Appendix to "Aggregate and Idiosyncratic Risk in a Frictional Labor Market"

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This document contains Appendixes C-F (Appendixes A and B can be found in the paper).

Appendix C: Proofs

The details of the definition of equilibrium are as follows:

Definition C.1. *A competitive search equilibrium consists of, for all $\mathcal{F}_t, t \geq 0$, the price of consumption $p(t)$, value of unemployment $V^u(t)$, and for each entrepreneur: consumption $c_i(t)$, measure of vacancies posted $v_i(t)$, and, whenever $v_i(t) > 0$, the vacancy-applicant ratio $\theta_i(t)$ and contract offered $\sigma_i(t) \in \Sigma(t)$, as well as resulting workforce sizes $n_i(\tau, t)$ for $\tau \leq t$ and economy-wide unemployment $n_u(t) = 1 - \sum_i \int_{\tau \leq t} n_i(\tau, t) d\tau$ such that*

1. *Entrepreneurs optimize: Given the processes for prices $p(t)$ and value of unemployment $V^u(t)$, the allocation $\{c_i(t), v_i(t), \sigma_i(t), \theta_i(t), n_i(\tau, t)\}$ solves the entrepreneur's problem (P) for all $i, \mathcal{F}_t, t \geq 0$.*
2. *Workers optimize: $V^u(t), \sigma_i(t), \theta_i(t)$ satisfy equation (1) for all $i, \mathcal{F}_t, t \geq 0$.*
3. *Goods market clearing: $\sum_i [\int_{-\infty}^t n_i(\tau, t)(z(t) - w_i(\tau, t)) d\tau - \kappa v_i(t) - c_i(t)] = 0$ for all $\mathcal{F}_t, t \geq 0$, where $w_i(\tau, t)$ is specified by $\sigma_i(\tau)$.*
4. *Labor market clearing: $\sum_i v_i(t)/\theta_i(t) = n_u(t)$ for all $\mathcal{F}_t, t \geq 0$.*

Lemma C.1. *If an equilibrium exists, it has the properties:*

1. *All entrepreneurs posting vacancies offer the same contract: $\sigma_i(t) = \sigma(t)$, $\theta_i(t) = \theta(t)$ for all $i, \mathcal{F}_t, t \geq 0$.*
2. *Marginal rates of substitution are equated with relative prices for both entrepreneurs and workers remaining in the same job:*
$$e^{-\rho s} \left(\frac{c_i(t+s)}{c_i(t)} \right)^{-\gamma} = e^{-\rho s} \left(\frac{w(\tau, t+s)}{w(\tau, t)} \right)^{-\gamma} = \frac{p(t+s)}{p(t)},$$
for all $i, 0 \leq \tau \leq t \leq t+s$, and $\mathcal{F}_\tau \subset \mathcal{F}_t \subset \mathcal{F}_{t+s}$.
3. *Posting vacancies yields zero profit.*
4. *Only aggregate measures of vacancies posted and workforce size are determined in equilibrium, not their distribution among entrepreneurs.*

Proof 1) The first part of problem (P1) involves finding the Pareto-frontier of contracts: $\max_{\sigma_i(t)} E_t \int_0^\infty e^{-\delta s} p(t+s)[z(t+s) - w_i(t, t+s)] ds$ s.t. $\int_0^\infty e^{-(\rho+\delta)s}[u(w_i(t, t+s)) + \delta V^u(t+s)] ds \geq V$. With the process for prices $p(t)$ taken as given, the efficient contract must satisfy equation (6), which implies that once one knows the initial wage $w_i(t, t)$ in a contract, future wages are pinned down as $w_i(t, t+s) = w_i(t, t)(e^{\rho s} \frac{p(t+s)}{p(t)})^{-\frac{1}{\gamma}}$ by prices.

The second part of problem (P1) involves choosing an optimal contract on the Pareto-frontier. Using the shorthand $w_i(t, t+s) = w_i(t, t) f(t, t+s)$ for the above relationship, $\hat{w}$ to denote the starting wage and $\hat{\theta}$ the corresponding

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vacancy-applicant ratio, problem (P1) can be written as

$$\max_{\hat\theta,\hat w} - p(t)\kappa + q(\hat\theta)E_t\int_0^\infty e^{-\delta s}p(t+s)[z(t+s)-\hat w f(t,t+s)]ds$$

$$\text{s.t. } \rho V^u(t) = u(b) + \mu(\hat\theta)\big[E_t\int_0^\infty e^{-(\rho+\delta)s}[u\big(\hat w f(t,t+s)\big)+\delta V^u(t+s)]ds - V^u(t)\big] + \eta\big[E_{t,+}V^u(t)-V^u(t)\big] + \frac{d}{dt}V^u(t).$$

The constraint implicitly defines $\hat\theta$ as a function of $\hat w$. Defining the utility gain to a worker from a new contract as $g(\hat w,t) := E_t\int_0^\infty e^{-(\rho+\delta)s}[u\big(\hat w f(t,t+s)\big)+\delta V^u(t+s)]ds - V^u(t)$, a strictly increasing function of the wage $\hat w$, the constraint reads $\rho V^u(t) = u(b)+\mu\big(\hat\theta\big)g(\hat w,t)+\eta\big[E_{t,+}V^u(t)-V^u(t)\big]+\frac{d}{dt}V^u(t)$.

Substituting the implicitly defined $\hat\theta$ into the objective and differentiating w.r.t $\hat w$ gives

$$\bar V(t)^{-\frac{\alpha}{1-\alpha}}g(\hat w,t)^{\frac{\alpha}{1-\alpha}}p(t)h(\hat w,t)\times\big[\frac{\alpha}{1-\alpha}\frac{g_w(\hat w,t)}{g(\hat w,t)} - \frac{1}{h(\hat w,t)}E_t\int_0^\infty e^{-\delta s}\frac{p(t+s)}{p(t)}f(t,t+s)ds\big],$$

where $h(\hat w,t) := \int_0^\infty e^{-\delta s}\frac{p(t+s)}{p(t)}[z(t+s)-\hat w f(t,t+s)]ds$ is the present value of profit to the entrepreneur, a strictly decreasing function of the wage $\hat w$, and $\bar V(t) := \rho V^u(t) - u(b) - \eta[E_{t,+}V^u(t)-V^u(t)] - \frac{d}{dt}V^u(t)$. For an equilibrium contract we must have $g(\hat w,t)>0$ and $h(\hat w,t)>0$, and hence also $\bar V(t)>0$.

With algebra, the derivative can be written as

$$\bar V(t)^{-\frac{\alpha}{1-\alpha}}g(\hat w,t)^{\frac{\alpha}{1-\alpha}}p(t)h(\hat w,t)E_t\int_0^\infty e^{-\delta s}\frac{p(t+s)}{p(t)}f(t,t+s)ds\times\big[\frac{\alpha}{1-\alpha}\frac{u'(\hat w)}{g(\hat w,t)} - \frac{1}{h(\hat w,t)}\big],$$

where the term in brackets is strictly decreasing in $\hat w$. Consider the interval for $\hat w$ where both $g(\hat w,t)>0$, so the wage is high enough that the worker gains from the contract, but also $h(\hat w,t)>0$, so the wage is low enough that the present value of profits to the entrepreneur is positive. As $\hat w$ approaches the lower end of this interval, the worker's percentage gain from a wage increase, $\frac{g_w(\hat w,t)}{g(\hat w,t)}$, becomes increasingly large, making the derivative positive. As $\hat w$ approaches the upper end of the interval, the present value of profits, $h(\hat w,t)$, approaches zero, making the derivative negative. The unique point where the term equals zero is characterized by equation (7).

2) Because the entrepreneur faces complete markets, part of the entrepreneur problem (P) is a standard consumption problem: $\max_{c_i(t)} E_0\int_0^\infty e^{-\rho t}u(c_i(t))dt$ s.t. $E_0\int_0^\infty p(t)c_i(t)dt = W_i,$. Optimality requires that marginal rates of substitution are consistent with the relative prices of consumption $e^{-\rho s}\frac{u'(c_i(t+s))}{u'(c_i(t))} = \frac{p(t+s)}{p(t)}$ for all $0\le t\le t+s$ and continuations $\mathcal{F}_{t+s}$ of $\mathcal{F}_t$.

3) If a contract is offered in equilibrium, optimality requires that posting it has to yield zero profit.

4) Because of the linearity of the production technology and vacancy creation costs, vacancy creation is not pinned down on the entrepreneur level.

Proof of Proposition 1 ***(Aggregation)*** 1) Aggregating up the multi-agent problem allocations for c_i, v_i, n_i along with the economy-wide θ,σ one obtains a feasible candidate solution to the representative entrepreneur's problem with initial conditions $n(\tau,0) := \sum_i n_i(\tau,0)$, $\sigma(\tau) := \{w(\tau,t)\forall t\ge\tau\}$, where $w(\tau,t) := \frac{1}{n(\tau,0)}\sum_i n_i(\tau,0)w_i(\tau,t)$ and $W_0 := 0$. This candidate solution satisfies the optimality conditions and other equilibrium conditions of the representative entrepreneur equilibrium.

2) The individual initial conditions $\{\sigma_i(\tau), n_i(\tau,0), W_i^0, \forall\tau<0, i=1\dots N\}$ must i) integrate up to the initial conditions of the representative entrepreneur equilibrium, and ii) satisfy $E_0\int_0^\infty p(t)\big[\int_{-\infty}^0 n_i(\tau,t)[z(t)-w_i(\tau,t)]d\tau - \kappa v_i(t)\big]dt + W_i^0 > 0$ for all i.

Each entrepreneur in the multi-agent problem consumes a constant fraction of the aggregate over time: $c_i(t) = d_i c(t)$. The budget constraint pins down the individual d_i according to wealth and income: $d_iE_0\int_0^\infty p(t)c(t)dt = W_i^0 + E_0\int_0^\infty\int_{-\infty}^0 p(t)n_i(\tau,t)[z(t)-w_i(\tau,t)]d\tau dt$. Given a representative entrepreneur equilibrium and individual initial conditions, one can then back out an equilibrium for the multi-agent case.

Lemma C.2. ***(Characterization)***

1. *If a representative-entrepreneur equilibrium exists, then it satisfies the following for all $0\le t\le t+s$ and information sets:*
   (a) *Wage contract form: $\sigma(t)$ takes the form $w(t,t+s) = a(t)c(t+s)$ for any continuation history $\mathcal{F}_{t+s}$ of $\mathcal{F}_t$*

(b) *Optimal wage level:* $1/F(t) = \alpha/(1-\alpha) \times u'(w(t,t))/(V(t) - V^u(t))$

(c) *Zero profit:* $\kappa = q\big(\theta(t)\big)F(t)$

(d) *Resource constraint:* $c(t) = \int_{-\infty}^{t} n(\tau,t)[z(t) - w(\tau,t)]d\tau - \kappa\theta(t)n_u(t)$

(e) *Value of unemployment:*
$\rho V^u(t) = u(b) + \mu\big(\theta(t)\big)\big[V(t) - V^u(t)\big] + \eta\big[E_{t,+}V^u(t) - V^u(t)\big] + \dot{V}^u(t)$

(f) *Law of motion for unemployment:* $\dot{n}_u(t) = -\mu\big(\theta(t)\big)n_u(t) + \delta(1 - n_u(t))$

where $F(t) := E_t \int_0^\infty e^{-(\rho+\delta)s}u'(c(t+s))/u'(c(t))[z(t+s) - w(t,t+s)]ds$ *is the present value of profits, and* $V(t) := E_t \int_0^\infty e^{-(\rho+\delta)s}[u(w(t,t+s)) + \delta V^u(t+s)]ds$ *the present value of utility, from a new contract at time* t.

2. *If* $V^u(t), \theta(t) > 0, \sigma(t), n(t,t+s), n_u(t), c(t) > 0$, *for all* $0 \le t \le t+s$, $\mathcal{F}_t \subset \mathcal{F}_{t+s}$ *satisfy conditions 1)-6) above, then they correspond to a unique representative-entrepreneur equilibrium (up to price level scaling), with* $v(t) := \theta(t)n_u(t)$ *and* $p(t+s)/p(t) := e^{-\rho s}u'(c(t+s))/u'(c(t))$ *(for the corresponding continuation histories).*

Proof 1) Assume a representative entrepreneur equilibrium exists. a) This contract form follows from Lemma C.1, part 2), which states that the consumption growth of the entrepreneur and worker are equal as long as the worker remains employed. b) This is equation (7). c) Lemma C.1, part 3). d) This combines the goods market clearing condition with the labor market clearing condition from the definition of equilibrium. e) This is implied by the entrepreneur problem. f) This is implied by the hiring and separation rates.

2) Given variables $V^u(t), c(t), \theta(t), \sigma(t), n_u(t)$ satisfying the conditions of Lemma C.2 for all $t \ge 0$, one can use the consumption path $c(t)$ to construct prices for which $c(t), \theta(t), \sigma(t)$ satisfy the optimality conditions of the agent ($\frac{p(t+s)}{p(t)} := e^{-\rho s}\frac{u'(c(t+s))}{u'(c(t))}$) and with $v(t) := \theta(t)n_u(t)$, these are budget feasible for the entrepreneur and clear markets.

Lemma C.3. *The incomplete markets wedge* $\Delta(w;\gamma)$ *is strictly decreasing in* γ *and* w.

Proof To show these properties, consider $\hat{\Delta}(x,\gamma) := b\Delta(w;\gamma) = \frac{x^{-\gamma}}{\frac{[x^{1-\gamma}-1]}{1-\gamma}}[x-1]$, where $x = \frac{w}{b} \ge 1$. To see that $\hat{\Delta}(x,\gamma)$ decreases in γ, note first that $x^{1-\gamma} = e^{(1-\gamma)\ln x} > 1 + (1-\gamma)\ln x$ for all $\gamma \neq 1$. It follows that for $x > 1$, $\frac{d}{d\gamma}ln\hat{\Delta} = -\frac{1}{1-\gamma} - \ln x + \frac{x^{1-\gamma}\ln x}{x^{1-\gamma}-1} = -\frac{1}{1-\gamma} + \frac{\ln x}{x^{1-\gamma}-1} < 0$.

To see that $\hat{\Delta}(x,\gamma)$ decreases in $x > 1$, note first that $\frac{x^{1-\gamma}-1}{1-\gamma} < x - 1$ for $x > 1$, because the left hand side is concave in x. It follows that $\frac{d}{dx}ln\hat{\Delta} = \frac{1}{x-1} - \frac{\gamma}{x} - \frac{(1-\gamma)x^{-\gamma}}{x^{1-\gamma}-1} = \frac{1}{x(x-1)}[1 - (x-1)\frac{1-\gamma}{x^{1-\gamma}-1}] < 0$.

Proof of Lemma 1 1) Reduce the steady-state equations into one equation in the wage:

$$\frac{z-w}{\rho+\delta}[\rho+\delta+k(\frac{z-w}{\rho+\delta}\frac{k}{\kappa})^{\frac{1-\alpha}{\alpha}}] = \frac{1-\alpha}{\alpha}w\ln(w/b).$$

This equation has a unique solution $w \in (b,z)$. The other variables can be expressed as functions of the wage:$\theta = q^{-1}(\frac{\kappa(\rho+\delta)}{z-w})$, $m = \frac{\mu(\theta)}{\mu(\theta)+\delta}$, $c = \frac{\rho\kappa}{q(\theta)+\delta/\theta}$, $a = \Phi = \frac{w}{c}$.
2) Equation (12) determines a: $\frac{Y}{a} = \frac{1-\alpha}{\alpha}(\frac{\ln a}{\rho+\delta} + X) + \frac{1}{\rho+\delta}$. Given that Y is positive, there is a unique strictly positive solution for any X. In steady state $\bar{Y} - \frac{\bar{a}}{\rho+\delta} = \frac{1}{\bar{c}}\frac{z-\bar{w}}{\rho+\delta} > 0$ and hence $Y - \frac{a}{\rho+\delta}$ is positive when Y, X are close to steady state. The equation is continuously differentiable in a, X, Y and the derivative w.r.t a is nonzero, so that according to the implicit function theorem one can solve for a differentiable $\tilde{a}(X,Y)$ with continuous partial derivatives.

From equation (13), we have $c = \frac{\kappa}{q(\theta)(Y - \frac{a(\tilde{X},Y)}{\rho+\delta})}$ and plugging this into equation (11): $c(1+m\Phi) = mz - \kappa\theta(1-m)$, one can show that a unique solution $\theta > 0$ exists. Because the expression is continuously differentiable, and the partial w.r.t. θ is nonzero, there exists a continuously differentiable $\theta(m,\Phi,X,Y)$. This then extends to c by the expression above.

3) The result on the eigenvalues is shown by considering the characteristic polynomial $\pi(x) = \det(A - xI)$, where A is the system matrix. We know that $\pi(x) = (\lambda_1 - x)(\lambda_2 - x)(\lambda_3 - x)(\lambda_4 - x)$. By tedious algebra (see file `eigenvalues.nb`), one can show that $\pi(0) > 0, \pi(\rho) = 0, \pi(\rho+\delta) < 0, \pi(-\delta) < 0$.

Proof of Proposition 2 ***(Unique Equilibrium Path)*** As the system of differential equations is smooth and the eigenvalues of the linearized system satisfy property 3) of Lemma 1, this follows from the theory of dynamic systems.

Proof of Proposition 3 See Appendix D.

Appendix D: Relation to the Mortensen-Pissarides Model

If workers and entrepreneurs both have linear preferences, the model generates the same unemployment dynamics as the Mortensen-Pissarides model. To see this, consider the system given by equations (9), (16)-(22), imposing $\hat{\rho} = \rho$ and $\hat{\gamma} = \gamma$. Then, let $\gamma = 0$.

Defining $S(t) := X^2(t) + Y(t)$, equation (18) implies that $S(t) - (a(t)X^1(t) + X^2(t)) = \frac{1-\alpha}{\alpha}(a(t)X^1(t) + X^2(t))$, or $a(t)X^1(t) + X^2(t) = \alpha S(t)$. Together with the laws of motion for $X^2(t)$ and $Y(t)$, this leads to the law of motion $\dot{S}(t) = b - z + (\rho + \delta + \mu(\theta(t))\alpha)S(t)$. The variable $\theta(t)$ can further be substituted out: equation (19) implies that $\kappa = q(\theta(t))(1-\alpha)S(t)$, which determines $\theta(t)$ as a function of $S(t)$. The resulting law of motion for $S(t)$ is an unstable differential equation, implying that $S(t)$ (and hence also $\theta(t)$) must jump to steady state immediately. Just as in the Mortensen-Pissarides model, the dynamics of employment are thus characterized by the law of motion $\dot{m}(t) = -\delta m(t) + \mu(\theta)(1 - m(t))$, where θ is the steady-state vacancy-unemployment ratio. Moreover, it is easy to see that the value of θ is the same across the two models if the bargaining power of the worker equals α: The steady-state values of S and θ are determined by $\rho S = z - b - \mu(\theta)\alpha S - \delta S$ and $\kappa = q(\theta)(1-\alpha)S$. But with S corresponding to the match surplus, these equations coincide with the steady-state equations of the Mortensen-Pissarides model if the bargaining power of the worker equals α.

To recover the dynamics of $\Phi(t)$ and $a(t)$ – necessary for analyzing wage dynamics – one needs to keep track of an additional variable, for example $X^1(t)$. Notice that because $Y(t)$ also jumps to steady state immediately, so must $X^2(t)$ and $(aX^1)(t)$. We can then characterize the dynamics of $m(t)$ and $\Phi(t)$ with the system

$$\begin{aligned}
\dot{m}(t) &= -\delta m(t) + \mu(\theta)(1 - m(t)) \\
\dot{\Phi}(t) &= \frac{\mu(\theta)(1 - m(t))}{m(t)}\left[\frac{(aX^1)}{X^1(t)} - \Phi(t)\right] \\
\dot{X^1}(t) &= -\frac{m(t)z - \kappa\theta(1 - m(t))}{1 + m(t)\Phi(t)} + (\rho + \delta)X^1(t)
\end{aligned}$$

where θ and (aX^1) are immediately at steady state. The Jacobian evaluated at the steady state is

$$\begin{pmatrix} -\delta - \mu(\theta) & 0 & 0 \\ 0 & -\frac{\mu(\theta)(1-m)}{m} & -\frac{\mu(\theta)(1-m)}{m}\frac{(aX^1)}{(X^1)^2} \\ \dots & c\frac{m}{1+m\Phi} & \rho + \delta \end{pmatrix} = \begin{pmatrix} -\delta - \mu(\theta) & 0 & 0 \\ 0 & -\delta & -\delta(\rho + \delta)\frac{\Phi}{c} \\ \dots & c\frac{m}{1+m\Phi} & \rho + \delta \end{pmatrix},$$

with eigenvalues $-\delta - \mu(\theta)$ and $\frac{1}{2}[\rho \pm \sqrt{\rho^2 + \frac{4\delta(\rho+\delta)}{1+m\Phi}}]$.

With linear preferences and the right bargaining shares, the two models differ only in the dynamics of wages. The difference has no effect on employment, however, because the present value of wages for new workers remains the same across the models.

Appendix E: Solving the Model

This section describes an approach for simulating the model and producing impulse responses. I focus on the log-utility case here, but the same strategy extends to other CRRA preferences.

Calibrating the Productivity Process As discussed in Section 1, aggregate shocks arrive at Poisson rate η and when the shock hits, the value of z jumps. I assume the new value of productivity z', conditional on the prevailing value z, satisfies $z' - \bar{z}|z \sim N(\xi(z - \bar{z}), \sigma_\epsilon^2)$, where the draws are independent, $\xi \in (0, 1)$ and the variance σ_ϵ^2 is small.

The target series for the productivity process is the non-farm business sector labor productivity series reported by the Bureau of Labor Statistics. Taking logs and $HP(10^5)$-filtering, the quarterly series has autocorrelation 0.89 and standard deviation 0.02. The parameters of the productivity process $(\xi, \sigma_\epsilon, \eta)$, are set such that the simulated moments of autocorrelation and standard deviation match the numbers above. The simulated moments for given values of $(\xi, \sigma_\epsilon, \eta)$ are averages over 100 55-year samples of simulated monthly data, where each 55 year data series is first aggregated to quarterly, logged and $HP(10^5)$-filtered, before calculating the two moments. I set $\eta = 0.1$, which leaves ξ and σ_ϵ to be pinned down by the two empirical moments. To match these targets, I use input values $\xi = 0.768$, $\sigma_\epsilon = 0.0191$.

Computing Simulated Moments The simulated moments are computed by first feeding into the model simulated monthly productivity data. Each 55 year data series is first aggregated to quarterly, logged and $HP(10^5)$-filtered, before calculating moments. These moments are then averaged over 100 55-year samples.

Solving the Model I consider equilibria where the state of the economy is fully characterized by the current values of $\psi := (m, \Phi, z)$. With no aggregate shocks we know that if m and Φ are close to steady state, generically there exist unique values of X and Y that guarantee convergence to the steady state. The equilibrium path can thus be represented by $(m, \Phi, \hat{X}(m, \Phi), \hat{Y}(m, \Phi))$, where $\hat{X}(m, \Phi)$ and $\hat{Y}(m, \Phi)$ take on these unique values. Analogously, postulate that for each ψ, there is a unique pair $\hat{X}(\psi)$ and $\hat{Y}(\psi)$ corresponding to equilibrium in the presence of shocks. These $\hat{X}(\psi), \hat{Y}(\psi))$ must satisfy the system (analogously to equations (9)-(15)):

$$
\begin{aligned}
0 =& F(\psi, \hat{X}(\psi), \hat{Y}(\psi)) + \eta E[\hat{X}(\psi') - \hat{X}(\psi)|\psi] + \hat{X}_m(\psi) f(\psi, \hat{X}(\psi), \hat{Y}(\psi)) + \hat{X}_\Phi(\psi) g(\psi, \hat{X}(\psi), \hat{Y}(\psi)), \qquad \text{(E1)}\\
0 =& G(\psi, \hat{X}(\psi), \hat{Y}(\psi)) + \eta E[\hat{Y}(\psi') - \hat{Y}(\psi)|\psi] + \hat{Y}_m(\psi) f(\psi, \hat{X}(\psi), \hat{Y}(\psi)) + \hat{Y}_\Phi(\psi) g(\psi, \hat{X}(\psi), \hat{Y}(\psi)),
\end{aligned}
$$

where $\psi' := (m, \Phi, z')$ and

$$
\begin{aligned}
f(\psi, X, Y) :=& -\delta m + \mu\big(\tilde{\theta}(\psi, X, Y)\big)(1-m),\\
g(\psi, X, Y) :=& \frac{\mu\big(\tilde{\theta}(\psi, X, Y)\big)(1-m)}{m}[\tilde{a}(\psi, X, Y) - \Phi],\\
F(\psi, X, Y) :=& -(\delta+\rho)X + \ln \tilde{c}(\psi, X, Y) - \ln b - \mu\big(\tilde{\theta}(\psi, X, Y)\big)[\frac{\ln \tilde{a}(\psi, X, Y)}{\rho+\delta} + X],\\
G(\psi, X, Y) :=& -(\delta+\rho)Y + \frac{z}{\tilde{c}(\psi, X, Y)}.
\end{aligned}
$$

Here the functions $\tilde{a}$, $\tilde{c}$ and $\tilde{\theta}$ are based on Lemma 1.

Because the labor productivity shocks associated with business cycles are small, I use a linearization approach to solve this system. Denoting the steady state corresponding to average productivity as $\bar{m}, \bar{\phi}, \bar{z}$, we can approximate: $\hat{X}(m, \Phi, z) \approx \bar{X} + \bar{X}_m(m-\bar{m}) + \bar{X}_\Phi(\Phi-\bar{\Phi}) + \bar{X}_z(z-\bar{z})$, and $\hat{Y}(m, \Phi, z) \approx \bar{Y} + \bar{Y}_m(m-\bar{m}) + \bar{Y}_\Phi(\Phi-\bar{\Phi}) + \bar{Y}_z(z-\bar{z})$, where $\bar{X} := \hat{X}(\bar{m}, \bar{\Phi}, \bar{z}), \bar{Y} := \hat{Y}(\bar{m}, \bar{\Phi}, \bar{z})$ and $\bar{X}_m, \bar{X}_\Phi, \bar{X}_z, \bar{Y}_m, \bar{Y}_\Phi, \bar{Y}_z$ are partial derivatives evaluated at the above steady state. One needs to solve for these partial derivatives, as well as the steady state. The derivatives can be found by differentiating the above system with respect to m, Φ and z, and imposing the linear approximation for $\hat{X}$ and $\hat{Y}$. Given the assumptions about the distribution of new productivity levels, we have $E[\hat{X}(m, \Phi, z') - \hat{X}(m, \Phi, z)|m, \Phi, z] \approx \bar{X}_z E[(z'-z)|m, \Phi, z] = \bar{X}_z(\xi-1)(z-\bar{z})$ and similarly for $\hat{Y}$.

Simulating Between aggregate shocks, the linearized system for m and Φ reads:

$$
\begin{aligned}
\dot{m} = \bar{f}_m(m-\bar{m}) + \bar{f}_\Phi(\Phi-\bar{\Phi}) + \bar{f}_z(z-\bar{z}) &+ \bar{f}_X[\bar{X}_m(m-\bar{m}) + \bar{X}_\Phi(\Phi-\bar{\Phi}) + \bar{X}_z(z-\bar{z})] \qquad \text{(E2)}\\
&+ \bar{f}_Y[\bar{Y}_m(m-\bar{m}) + \bar{Y}_\Phi(\Phi-\bar{\Phi}) + \bar{Y}_z(z-\bar{z})]\\
\dot{\Phi} = \bar{g}_m(m-\bar{m}) + \bar{g}_\Phi(\Phi-\bar{\Phi}) + \bar{g}_z(z-\bar{z}) &+ \bar{g}_X[\bar{X}_m(m-\bar{m}) + \bar{X}_\Phi(\Phi-\bar{\Phi}) + \bar{X}_z(z-\bar{z})]\\
&+ \bar{g}_Y[\bar{Y}_m(m-\bar{m}) + \bar{Y}_\Phi(\Phi-\bar{\Phi}) + \bar{Y}_z(z-\bar{z})]
\end{aligned}
$$

As a linear system, this has a closed form solution. Simulating can be done by drawing arrival times from an exponential distribution, drawing new productivity realizations from the appropriate normal distribution and using the closed form solution of the system to calculate paths of $m(t)$, $\Phi(t)$, as well as $a(t), c(t), \theta(t)$.

Impulse Responses Suppose $z(0) \neq \bar{z}$, and define $\hat{z}(t) := E[z(t)|z(0)] - \bar{z}$. We know that $E[z(t+\Delta t) - \bar{z}|z(t)] = \eta\Delta t\xi(z(t)-\bar{z}) + (1-\eta\Delta t)(z(t)-\bar{z})$. Hence $\dot{\hat{z}} = -\eta(1-\xi)\hat{z}$ and $\hat{z}(t) = z(0)e^{-\eta(1-\xi)t}$. To find the responses of endogenous variables $\hat{m}(t) := E[m(t)|z(0)] - \bar{m}$, $\hat{\Phi}(t) := E[\Phi(t)|z(0)] - \bar{\Phi}$, note that the linearized system above yields the linear system

$$
\dot{\begin{pmatrix} m \\ \Phi \end{pmatrix}} = \bar{A}\begin{pmatrix} m-\bar{m} \\ \Phi-\bar{\Phi} \end{pmatrix} + \bar{b}(z-\bar{z}) \Rightarrow \dot{\begin{pmatrix} \hat{m} \\ \hat{\Phi} \end{pmatrix}} = \bar{A}\begin{pmatrix} \hat{m} \\ \hat{\Phi} \end{pmatrix} + \bar{b}\hat{z} \Rightarrow \dot{\begin{pmatrix} \hat{m} \\ \hat{\Phi} \\ \hat{z} \end{pmatrix}} = \begin{pmatrix} \bar{A} & \bar{b} \\ 0 & -\eta(1-\xi) \end{pmatrix}\begin{pmatrix} \hat{m} \\ \hat{\Phi} \\ \hat{z} \end{pmatrix}.
$$

Quality of Linearized Solution In the absence of aggregate shocks, solving the original non-linear problem numerically is relatively straightforward, and comparing the solution of this problem with the corresponding linearization based solution allows examining the accuracy of the linearization approach. Computing a solution to the nonlinear deterministic system involves a shooting problem with two predetermined variables and either two or three jump variables depending on the utility function used for workers. Perturbing the initial conditions $(m(0), \Phi(0))$ from steady state to the steady state of an economy with 2% higher or lower productivity and examining the adjustment paths of the two solutions, I find that the differences are practically indistinguishable to the eye.[1]

[1] I used Matlab's collocation based bvp4c-function with continuation in productivity.

Appendix F: A Complete Markets Economy

First, assume identical preferences across workers and entrepreneurs. Then, consider altering the model environment such that: i) also workers have access to the Arrow-Debreu markets spanning aggregate risk, and ii) workers are able to insure away the idiosyncratic unemployment risk by pooling their labor income.

In such circumstances workers' job search decisions are based on maximizing the present value of labor income, valued at the Arrow-Debreu prices. Defining these prices (probability normalized) as $p(t) := e^{-\rho t}u'(c^A(t))$, where $c^A(t)$ refers to aggregate consumption, the present value of wages to a new worker hired at time t can be written as $W(t) = E_t \int_0^\infty e^{-\delta s} \frac{p(t+s)}{p(t)} w(t+s)ds$. It is convenient to define also $\tilde{W}(t) := u'(c^A(t))W(t)$. The present value of labor income for an unemployed worker $V^u(t)$, or rather $\tilde{V}^u(t) := u'(c^A(t))V^u(t)$, satisfies the Bellman equation

$$\rho\tilde{V}^u(t) = u'(c^A(t))b + \mu(\theta(t))[\tilde{W}(t) + \tilde{X}(t)] + \eta[E_{t,+}\tilde{V}^u(t) - \tilde{V}^u(t)] + \dot{\tilde{V}}^u(t),$$

where $X(t) := E_t \int_0^\infty e^{-\delta s}\delta \frac{p(t+s)}{p(t)} V^u(t+s)ds - V^u(t)$, and $\tilde{X}(t) := u'(c^A(t))X(t)$. This implies also that

$$(\rho+\delta)\tilde{X}(t) = -u'(c^A(t))b - \mu(\theta(t))[\tilde{W}(t) + \tilde{X}(t)] + \eta[E_{t,+}\tilde{X}(t) - \tilde{X}(t)] + \dot{\tilde{X}}(t).$$

Entrepreneurs' contract posting decisions are based on maximizing the value of a vacancy. The present value of output from a new worker hired at time t can be written as $Z(t) := E_t \int_0^\infty e^{-\delta s} \frac{p(t+s)}{p(t)} z(t+s)ds$, with $\tilde{Z}(t) := u'(c^A(t))Z(t)$. The contract posting problem (corresponding to problem (P1)) then reads

$$\begin{aligned} &\max_{\theta(t),\tilde{W}(t)} q(\theta(t))[\tilde{Z}(t) - \tilde{W}(t)] \text{ s.t.} \\ &\rho\tilde{V}^u(t) = u'(c^A(t))b + \mu(\theta(t))[\tilde{W}(t) + \tilde{X}(t)] + \eta[E_{t,+}\tilde{V}^u(t) - \tilde{V}^u(t)] + \dot{\tilde{V}}^u(t). \end{aligned}$$

The optimum is characterized by $\tilde{Z}(t) - \tilde{W}(t) = \frac{1-\alpha}{\alpha}[\tilde{W}(t) + \tilde{X}(t)]$, while the zero profit condition on vacancy-posting reads $q(\theta(t))\frac{[\tilde{Z}(t)-\tilde{W}(t)]}{u'(c^A(t))} = \kappa$. Finally, aggregate consumption satisfies $c^A(t) = m(t)z(t) + (1-m(t))b - \kappa\theta(t)(1-m(t))$. This yields a dynamic system characterizing equilibrium

$$\begin{aligned} \dot{m}(t) &= -\delta m(t) + \mu(\theta(t))(1-m(t)), \\ \dot{\tilde{X}}(t) &= (\rho+\delta)\tilde{X}(t) + u'(c^A(t))b + \mu(\theta(t))[\tilde{W}(t) + \tilde{X}(t)] - \eta[E_{t,+}\tilde{X}(t) - \tilde{X}(t)], \\ \dot{\tilde{Z}}(t) &= (\rho+\delta)\tilde{Z}(t) - u'(c^A(t))z(t) - \eta[E_{t,+}\tilde{Z}(t) - \tilde{Z}(t)], \end{aligned}$$

where $c^A(t)$, $\theta(t)$ and $\tilde{W}(t)$ are determined by $c^A(t) = m(t)z(t) + (1-m(t))b - \kappa\theta(t)(1-m(t))$, $\tilde{Z}(t) - \tilde{W}(t) = \frac{1-\alpha}{\alpha}[\tilde{W}(t) + \tilde{X}(t)]$, and $q(\theta(t))\frac{[\tilde{Z}(t)-\tilde{W}(t)]}{u'(c^A(t))} = \kappa$.

To simplify, one can further define $\tilde{S}(t) := \tilde{X}(t) + \tilde{Z}(t)$ to arrive at the system

$$\begin{aligned} \dot{m}(t) &= -\delta m(t) + \mu(\theta(t))(1-m(t)), \\ \dot{\tilde{S}}(t) &= u'(c^A(t))(b - z(t)) + (\rho + \delta + \mu(\theta(t))\alpha)\tilde{S}(t) - \eta[E_{t,+}\tilde{S}(t) - \tilde{S}(t)], \end{aligned}$$

where $c^A(t)$ and $\theta(t)$ are determined by $c^A(t) = m(t)z(t) + (1-m(t))b - \kappa\theta(t)(1-m(t))$, and $q(\theta(t))(1-\alpha)\frac{\tilde{S}(t)}{u'(c^A(t))} = \kappa$.

The timing of wage payments is not pinned down in this complete markets environment. To derive wage implications, one possibility is to impose that all workers are paid the same wage at each instant.[2] Under this assumption, the present value of wages of existing workers equals that of newly hired workers, $W(t)$, at each instant. Given the path of $W(t)$, the wage must then satisfy

$$(\rho+\delta)\tilde{W}(t) = u'(c^A(t))w(t) + \eta[E_{t,+}\tilde{W}(t) - \tilde{W}(t)] + \dot{\tilde{W}}(t).$$

Using $\tilde{W}(t) = \tilde{Z}(t) - (1-\alpha)\tilde{S}(t)$ and the laws of motion we arrive at $w(t) = (1-\alpha)b + \alpha(z(t) + \kappa\theta(t))$.

[2] This corresponds to wages being determined by continual rebargaining, if the worker's share of the surplus equals the matching function elasticity.