta(1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z}(z_t - z_{t-1}),
\end{aligned}$$

and analogously:

$$\begin{aligned}
x_{F,k,t} &= z_t + \frac{\rho_z\beta(1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z}(z_t - z_{t-1}), \\
x^*_{H,k,t} &= z^*_t + \frac{\rho_z\beta(1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z}\left(z^*_t - z^*_{t-1}\right), \\
x^*_{F,k,t} &= z^*_t + \frac{\rho_z\beta(1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z}\left(z^*_t - z^*_{t-1}\right).
\end{aligned}$$

This implies that the country-sector price indices follow:

$$
\begin{aligned}
p_{H,k,t} &= (1-\alpha_k)\, p_{H,k,t-1} + \alpha_k \left(z_t + \frac{\rho_z \beta (1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z} (z_t - z_{t-1}) \right), \\
p_{F,k,t} &= (1-\alpha_k)\, p_{F,k,t-1} + \alpha_k \left(z_t + \frac{\rho_z \beta (1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z} (z_t - z_{t-1}) \right),
\end{aligned}
$$

$$
\begin{aligned}
p^*_{H,k,t} &= (1-\alpha_k)\, p^*_{H,k,t-1} + \alpha_k \left(z^*_t + \frac{\rho_z \beta (1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z} \left(z^*_t - z^*_{t-1}\right) \right), \\
p^*_{F,k,t} &= (1-\alpha_k)\, p^*_{F,k,t-1} + \alpha_k \left(z^*_t + \frac{\rho_z \beta (1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z} \left(z^*_t - z^*_{t-1}\right) \right),
\end{aligned}
$$

and that sectoral price indices evolve according to:

$$
\begin{aligned}
p_{k,t} &= (1-\alpha_k)\, p_{k,t-1} + \alpha_k \left[z_t + \frac{\rho_z \beta (1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z} (z_t - z_{t-1}) \right], \\
p^*_{k,t} &= (1-\alpha_k)\, p^*_{k,t-1} + \alpha_k \left[z^*_t + \frac{\rho_z \beta (1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z} \left(z^*_t - z^*_{t-1}\right) \right].
\end{aligned}
$$

Therefore, sectoral real exchange rates follow:

$$
\begin{aligned}
q_{k,t} &= e_t + p^*_{k,t} - p_{k,t} \\
&= e_t + \alpha_k \begin{pmatrix} z^*_t - z_t \\ + \frac{\rho_z \beta(1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z} \left(\Delta z^*_t - \Delta z_t\right) \end{pmatrix} + (1-\alpha_k)\, q_{k,t-1} - (1-\alpha_k)\, e_{t-1}.
\end{aligned} \tag{23}
$$

In turn, the nominal exchange rate can be written as:

$$
e_t = q_t + p_t - p^*_t = c_t - c^*_t + p_t - p^*_t = z_t - z^*_t. \tag{24}
$$

Substituting (24) into (23) and simplifying yields:

$$
q_{k,t} = (1-\alpha_k)\, q_{k,t-1} + \left(1 - \alpha_k - \alpha_k \frac{\rho_z \beta (1-\alpha_k)}{1-\beta(1-\alpha_k)\rho_z}\right) \Delta e_t.
$$

Finally, note that the nominal exchange rate evolves according to:

$$\begin{aligned} e_t &= z_t - z_t^* \\ &= (1+\rho_z)\left(z_{t-1} - z_{t-1}^*\right) - \rho_z\left(z_{t-2} - z_{t-2}^*\right) + \sigma_{\varepsilon_z}\left(\varepsilon_{z,t} - \varepsilon_{z,t}^*\right) \\ &= (1+\rho_z)\, e_{t-1} - \rho_z e_{t-2} + \sigma_{\varepsilon_z}\left(\varepsilon_{z,t} - \varepsilon_{z,t}^*\right), \end{aligned}$$

so that:

$$\Delta e_t = \rho_z \Delta e_{t-1} + u_t,$$

where $u_t \equiv \sigma_{\varepsilon_z}\left(\varepsilon_{z,t} - \varepsilon_{z,t}^*\right)$ is a white noise process. As a result, we can write:

$$(1-\rho_z L)(1-\lambda_k L)\, q_{k,t} = \varphi_k u_t,$$

where $\lambda_k \equiv 1 - \alpha_k$, and $\varphi_k \equiv \lambda_k - (1-\lambda_k)\frac{\rho_z \beta \lambda_k}{1-\rho_z\beta\lambda_k}$. ■

Corollary 1 *The aggregate real exchange rate follows an* $ARMA\,(K+1, K-1)$ *process:*

$$(1-\rho_z L)\prod_{k=1}^{K}(1-\lambda_k L)\, q_t = \left[\sum\nolimits_{k=1}^{K}\prod_{j\neq k}^{K}(1-\lambda_j L)\, f_k \varphi_k\right] u_t.$$

Proof. This is a standard result in aggregation of time-series processes (Granger and Morris 1976). The aggregate real exchange rate is given by:

$$q_t = \sum\nolimits_{k=1}^{K} f_k q_{k,t}.$$

From the result of **Proposition 1**, multiply each sectoral real exchange rate equation by its respective sectoral weight to obtain:

$$f_k (1-\rho_z L)(1-\lambda_k L)\, q_{k,t} = f_k \varphi_k u_t.$$

Multiplying each such equation by all $(K-1)$ L-polynomials of the form $(1-\lambda_m L)$, $m \neq k$ and adding them up yields:

$$(1-\rho_z L)\prod_{k=1}^{K}(1-\lambda_k L)\, q_t = \left[\sum\nolimits_{k=1}^{K}\prod_{m\neq k}^{K}(1-\lambda_m L)\, f_k\varphi_k\right] u_t,$$

so that q_t follows an $ARMA\,(K+1, K-1)$. ■

Corollary 2 *The aggregate real exchange rate of the counterfactual one-sector world economy follows an $AR\,(2)$ process:*

$$(1-\rho_z L)\left(1-\overline{\lambda}L\right) q_t^{1\,\mathrm{sec}} = \varphi u_t,$$

where $\overline{\lambda} \equiv \sum_{k=1}^{K} f_k\lambda_k$ and $\varphi \equiv \overline{\lambda} - \left(1-\overline{\lambda}\right)\frac{\rho_z\beta\overline{\lambda}}{1-\rho_z\beta\overline{\lambda}}$.

Proof. From **Corollary 1**, the real exchange rate in a one-sector world economy with frequency of price changes equal to $\overline{\alpha}$ – probability of no-adjustment equal to $\overline{\lambda} = 1-\overline{\alpha}$ – follows:

$$(1-\rho_z L)\left(1-\overline{\lambda}L\right) q_t = \left(1-\overline{\lambda}L\right)\left(\overline{\lambda} - \left(1-\overline{\lambda}\right)\frac{\rho_z\beta\overline{\lambda}}{1-\rho_z\beta\overline{\lambda}}\right) u_t.$$

■

Proposition 2 *For the measures of persistence $\mathcal{P} = \mathcal{CIR}$,$\mathcal{LAR}$,$\mathcal{SAC}$:*

$$\mathcal{P}\,(q) > \mathcal{P}\left(q^{1\,\mathrm{sec}}\right).$$

Proof. We prove separate results for each measure of persistence.

$\mathcal{CIR}$:

Recall that we denote the impulse response function of the q_t process to a unit impulse by $IRF_t\,(q)$. In turn, let $SIRF_t\,(q)$ denote the "scaled impulse response function," i.e. the impulse response function to one-standard-deviation shock. Since $q_t = \sum_{k=1}^{K} f_k q_{k,t}$, $SIRF_t\,(q) = \sum_{k=1}^{K} f_k SIRF_t\,(q_k)$. So, the impulse response function of the q_t process to a unit impulse, which is simply the scaled impulse response function normalized by the initial impact of the shock, can be written as:

$$IRF_t\left(q\right)=\frac{\sum_{k=1}^{K}f_kSIRF_t\left(q_k\right)}{\sum_{k=1}^{K}f_kSIRF_0\left(q_k\right)}. \tag{25}$$

From (25), the cumulative impulse response for q_t is:

$$\mathcal{CIR}\left(q\right)=\sum\nolimits_{t=0}^{\infty}IRF_t\left(q\right)=\frac{\sum_{k=1}^{K}f_k\sum_{t=0}^{\infty}SIRF_t\left(q_k\right)}{\sum_{k=1}^{K}f_kSIRF_0\left(q_k\right)}. \tag{26}$$

From the processes in **Proposition 1** we can compute $\sum_{t=0}^{\infty}SIRF_t\left(q_k\right)$, and $SIRF_0\left(q_k\right)$:

$$\sum\nolimits_{t=0}^{\infty}SIRF_t\left(q_k\right) = \frac{\lambda_k\left(1-\rho_z\beta\right)}{1-\rho_z\beta\lambda_k}\frac{1}{\left(1-\lambda_k\right)\left(1-\rho_z\right)}, \tag{27}$$

$$SIRF_0\left(q_k\right) = \frac{\lambda_k\left(1-\rho_z\beta\right)}{1-\rho_z\beta\lambda_k}. \tag{28}$$

Substituting (27) and (28) into (26) yields:

$$\mathcal{CIR}\left(q\right)=\frac{\sum_{k=1}^{K}f_k\frac{\lambda_k(1-\rho_z\beta)}{1-\rho_z\beta\lambda_k}\frac{1}{(1-\lambda_k)(1-\rho_z)}}{\sum_{k=1}^{K}f_k\frac{\lambda_k(1-\rho_z\beta)}{1-\rho_z\beta\lambda_k}}.$$

Note that $\frac{\lambda_k(1-\rho_z\beta)}{1-\rho_z\beta\lambda_k}$ is increasing in λ_k, so that $\widetilde{f_k}\equiv\frac{f_k\frac{\lambda_k(1-\rho_z\beta)}{1-\rho_z\beta\lambda_k}}{\sum_{k=1}^{K}f_k\frac{\lambda_k(1-\rho_z\beta)}{1-\rho_z\beta\lambda_k}}$ are sectoral weights obtained through a transformation of f_k, which attaches higher weight to higher λ_ks. The fact that $\frac{1}{(1-\lambda_k)(1-\rho_z)}$ is also increasing, and moreover convex, in λ_k, thus implies the following inequalities:

$$\underbrace{\sum\nolimits_{k=1}^{K}\widetilde{f_k}\frac{1}{\left(1-\lambda_k\right)\left(1-\rho_z\right)}}_{\mathcal{CIR}(q)}>\underbrace{\sum\nolimits_{k=1}^{K}f_k\frac{1}{\left(1-\lambda_k\right)\left(1-\rho_z\right)}}_{\sum_{k=1}^{K}f_k\mathcal{CIR}(q_k)}>\underbrace{\frac{1}{\sum_{k=1}^{K}f_k\left(1-\lambda_k\right)\left(1-\rho_z\right)}}_{\mathcal{CIR}(q^{1\,\text{sec}})}. \tag{29}$$

This proves that $\mathcal{CIR}\left(q\right)>\mathcal{CIR}\left(q^{1\,\text{sec}}\right)$.

$\mathcal{LAR}$:

We order the sectors in terms of price stickiness, starting from the most flexible: $\alpha_k>\alpha_{k+1}$ $\left(\lambda_k<\lambda_{k+1}\right)$. Moreover, recall that we assume $\rho_z\in\left(1-\alpha_1,1-\alpha_K\right)$. Thus, based on **Proposition**

1 and **Corollaries 1** and **2**, we obtain directly the following results:

$$\begin{aligned} \mathcal{LAR}(q) &= \lambda_K, \\ \mathcal{LAR}(q_k) &= \max\{\lambda_k, \rho_z\}, \\ \mathcal{LAR}\left(q^{1\,\text{sec}}\right) &= \max\left\{\overline{\lambda}, \rho_z\right\} = \max\left\{\sum\nolimits_{k=1}^{K} f_k \lambda_k, \rho_z\right\}. \end{aligned}$$

Therefore:

$$\mathcal{LAR}(q) > \sum\nolimits_{k=1}^{K} f_k \mathcal{LAR}(q_k) > \mathcal{LAR}\left(q^{1\,\text{sec}}\right). \tag{30}$$

$\mathcal{SAC}$:

From **Corollary 1**:

$$\begin{aligned} \mathcal{SAC}(q) &= 1 - (1-\rho_z) \prod_{k=1}^{K} (1-\lambda_k) \qquad (31) \\ &> 1 - (1-\rho_z) \prod_{k=1}^{K} (1-\lambda_k)^{f_k} \\ &> 1 - (1-\rho_z) \sum\nolimits_{k=1}^{K} f_k (1-\lambda_k) \\ &= \sum\nolimits_{k=1}^{K} f_k (1 - (1-\rho_z)(1-\lambda_k)) = \sum\nolimits_{k=1}^{K} f_k \mathcal{SAC}(q_k) \\ &= 1 - (1-\rho_z)\left(1 - \sum\nolimits_{k=1}^{K} f_k \lambda_k\right) \\ &= 1 - (1-\rho_z)\left(1 - \overline{\lambda}\right) = \mathcal{SAC}\left(q^{1\,\text{sec}}\right). \end{aligned}$$

■

Proposition 3 *Let $\mathcal{V}(q)$ denote the variance of the q_t process. Then:*

$$\mathcal{V}(q) > \mathcal{V}\left(q^{1\,\text{sec}}\right).$$

Proof. We first construct an auxiliary $AR(K+1)$ process, which we denote by $\widetilde{\widetilde{q}}_t$, by dropping the moving average component of the process for the aggregate real exchange rate in the multi-sector economy (**Corollary 1):**

$$(1-\rho_z L)\prod_{k=1}^{K}(1-\lambda_k L)\,\widetilde{\widetilde{q}}_t = \sum\nolimits_{k=1}^{K} f_k \varphi_k u_t.$$

It is clear that $\mathcal{V}(q) > \mathcal{V}\left(\widetilde{\widetilde{q}}\right)$. The next steps will show that $\mathcal{V}\left(\widetilde{\widetilde{q}}\right) > \mathcal{V}\left(q^{1\,\text{sec}}\right)$, and thus establish the result. From **Corollary 2**, recall that the process for $q^{1\,\text{sec}}$ is:

$$(1-\rho_z L)\left(1-\overline{\lambda}L\right) q_t^{1\,\text{sec}} = \varphi u_t, \tag{32}$$

with $\overline{\lambda} \equiv \sum_{k=1}^{K} f_k \lambda_k$ and $\varphi \equiv \overline{\lambda} - \left(1-\overline{\lambda}\right)\frac{\rho_z \beta \overline{\lambda}}{1-\rho_z \beta \overline{\lambda}}$. Since φ_k is convex in λ_k ($\frac{\partial^2 \varphi_k}{\partial \lambda_k^2} = \frac{2\beta\rho(\beta\rho-1)}{(\beta\lambda\rho-1)^3} > 0$), $\sum_{k=1}^{K} f_k \varphi_k > \varphi$. Thus, defining another auxiliary process $\widetilde{q}$ such that:

$$(1-\rho_z L)\prod_{k=1}^{K}(1-\lambda_k L)\,\widetilde{q}_t = \varphi u_t, \tag{33}$$

it suffices to show that $\mathcal{V}(\widetilde{q}) > \mathcal{V}\left(q^{1\,\text{sec}}\right)$. We consider two cases:

i) $\exists\, k' \in \{1,...,K\} \mid \lambda_{k'} = \overline{\lambda}$. Since $\lambda_k \geq 0$ for all k, it is easy to check that $\mathcal{V}(\widetilde{q}) > \mathcal{V}\left(q^{1\,\text{sec}}\right)$.[40] This follows directly from the fact that in the $MA(\infty)$ representation of $\widetilde{q}$, each coefficient is equal to the corresponding coefficient in the $MA(\infty)$ representation of $q^{1\,\text{sec}}$ *plus positive terms* that originate from all the additional λ_k roots, $k \neq k'$.

ii) $\forall\ k \in \{1,...,K\}$, $\lambda_k \neq \overline{\lambda}$. In that case $\exists\, k'' \in \{1,...,K-1\} \mid \lambda_{k''} < \overline{\lambda} < \lambda_{k''+1}$. We construct an auxiliary process $\widetilde{q}^{1\,\text{sec}}$ such that:

$$(1-\rho_z L)(1-\lambda_{k''+1}L)\,\widetilde{q}_t^{1\,\text{sec}} = \varphi u_t,$$

and note that $\mathcal{V}\left(\widetilde{q}^{1\,\text{sec}}\right) = \frac{1+\rho\lambda_{k''+1}}{(1-\rho^2)(1-\rho\lambda_{k''+1})\left(1-\lambda_{k''+1}^2\right)} > \frac{1+\rho\overline{\lambda}}{(1-\rho^2)(1-\rho\overline{\lambda})\left(1-\overline{\lambda}^2\right)} = \mathcal{V}\left(q^{1\,\text{sec}}\right)$. Thus, the same argument as in case i) shows that $\mathcal{V}(\widetilde{q}) > \mathcal{V}\left(\widetilde{q}^{1\,\text{sec}}\right)$. This completes the proof. ■

[40] The strict inequality comes from the fact that, as long as prices are sticky in at least one sector, for the average frequency of price changes to be equal to the frequency in one of the sectors, there must be at least one more sector in which prices are sticky.

Proposition 4 *Under the simplified model of Section II, for the measures of persistence $\mathcal{P} = \mathcal{CIR}$, $\mathcal{LAR}$:*

$$\begin{aligned} \textit{aggregation effect under } \mathcal{P} &> 0, \\ \textit{counterfactuality effect under } \mathcal{P} &> 0. \end{aligned}$$

Proof. The proof is a by-product of the proof of **Proposition 2**, equations (29) and (30). ■

Proposition 5 *Under the assumptions of Section II and equal sectoral weights, application of the Mean Group estimator to the sectoral real exchange rates from the multi-sector world economy yields the dynamics of the real exchange rate in the corresponding counterfactual one-sector world economy.*

Proof. From **Proposition 1** sectoral exchange rates follow $AR(2)$ processes:

$$q_{k,t} = (\rho_z + \lambda_k)\, q_{k,t-1} - \rho_z \lambda_k q_{k,t-2} + \varphi_k u_t.$$

Applying the MG estimator to these processes yields $\rho_z + \frac{1}{K}\sum_{k=1}^{K} \lambda_k$ as the cross-sectional average of the first autoregressive coefficients, and $-\rho_z \frac{1}{K}\sum_{k=1}^{K} \lambda_k$ as the cross-sectional average of the second autoregressive coefficients. An application of **Corollary 2** to the case of equal sectoral weights shows that these are exactly the autoregressive coefficients of the $AR(2)$ process followed by the aggregate real exchange rate in the corresponding counterfactual one-sector world economy.

■

Lemma 1 *The measures of persistence $\mathcal{P} = \mathcal{CIR}$,$\mathcal{LAR}$,$\mathcal{SAC}$ are (weakly) increasing in the degree of sectoral price rigidity:*

$$\frac{\partial \mathcal{CIR}(q_k)}{\partial \lambda_k} > 0, \frac{\partial \mathcal{LAR}(q_k)}{\partial \lambda_k} \geq 0, \frac{\partial \mathcal{SAC}(q_k)}{\partial \lambda_k} > 0,$$

Moreover, the variance of sectoral real exchange rates is increasing in the degree of sectoral price *rigidity:*

$$\frac{\partial \mathcal{V}(q_k)}{\partial \lambda_k} > 0.$$

Proof. From the proof of **Proposition 2**, $\mathcal{CIR}(q_k) = \frac{1}{(1-\lambda_k)(1-\rho_z)}$, and $\mathcal{LAR}(q_k) = \max\{\lambda_k, \rho_z\}$. From **Proposition 1**, $\mathcal{SAC}(q_k) = \rho_z + \lambda_k - \rho_z\lambda_k$. Direct differentiation of these three expressions with respect to λ_k proves the first part of the lemma. As for the variance of sectoral real exchange rates, standard time-series calculations yield:

$$\mathcal{V}(q_k) = \underbrace{\frac{1+\rho_z\lambda_k}{(1-\rho_z\lambda_k)\left((1+\rho_z\lambda_k)^2 - (\rho_z+\lambda_k)^2\right)}}_{\mathcal{V}_1(q_k)} \underbrace{\left(\lambda_k - (1-\lambda_k)\frac{\rho_z\beta\lambda_k}{1-\rho_z\beta\lambda_k}\right)^2}_{\mathcal{V}_2(q_k)} \sigma^2_{\varepsilon_z}.$$

Differentiating each of $\mathcal{V}_1(q_k)$ and $\mathcal{V}_2(q_k)$ with respect to λ_k yields:

$$\begin{aligned}
\frac{\partial \mathcal{V}_1(q_k)}{\partial \lambda_k} &= 2\frac{\rho_z\lambda_k^2 + \rho_z^2\lambda_k^3 - \rho_z - \lambda_k}{\left(\lambda_k^2-1\right)^2(\rho_z\lambda_k-1)^2(\rho_z^2-1)} > 0 \\
\frac{\partial \mathcal{V}_2(q_k)}{\partial \lambda_k} &= -2\frac{\lambda_k(\rho_z-1)^2}{(\rho_z\lambda_k-1)^3} > 0.
\end{aligned}$$

Since $\mathcal{V}_1(q_k), \mathcal{V}_2(q_k) > 0$, application of the product rule yields:

$$\frac{\partial \mathcal{V}(q_k)}{\partial \lambda_k} = \left(\frac{\partial \mathcal{V}_1(q_k)}{\partial \lambda_k}\mathcal{V}_2(q_k) + \frac{\partial \mathcal{V}_2(q_k)}{\partial \lambda_k}\mathcal{V}_1(q_k)\right)\sigma^2_{\varepsilon_z} > 0.$$

■

B A limiting result

We show that a "suitably heterogeneous" multi-sector world economy can generate an aggregate real exchange rate that is arbitrarily more volatile and persistent than the real exchange rate in the counterfactual one-sector world economy.[41] We consider the effects of progressively adding more sectors, and assume that the frequency of price changes for each new sector is drawn from

[41]We build on the work of Granger (1980), Granger and Roselyne Joyeux (1980), and Zaffaroni (2004).

$(0, 1-\delta)$ for arbitrarily small $\delta > 0$, according to some distribution with density $g(\alpha|b)$, where α is the frequency of price changes and $b > 0$ is a parameter. For $\alpha \approx 0$ such density is assumed to be approximately proportional to α^{-b}, with $b \in \left(\frac{1}{2}, 1\right)$.[42] The shape of this distribution away from zero need not be specified. It yields a strictly positive average frequency of price changes: $\overline{\alpha} = \int_0^{1-\delta} g(\alpha|b)\, \alpha d\alpha > 0$. We prove the following:

Proposition 6 *Under the assumptions above:*

$$\mathcal{V}\left(\frac{1}{K}\sum\nolimits_{k=1}^{K} q_{k,t}\right) \underset{K\to\infty}{\longrightarrow} \infty,$$
$$CIR\left(\frac{1}{K}\sum\nolimits_{k=1}^{K} q_{k,t}\right) \underset{K\to\infty}{\longrightarrow} \infty,$$
$$\mathcal{V}\left(q^{1\,\text{sec}}\right), CIR\left(q^{1\,\text{sec}}\right) < \infty.$$

Proof. We start with the case of $\rho_z = 0$. For each $q_{k,t}$ process $(1-\lambda_k L)\, q_{k,t} = \varphi_k u_t$ with $\varphi_k \equiv \lambda_k$ and $\alpha_k = 1-\lambda_k$ drawn from $g(\alpha|b)$, define an auxiliary $\widetilde{q}_{k,t}$ process satisfying:

$$(1-\lambda_k L)\, \widetilde{q}_{k,t} = \widetilde{\varphi} u_t,$$

where $\widetilde{\varphi} < \delta$ is a constant. Since $\widetilde{\varphi}$ is independent of λ_k, these $\widetilde{q}_{k,t}$ processes satisfy the assumptions in Zaffaroni (2004), and application of his Theorem 4 yields:

$$\mathcal{V}\left(\frac{1}{K}\sum\nolimits_{k=1}^{K} \widetilde{q}_{k,t}\right) \underset{K\to\infty}{\longrightarrow} \infty.$$

Since the α_k's have support $(0, 1-\delta)$ for small $\delta > 0$, $\mathcal{V}\left(\frac{1}{K}\sum_{k=1}^{K} q_{k,t}\right) > \mathcal{V}\left(\frac{1}{K}\sum_{k=1}^{K} \widetilde{q}_{k,t}\right)$, which proves that $\mathcal{V}\left(\frac{1}{K}\sum_{k=1}^{K} q_{k,t}\right) \underset{K\to\infty}{\longrightarrow} \infty$. Analogously, application of Zaffaroni's (2004) result to the spectral density of the limiting process at frequency zero shows that it is unbounded. In turn, the fact that the spectral density at frequency zero for $AR(p)$ processes is an increasing monotonic transformation of the cumulative impulse response (e.g., Donald W.K. Andrews and Hong-Yuan

[42] Thus, we approximate a large number of potential new sectors by a continuum, and replace the general f_k distribution by this semi-parametric specification for $g(\alpha|b)$, based on Zaffaroni (2004). An example of a parametric distribution that satisfies this restriction is a Beta distribution with suitably chosen support and parameters.

Chen, 1994) implies that $CIR\left(\frac{1}{K}\sum_{k=1}^{K} q_{k,t}\right) \underset{K\rightarrow\infty}{\longrightarrow} \infty$. The results for the real exchange rate in the limiting counterfactual one-sector world economy follow directly from the fact that $\overline{\alpha} = \int_0^{1-\delta} g(\alpha|b)\,\alpha d\alpha > 0$, so that it follows a stationary $AR(1)$ process. Finally, Zaffaroni's (2004) extension of his results to $ARMA(p,q)$ processes implies that **Proposition 6** also holds for $\rho_z > 0$.

■

The results in **Proposition 6** follow from the fact that, under suitable assumptions, the aggregate real exchange rate converges to a non-stationary process. It inherits some features of unit-root processes, such as infinite variance and persistence, due to the relatively high density of very persistent sectoral real exchange rates embedded in the distributional assumption for the frequencies of price changes. However, the process does not have a unit root, since none of the sectoral exchange rates actually has one. Moreover, the limiting process remains mean reverting in the sense that its impulse response function converges to zero as $t \longrightarrow \infty$.[43] In contrast, since $\overline{\alpha} > 0$, the limiting process for the real exchange rate in the counterfactual one-sector world economy remains stationary, and as such it has both finite variance and persistence.

C Robustness

C.1 Strategic neutrality in price setting

As a first robustness check of our parameterization, we redo the quantitative analysis imposing the restrictions on parameter values that underscore our analytical results from Section II.[44] That is, we look at the quantitative implications of our model in the case of strategic neutrality in price setting.

The outcomes of the models are summarized in Table 6. Note that in this case the results are exact, since we know the processes followed by each of the variables from **Proposition 1,** and

[43]Such properties characterize the so-called *fractionally integrated processes*. See, for example, Granger and Joyeux (1980).

[44]Recall that these are $\sigma = 1$, $\gamma = 0$, and $\chi = 1$. Under these assumptions, the additional structural parameters have no effect on the dynamics of real exchange rates.

Corollaries 1 and **2**.[45] Despite the change in the parameterization, the essence of our results is not affected: the aggregate real exchange rate in the heterogeneous economy is still more volatile and persistent than in the counterfactual one-sector world economy.

Table 6: Results under Strategic Neutrality in Price Setting

Persistence measures:	$\mathcal{P}(q)$	$\mathcal{P}\left(q^{1\,\mathrm{sec}}\right)$
$\mathcal{CIR}$	79.8	23.7
$\mathcal{SAC}$	$\simeq 1$	0.96
$\mathcal{LAR}$	.98	0.80
ρ_1	0.99	0.97
$\mathcal{HL}$	44	16
$\mathcal{UL}$	29	10
$\mathcal{QL}$	59	20
Volatility measure:	$\mathcal{V}(q)^{1/2}$	$\mathcal{V}\left(q^{1\,\mathrm{sec}}\right)^{1/2}$
	0.03	0.01

C.2 Interest-rate rule and different shocks

We consider a specification with an explicit description of monetary policy, and later also add productivity shocks. We assume that in each country monetary policy is conducted according to an interest-rate rule subject to persistent shocks:

$$I_t = \beta^{-1} \left(\frac{P_t}{P_{t-1}} \right)^{\phi_\pi} \left(\frac{GDP_t}{GDP_t^n} \right)^{\phi_Y} e^{\upsilon_t},$$

where I_t is the short-term nominal interest rate in Home, GDP_t is gross domestic product, GDP_t^n denotes gross domestic product when all prices are flexible, ϕ_π and ϕ_Y are the parameters associated with Taylor-type interest-rate rules, and υ_t is a persistent shock with process $\upsilon_t = \rho_\upsilon \upsilon_{t-1} + \sigma_{\varepsilon_\upsilon} \varepsilon_{\upsilon,t}$, where $\varepsilon_{\upsilon,t}$ is a zero-mean, unit-variance $i.i.d.$ shock, and $\rho_\upsilon \in [0, 1)$. The policy rule in Foreign is analogous, and we assume that the shocks are uncorrelated across countries. We set $\phi_\pi = 1.5$, $\phi_y = .5/12$, and $\rho_\upsilon = 0.965$.[46] The remaining parameter values are unchanged from the baseline

[45]The only exceptions are the first autocorrelation for the aggregate real exchange rate and the volatilities, which for simplicity are calculated through simulations, as outlined in footnote 14 in the paper.

[46]Recall that the parameters are calibrated to the monthly frequency, and so this value for ρ_υ corresponds to an autoregressive coefficient of roughly 0.9 at a quarterly frequency. We specify the size of the shocks to be consistent

specification.

The results are presented in Table 7.[47] The model with heterogeneity still produces a significantly more volatile and persistent real exchange rate than the counterfactual one-sector world economy.

Table 7: Results under interest-rate rule

Persistence measures:	$\mathcal{P}(q)$	$\mathcal{P}\left(q^{1\,\text{sec}}\right)$
$\mathcal{CIR}$	49.6	28.4
$\mathcal{SAC}$	0.98	0.96
$\mathcal{LAR}$	0.96	0.91
ρ_1	0.98	0.96
$\mathcal{HL}$	39	20
$\mathcal{UL}$	14	0
$\mathcal{QL}$	60	39
Volatility measure:	$\mathcal{V}(q)^{1/2}$	$\mathcal{V}\left(q^{1\,\text{sec}}\right)^{1/2}$
	0.07	0.01

We also consider a version of the model with interest-rate and productivity shocks. We introduce the latter by changing the production function in (18) to:

$$Y_{H,k,j,t} + Y^*_{H,k,j,t} = A_t N^{\chi}_{k,j,t},$$

where A_t is a productivity shock. It evolves according to:

$$\log A_t = \rho_A \log A_{t-1} + \sigma_{\varepsilon_A} \varepsilon_{A,t},$$

where $\rho_A \in [0,1)$ and $\varepsilon_{A,t}$ is a zero-mean, unit-variance $i.i.d.$ shock. An analogous process applies to A^*_t, and once more we assume that the shocks are uncorrelated across countries.

We keep the same specification for the monetary policy rule, and set $\rho_A = 0.965$. To determine the relative size of the shocks we rely on the estimates obtained by Alejandro Justiniano, Giorgio Primiceri, and Andrea Tambalotti (2010), and set $\sigma_{\varepsilon_v} = 0.12\%$, and $\sigma_{\varepsilon_A} = 0.52\%$. The

with the estimates of Justiniano et al. (2010), and thus set the standard deviation to 0.2% at a quarterly frequency.

[47] We compute these statistics based on simulations, following the methodology outlined in footnote 14 in the paper.

remaining parameter values are unchanged from the baseline parameterization. The heterogeneous world economy still produces a significantly more volatile and persistent real exchange rate than the counterfactual one-sector world economy. The half-life of the aggregate real exchange rate in the multi-sector world economy is around 33.5 months, while in the counterfactual one-sector world economy it is around 19 months.

We also considered additional parameterizations. We found that the results with shocks to the interest-rate rule and productivity shocks are somewhat more sensitive to the details of the specification than under nominal aggregate demand shocks. On the one hand, they still hold under strategic neutrality in price setting. On the other, they are more sensitive to the source of persistence in the interest-rate rule – persistent shocks versus interest-rate smoothing.[48] Uncovering the reasons for such differences in results is an interesting endeavor for future research. In particular, it would be valuable to investigate the "demand block" of the model further. The reason is that the forward-looking "IS curve" that enters the demand side of the model has only weak empirical support (e.g., Jeffrey Fuhrer and Glenn Rudebusch 2004). Thus, in circumstances in which the model struggles to produce realistic real exchange rate dynamics, this should help us assess whether the problem originates in the nature of price setting – which is the focus of the paper – or in other parts of the model.

C.3 Additional sensitivity analysis

Our findings are robust to changes in the values of the elasticities of substitution between varieties of the intermediate goods, and in the share of imported goods. In particular, departing from our baseline parameterization we analyze the effects of increasing the value of the elasticity of substitution between Home and Foreign goods to as much as 10 (equal to the baseline value for the elasticity of substitution between varieties of the same sector in a given country), and the share of imported inputs to as much as 50%. Despite these extreme values, the half-life of deviations from PPP in the multi-sector model drops only modestly, to 31 months. We also analyze the sensitivity

[48] Chari et al.(2002) find that their sticky-price model fails to generate reasonable business cycle behavior under a policy rule with interest-rate smoothing, in particular in terms of real exchange rate persistence.

of our findings to changes in the time-discount factor β, and find only negligible effects. Finally, our conclusions are also robust to alternative aggregation schemes leading to different numbers of sectors in the heterogeneous economy, as in Carvalho and Nechio (2008).

References

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