Web Appendix for "Expectations, Learning and Business Cycle Fluctuations"

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This document provides additional calculations and results for the paper.

1 Model

This section shows the solution of the model that includes capacity utilization, non-separability between consumption and leisure and externalities of production. Consumers choose consumption, leisure and capital in order to maximize

$$E_t \sum_{T=t}^{\infty} \beta^{T-t} u\left(C_T, L_T\right)$$

subject to

$$\begin{aligned} C_t + K_{t+1} &= R_t^K \left(u_t K_t\right) + W_t H_t + \left(1 - \delta\left(U_t\right)\right) K_t \\ L_t &= 1 - H_t. \end{aligned}$$

The first-order conditions are

$$\begin{aligned} C_t &: u_c(C_t, L_t) = \Lambda_t \\ K_{t+1} &: \beta \hat{E}_t \Lambda_{t+1} R_{t+1}^K u_{t+1} - \Lambda_t + \beta \hat{E}_t \left[\Lambda_{t+1}\left(1 - \delta\left(U_{t+1}\right)\right)\right] = 0 \\ L_t &: u_L(C_t, L_t) = \Lambda_t W_t \\ U_t &: R_t^K = \delta'\left(U_t\right). \end{aligned}$$

In the sequel we assume

$$u\left(C_t, L_t\right) = \frac{C_t^{1-\sigma} v\left(1 - L_t\right)}{1 - \sigma}$$

where $\nu'\left(\bar{H}\right) > 0$, ($\bar{H}$ is steady state hours worked) and $\epsilon_\nu = \nu'' \bar{H} / \nu' > 0$. Also, we assume

$$\delta\left(U_t\right) = \frac{1}{\theta} U_t^{\theta}.$$

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Non-stationary variables (expressed in efficiency units) are denoted in lower case letters. Stationary variables are left unchanged. Hence, for any trending variable G_t define $g_t = G_t/X_t$ as the corresponding variable in efficiency units, where X_t is the level of technology in period t described further below. The model is then studied in log deviation from a non-stochastic steady state in these normalized variables so that $\hat{g}_t = \ln(g_t/\bar{g})$, with $\bar{g}$ denoting the steady state value of g_t.

1.1 Households

In terms of normalized variables the first-order conditions are as follows. For consumption:

$$\lambda_t \equiv X_t^\sigma \Lambda_t = X_t^\sigma u_c(C_t, L_t) = X_t^\sigma C_t^{-\sigma} v(H_t) = c_t^{-\sigma} v(H_t).$$

For capital:

$$\begin{aligned}
1 &= \beta \hat{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1}^K U_{t+1} + \frac{\Lambda_{t+1}}{\Lambda_t} (1 - \delta(U_{t+1})) \right] \\
&= \beta \hat{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \frac{X_{t+1}^\sigma}{X_{t+1}^\sigma} \frac{X_t^\sigma}{X_t^\sigma} R_{t+1}^K U_{t+1} + \frac{\Lambda_{t+1}}{\Lambda_t} \frac{X_{t+1}^\sigma}{X_{t+1}^\sigma} \frac{X_t^\sigma}{X_t^\sigma} (1 - \delta(U_{t+1})) \right] \\
&= \beta \hat{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\gamma_{t+1}^\sigma} R_{t+1}^K U_{t+1} + \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\gamma_{t+1}^\sigma} (1 - \delta(U_{t+1})) \right] \\
& \quad \beta \hat{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\gamma_{t+1}^\sigma} \left(R_{t+1}^K U_{t+1} + (1 - \delta(U_{t+1})) \right) \right]
\end{aligned}$$

For Leisure:

$$\lambda_t w_t = X_t^{\sigma-1} \Lambda_t W_t = X_t^{\sigma-1} u_L(C_t, L_t) = X_t^{\sigma-1} \frac{C_t^{1-\sigma}}{1-\sigma} \frac{v'(H_t)}{v(H_t)} = \frac{c_t^{1-\sigma}}{1-\sigma} \frac{v'(H_t)}{v(H_t)}.$$

A log-linear approximation to these relations around a balanced growth path provides:

1. Marginal utility of consumption:

$$\hat{\lambda}_t = -\sigma \hat{c}_t - \psi (1 - \sigma) \hat{H}_t$$

where in steady state

$$\psi \equiv \frac{\bar{H} v'(\bar{H})}{v(\bar{H})} (1 - \sigma)^{-1} = \frac{\bar{w}\bar{H}}{\bar{c}}.$$

2. Euler equation:

$$\beta \hat{E}_t \left[\beta^{-1} \left(\hat{\lambda}_{t+1} - \hat{\lambda}_t - \sigma \hat{\gamma}_{t+1} \right) + \left(\beta^{-1} - \frac{(1-\delta)}{\bar{\gamma}^\sigma} \right) \left(\hat{R}_{t+1}^K + \hat{U}_{t+1} \right) - \frac{\delta}{\bar{\gamma}^\sigma} \theta \hat{U}_{t+1} \right] = 0$$

which on using the steady-state relation

$$\frac{\bar{R}^K \bar{U}}{\bar{\gamma}^\sigma} = \left(\beta^{-1} - \frac{(1-\delta)}{\bar{\gamma}^\sigma} \right) = \frac{\theta \delta}{\bar{\gamma}^\sigma}$$

becomes

$$\beta \hat{E}_t \left[\beta^{-1} \left(\hat{\lambda}_{t+1} - \hat{\lambda}_t - \sigma \hat{\gamma}_{t+1} \right) + \left(\beta^{-1} - \frac{(1-\delta)}{\gamma^\sigma} \right) \hat{R}_{t+1}^K \right] = 0.$$

3. Labor-leisure choice:

$$(1-\sigma)\,\hat{c}_t + \epsilon_\nu \hat{H}_t = \hat{\lambda}_t + \hat{w}_t,$$

which, combined with the expression for marginal utility, gives:

$$\sigma^{-1}\hat{\lambda}_t + \hat{w}_t = \epsilon_H \hat{H}_t$$

where

$$\epsilon_H = \epsilon_\nu - \frac{(\sigma-1)^2}{\sigma}\psi > 0$$

is the inverse Frisch elasticity of labor supply.[1]

4. Capacity utilization:

$$\hat{U}_t = \frac{1}{(\theta-1)}\hat{R}^K_t.$$

Using the expressions for capital utilization and the marginal utility of consumption, the Euler equation can be expressed in familiar form

$$\begin{aligned} 0 &= \beta \hat{E}_t \left[\beta^{-1}\left(\hat{\lambda}_{t+1} - \hat{\lambda}_t - \sigma\hat{\gamma}_{t+1}\right) + \left(\beta^{-1} - \frac{(1-\delta)}{\gamma^\sigma}\right)\hat{R}^K_{t+1}\right] \\ &= \hat{E}_t\left[\left(\hat{\lambda}_{t+1} - \hat{\lambda}_t - \sigma\hat{\gamma}_{t+1}\right) + \beta\left(\beta^{-1} - \frac{(1-\delta)}{\gamma^\sigma}\right)\hat{R}^K_{t+1}\right]. \end{aligned}$$

Rearranging provides

$$\begin{aligned} \hat{\lambda}_t &= \hat{E}_t\left[\left(\hat{\lambda}_{t+1} - \sigma\hat{\gamma}_{t+1}\right) + \beta\bar{R}\hat{R}^K_{t+1}\right] = \\ -\sigma\hat{c}_t - \psi(1-\sigma)\hat{H}_t &= \hat{E}_t\left[-\sigma\hat{c}_{t+1} - \psi(1-\sigma)\hat{H}_{t+1}\right] - \sigma\hat{E}_t\hat{\gamma}_{t+1} + \hat{E}_t\beta\bar{R}\hat{R}^K_{t+1} = \\ \hat{Q}_t &= -\hat{E}_t\beta\bar{R}\left(\hat{R}^K_{t+1}\right) + \hat{E}_t\hat{Q}_{t+1} + \sigma\hat{E}_t\hat{\gamma}_{t+1} \end{aligned}$$

where

$$\bar{R} = \left(\beta^{-1} - \frac{(1-\delta)}{\bar{\gamma}^\sigma}\right)$$

and

$$\hat{Q}_t = \sigma\hat{c}_t + \psi(1-\sigma)\hat{H}_t.$$

1.2 Firms

The firms's problem is

$$\max_{u_t K_t, H_t} Y_T - W_T H_T - R^K_T(U_t K_t)$$

subject to the production technology

$$Y_t = \Psi_t (U_t K_t)^\alpha (X_t H_t)^{1-\alpha}$$

where

$$\Psi_t = \left[(U_t K_t)^\alpha (X_t H_t)^{1-\alpha}\right]^\eta \; X_t^{-\eta}$$

[1] The restriction $\epsilon_N > 0$ guarantees the concavity of the utility function.

denotes the external effects of aggregate capital. The term $X_t^{-\eta}$ guarantees that a balanced growth path with exogenous growth exists in this model. Output is made stationary by the following transformation

$$\Psi_t \gamma_t^{-\alpha} k_t^{\alpha} U_t^{\alpha} H_t^{1-\alpha} = y_t$$

which is log-linearized to

$$\hat{\Psi}_t - \alpha\hat{\gamma}_t + \alpha\hat{k}_t + \alpha\hat{U}_t + (1-\alpha)\,\hat{H}_t = \hat{y}_t. \tag{1}$$

The external effects can be expressed in terms of stationary variables

$$\begin{aligned}\Psi_t &= \left[(U_t K_t)^{\alpha}\,(X_t H_t)^{1-\alpha}\right]^{\eta}\; X_t^{-\eta} = \left[\left(U_t \frac{K_t}{X_t}\right)^{\alpha} H_t^{1-\alpha}\right]^{\eta} \\ &= \left[\left(\frac{U_t}{\gamma_t} k_t\right)^{\alpha} H_t^{1-\alpha}\right]^{\eta},\end{aligned}$$

and after log-linearizing

$$\hat{\Psi}_t = \eta\alpha\left(\hat{U}_t + \hat{k}_t - \hat{\gamma}_t\right) + (1-\alpha)\,\eta\hat{H}_t. \tag{2}$$

The first order condition with respect to hours:

$$-W_t + (1-\alpha)\,\Psi_t\,(U_t K_t)^{\alpha}\,(X_t)^{1-\alpha}\,H_t^{-\alpha} = 0$$

becomes

$$(1-\alpha)\,\Psi_t X_{t-1}^{\alpha}\left(U_t \frac{K_t}{X_{t-1}}\right)^{\alpha} \frac{(X_t)^{1-\alpha}}{X_t} H_t^{-\alpha} = \frac{W_t}{X_t}$$

and hence

$$(1-\alpha)\,\Psi_t \gamma_t^{-\alpha}\,(U_t k_t)^{\alpha}\,H_t^{-\alpha} = w_t.$$

Combined with the definition of output gives

$$w_t = (1-\alpha)\,\frac{y_t}{H_t}$$

which in log-linear form becomes

$$\hat{w}_t = \hat{y}_t - \hat{H}_t. \tag{3}$$

The capital input decision gives:

$$\begin{aligned}0 &= -R_t^K + \alpha\Psi_t\,(U_t K_t)^{\alpha-1}\,(X_t H_t) \\ &= -R_t^K + \alpha\Psi_t\left(U_t \frac{K_t}{X_t}\right)^{\alpha-1} H_t^{1-\alpha} \\ &= -R_t^K + \alpha\Psi_t\left(\frac{U_t}{\gamma_t} k_t\right)^{\alpha-1} H_t^{1-\alpha}.\end{aligned}$$

Using the definition of output yields

$$R_t^K = \alpha\gamma_t \frac{y_t}{U_t k_t}$$

which in log-linear form is

$$\hat{R}_t^K = \hat{\gamma}_t + \hat{y}_t - \hat{U}_t - \hat{k}_t. \tag{4}$$

Finally, the evolution of capital in log-linear terms is described by:

$$\hat{k}_{t+1} = \frac{\bar{\imath}}{\bar{k}}\hat{\imath}_t + \frac{(1-\delta)}{\bar{\gamma}}\left(\hat{k}_t - \hat{\gamma}_t\right) - \frac{\delta\theta}{\bar{\gamma}}\hat{U}_t. \tag{5}$$

2 Consumption decision rule

The final task is to derive the optimal consumption decision rule under arbitrary expectations. Households choose a path for consumption, taking as given their initial capital holdings, capital and labor prices and their expectations about future prices. Let us start by defining the household's intertemporal budget constraint. The flow budget constraint can be expressed in terms of stationary variables

$$c_t + k_{t+1} = (\gamma_t)^{-1} R_t^K (U_t k_t) + w_t H_t + (1 - \delta(U_t))(\gamma_t)^{-1} k_t.$$

Log-linearization gives

$$\bar{\gamma}^{\sigma-1}\beta^{-1}\hat{k}_t = \left\{ \begin{array}{c} \frac{\bar{c}}{\bar{k}}\hat{c}_t + \hat{k}_{t+1} - \\ \bar{R}\bar{\gamma}^{\sigma-1}\left(\hat{R}_t^K + \hat{U}_t + \frac{1-\alpha}{\alpha}\hat{w}_t + \frac{1-\alpha}{\alpha}\hat{H}_t - \hat{\gamma}_t\right) - \\ \frac{(1-\delta)}{\bar{\gamma}}\left[-\hat{\gamma}_t - \frac{\delta}{(1-\delta)}\theta\hat{U}_t\right] \end{array} \right\}$$

where we use

$$\frac{\bar{w}\bar{H}}{\bar{k}} = \frac{1-\alpha}{\alpha}\frac{\bar{U}\bar{R}^k}{\bar{\gamma}} = \frac{1-\alpha}{\alpha}\bar{R}\bar{\gamma}^{\sigma-1}$$

and

$$\left[\frac{\bar{R}^k\bar{U}}{\bar{\gamma}} + \frac{1-\delta}{\bar{\gamma}}\right] = \bar{\gamma}^{\sigma-1}\beta^{-1} = \tilde{\beta}^{-1}$$

and where we define

$$\begin{aligned} \bar{R}\bar{\gamma}^{\sigma-1} &= \bar{\gamma}^{\sigma-1}\left(\beta^{-1} - \frac{(1-\delta)}{\bar{\gamma}^{\sigma}}\right) \\ &= \tilde{\beta}^{-1} - \frac{(1-\delta)}{\bar{\gamma}} \\ &= \tilde{R} \end{aligned}$$

from the steady-state condition of the Euler equation. The expression above can be further simplified by using $\frac{\bar{R}^k\bar{U}}{\bar{\gamma}} = \frac{\delta\theta}{\bar{\gamma}}$, which gets

$$\hat{k}_t = \tilde{\beta}\left[\frac{\bar{c}}{\bar{k}}\hat{c}_t + \hat{k}_{t+1} + \tilde{\beta}^{-1}\hat{\gamma}_t - \tilde{R}\left(\hat{R}_t^K + \frac{1-\alpha}{\alpha}\hat{w}_t + \frac{1-\alpha}{\alpha}\hat{H}_t\right)\right].$$

Using the labor supply

$$\sigma^{-1}\hat{\lambda}_t + \hat{w}_t = \epsilon_H\hat{H}_t$$

and the definition of marginal utility gives the following constant-consumption labor supply

$$\left[\epsilon_H - \frac{\sigma-1}{\sigma}\psi\right]\hat{H}_t = -\hat{c}_t + \hat{w}_t \tag{6}$$

We can substitute for labor supply decision $\hat{H}_t$ by using the household first-order conditions which gives

$$\hat{k}_t = \tilde{\beta}\left(\epsilon_c\hat{c}_t + \tilde{\beta}^{-1}\hat{\gamma}_t + \hat{k}_{t+1} - \epsilon_w\hat{w}_t - \tilde{R}\hat{R}^K_t\right)$$

where

$$\epsilon_c = \frac{\bar{c}}{\bar{\bar{k}}} + \left[\epsilon_H - \psi\frac{(\sigma-1)}{\sigma}\right]^{-1}\tilde{R}\frac{1-\alpha}{\alpha}$$

and

$$\epsilon_w = \left(1 + \left[\epsilon_H - \psi\frac{(\sigma-1)}{\sigma}\right]^{-1}\right)\tilde{R}\frac{1-\alpha}{\alpha}.$$

Iterating forward and taking expectations gives

$$\epsilon_c\hat{E}_t\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\hat{c}_T = \tilde{\beta}^{-1}\hat{k}_t + \hat{E}_t\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\left(\epsilon_w\hat{w}_T + \tilde{R}\hat{R}^K_T - \tilde{\beta}^{-1}\hat{\gamma}_T\right).$$

This expression defines households' expected intertemporal budget constraint, as of time t. Recall the Euler equation

$$\hat{Q}_t = -\hat{E}_t\beta\bar{R}\left(\hat{R}^K_{t+1}\right) + \hat{E}_t\hat{Q}_{t+1} + \sigma\hat{E}_t\hat{\gamma}_{t+1}$$

which can be re-written as

$$\begin{aligned}\hat{Q}_t &= -\hat{E}_t\beta\gamma^{1-\sigma}\gamma^{\sigma-1}\bar{R}\left(\hat{R}^K_{t+1}\right) + \hat{E}_t\hat{Q}_{t+1} + \sigma\hat{E}_t\hat{\gamma}_{t+1}\\ &= -\hat{E}_t\tilde{\beta}\tilde{R}\hat{R}^K_{t+1} + \hat{E}_t\hat{Q}_{t+1} + \sigma\hat{E}_t\hat{\gamma}_{t+1}.\end{aligned}$$

Solving backward from time T gives

$$\begin{aligned}\hat{E}_tQ_T &= \hat{Q}_t + \hat{E}_t\left[\sum_{T=t}^{T-1}\left(\tilde{\beta}\tilde{R}\hat{R}^K_{T+1} - \sigma\hat{\gamma}_{T+1}\right)\right]\\ \hat{E}_t\left(\sigma\hat{c}_T + \psi(1-\sigma)\hat{H}_T\right) &= \sigma\hat{c}_t + \psi(1-\sigma)\hat{H}_t + \hat{E}_t\left[\sum_{T=t}^{T-1}\left(\tilde{\beta}\tilde{R}\hat{R}^K_{T+1} - \sigma\hat{\gamma}_{T+1}\right)\right].\end{aligned}$$

Substituting for the constant-consumption labor supply yields

$$\begin{aligned}&\hat{E}_t\left(\sigma\hat{c}_T + \psi(1-\sigma)\left[\epsilon_H - \psi\frac{(\sigma-1)}{\sigma}\right]^{-1}(-\hat{c}_T + \hat{w}_T)\right)\\ &= \sigma\hat{c}_t + \psi(1-\sigma)\hat{H}_t + \hat{E}_t\left[\sum_{T=t}^{T-1}\left(\tilde{\beta}\tilde{R}\hat{R}^K_{T+1} - \sigma\hat{\gamma}_{T+1}\right)\right]\end{aligned}$$

which can be simplified to

$$\hat{E}_t\left(\left[(1-\chi)\sigma\hat{c}_T + \chi\sigma\hat{w}_T\right]\right) = \sigma\hat{c}_t + \psi(1-\sigma)\hat{H}_t + \hat{E}_t\left[\sum_{T=t}^{T-1}\left(\tilde{\beta}\tilde{R}\hat{R}^K_{T+1} - \sigma\hat{\gamma}_{T+1}\right)\right]$$

by using

$$\begin{aligned}\psi\left(1-\sigma\right)\left[\epsilon_{H}-\psi\frac{\left(\sigma-1\right)}{\sigma}\right]^{-1} &= \left[\frac{\epsilon_{H}}{\psi\left(1-\sigma\right)}+\sigma^{-1}\right]^{-1}\\ &= \sigma\left[\frac{\sigma\epsilon_{H}}{\psi\left(1-\sigma\right)}+1\right]^{-1}\\ &= \sigma\frac{\psi\left(1-\sigma\right)}{\sigma\epsilon_{H}+\psi\left(1-\sigma\right)}=\sigma\chi.\end{aligned}$$

Rearranging in terms of expected consumption

$$\hat{E}_{t}\hat{c}_{T}=\frac{1}{1-\chi}\left\{\hat{c}_{t}+\sigma^{-1}\psi\left(1-\sigma\right)\hat{H}_{t}+\hat{E}_{t}\left[\sum_{T=t}^{T-1}\left(\sigma^{-1}\tilde{\beta}\tilde{R}\hat{R}_{T+1}^{K}-\hat{\gamma}_{T+1}\right)\right]-\chi\hat{w}_{T}\right\}$$

and substituting into the intertemporal budget constraint we get

$$\begin{aligned}&\epsilon_{c}\hat{E}_{t}\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\left[\frac{1}{1-\chi}\left\{\hat{c}_{t}+\sigma^{-1}\psi\left(1-\sigma\right)\hat{H}_{t}+\hat{E}_{t}\left[\sum_{T=t}^{T-1}\left(\sigma^{-1}\tilde{\beta}\tilde{R}\hat{R}_{T+1}^{K}-\hat{\gamma}_{T+1}\right)\right]-\chi\hat{w}_{T}\right\}\right]\\ = \;&\tilde{\beta}^{-1}\hat{k}_{t}+\hat{E}_{t}\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\left(\epsilon_{w}\hat{w}_{T}+\tilde{R}\hat{R}_{T}^{K}-\tilde{\beta}^{-1}\hat{\gamma}_{T}\right).\end{aligned}$$

Further simplification leads to

$$\hat{c}_{t}+\sigma^{-1}\psi\left(1-\sigma\right)\hat{H}_{t}=$$

$$\begin{aligned}&\frac{\left(1-\chi\right)\left(1-\tilde{\beta}\right)}{\epsilon_{c}\tilde{\beta}}\hat{k}_{t}+\\ &\hat{E}_{t}\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\left[\frac{\left(1-\chi\right)\left(1-\tilde{\beta}\right)}{\epsilon_{c}}\left(\left(\epsilon_{w}+\tilde{\epsilon}_{c}\chi\right)\hat{w}_{T}+\tilde{R}\hat{R}_{T}^{K}-\tilde{\beta}^{-1}\hat{\gamma}_{T}\right)-\tilde{\beta}\left(\sigma^{-1}\tilde{\beta}\tilde{R}\hat{R}_{T+1}^{K}-\hat{\gamma}_{T+1}\right)\right].\end{aligned}$$

Finally, we obtain the consumption decision rule, depending only on forecast of prices that are beyond the control of the household,

$$\begin{aligned}\hat{c}_{t}+\sigma^{-1}\psi\left(1-\sigma\right)\hat{H}_{t} &= \frac{\left(1-\chi\right)\left(1-\tilde{\beta}\right)}{\epsilon_{c}}\left[\tilde{\beta}^{-1}\hat{k}_{t}+\tilde{R}\hat{R}_{t}^{K}-\tilde{\beta}^{-1}\hat{\gamma}_{t}+\left(\epsilon_{w}+\epsilon_{c}\frac{\chi}{1-\chi}\right)\hat{w}_{t}\right]\\ &+\hat{E}_{t}\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\left[\tilde{\beta}-\frac{\left(1-\chi\right)\left(1-\tilde{\beta}\right)}{\epsilon_{c}}\right]\hat{\gamma}_{T+1}\\ &+\hat{E}_{t}\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\left[\frac{\left(1-\chi\right)\left(1-\tilde{\beta}\right)}{\epsilon_{c}}-\tilde{\beta}\sigma^{-1}\right]\tilde{\beta}\tilde{R}\hat{R}_{T+1}^{K}\\ &+\hat{E}_{t}\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\frac{\left(1-\chi\right)\left(1-\tilde{\beta}\right)}{\epsilon_{c}}\tilde{\beta}\left(\epsilon_{w}+\epsilon_{c}\frac{\chi}{1-\chi}\right)\hat{w}_{T+1}.\end{aligned}$$

Setting $\sigma=1$ and $\chi=0$ we get back to the simple RBC model.

2.1 Perfectly elastic labor supply

In the benchmark calibration, we consider the case of a perfectly elastic labor supply. That is $\epsilon_H \to 0$. This case results in a well-defined consumption decision rule even when $\sigma = 1$. Recall

$$\epsilon_c = \frac{\bar{c}}{\bar{k}} + \left(\epsilon_H - \frac{\sigma-1}{\sigma}\psi\right)^{-1} \tilde{R}\frac{1-\alpha}{\alpha}$$

$$\epsilon_w = \left[1 + \left(\epsilon_H - \frac{\sigma-1}{\sigma}\psi\right)^{-1}\right] \tilde{R}\frac{1-\alpha}{\alpha}$$

which have limiting values of infinity when $\epsilon_H \to 0$. Consider the benchmark model with $\sigma = 1$. Then

$$\mu_1 = \frac{\epsilon_w}{\epsilon_c} = \frac{\tilde{R}\frac{1-\alpha}{\alpha} + \epsilon_H^{-1}\tilde{R}\frac{1-\alpha}{\alpha}}{\frac{\bar{c}}{\bar{k}} + \epsilon_H^{-1}\tilde{R}\frac{1-\alpha}{\alpha}}$$

and $\chi = 0$. This ratio has limiting value of unity. The consumption decision rule is

$$\begin{aligned}
\hat{c}_t \quad = \quad & \frac{\left(1-\tilde{\beta}\right)}{\epsilon_c}\left[\tilde{\beta}^{-1}\hat{k}_t + \tilde{R}\hat{R}_t^K - \tilde{\beta}^{-1}\hat{\gamma}_t + \epsilon_w \hat{w}_t\right] \\
& + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t}\left[\tilde{\beta} - \frac{\left(1-\tilde{\beta}\right)}{\epsilon_c}\right]\hat{\gamma}_{T+1} \\
& + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t}\left[\frac{\left(1-\tilde{\beta}\right)}{\epsilon_c} - \tilde{\beta}\sigma^{-1}\right]\tilde{\beta}\tilde{R}\hat{R}_{T+1}^K \\
& + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t}\frac{\left(1-\tilde{\beta}\right)}{\epsilon_c}\tilde{\beta}\epsilon_w \hat{w}_{T+1}.
\end{aligned}$$

Re-arranging

$$\begin{aligned}
\hat{c}_t \quad = \quad & \frac{\left(1-\tilde{\beta}\right)}{\epsilon_c}\left[\tilde{\beta}^{-1}\hat{k}_t + \tilde{R}\hat{R}_t^K - \tilde{\beta}^{-1}\hat{\gamma}_t\right] \\
& + \frac{\epsilon_w}{\epsilon_c}\left(1-\tilde{\beta}\right)\hat{w}_t \\
& - \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t}\left[\frac{\left(1-\tilde{\beta}\right)}{\epsilon_c} - \tilde{\beta}\sigma^{-1}\right]\tilde{\beta}\tilde{R}\hat{R}_{T+1}^K \\
& + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t}\tilde{\beta}\frac{\epsilon_w}{\epsilon_c}\left(1-\tilde{\beta}\right)\hat{w}_{T+1}.
\end{aligned}$$

For $\epsilon_H \to 0$ the consumption decision rule then becomes

$$\begin{aligned}
\hat{c}_t \quad = \quad & \left(1-\tilde{\beta}\right)\hat{w}_t \\
& - \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t}\sigma^{-1}\tilde{\beta}^2\tilde{R}\hat{R}_{T+1}^K \\
& + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t}\tilde{\beta}\left(1-\tilde{\beta}\right)\hat{w}_{T+1}.
\end{aligned}$$

3 Steady State

From the Euler equation we get

$$\begin{aligned}\frac{\bar{R}^k\bar{U}}{\bar{\gamma}} &= \bar{\gamma}^{\sigma-1}\beta^{-1}-\frac{1-\delta}{\bar{\gamma}}\\ &= \tilde{\beta}^{-1}-\frac{1-\delta}{\bar{\gamma}}\end{aligned}$$

and from the capacity utilization first-order condition

$$\frac{\bar{R}^k\bar{U}}{\bar{\gamma}}=\frac{\delta\theta}{\bar{\gamma}}\Longrightarrow\theta=\frac{\bar{R}^k\bar{U}}{\delta}$$

which defines θ, allowing to determine U and therefore R^k. The ratios

$$\frac{\bar{y}}{\bar{k}}=(\alpha)^{-1}\frac{\bar{R}^k\bar{U}}{\bar{\gamma}};\ \frac{\bar{\imath}}{\bar{k}}=1-\frac{1-\delta}{\bar{\gamma}};\ \frac{\bar{c}}{\bar{k}}=\frac{\bar{y}}{\bar{k}}-\frac{\bar{\imath}}{\bar{k}}\text{ and }\frac{\bar{c}}{\bar{y}}=\frac{\bar{c}}{\bar{k}}/\frac{\bar{y}}{\bar{k}}.$$

Finally the steady state level ψ, for a given choice of $\bar{H}$, is determined by

$$\begin{aligned}\psi &= \frac{\bar{H}v'\left(\bar{H}\right)}{v\left(\bar{H}\right)}(\sigma-1)^{-1}=\frac{\bar{w}\bar{H}}{\bar{k}}\frac{\bar{k}}{\bar{c}}\\ &= \frac{1-\alpha}{\alpha}\bar{R}\bar{\gamma}^{\sigma-1}\frac{\bar{k}}{\bar{c}}(\sigma-1)^{-1}\\ &= \frac{1-\alpha}{\alpha}\tilde{R}\frac{\bar{k}}{\bar{c}}(\sigma-1)^{-1}.\end{aligned}$$

4 The model with costly participation

The preferences described above suffer from the problem that, for a given σ, as the Frisch elasticity of labor supply increases consumption becomes an inferior good. In this section we show how a simple model of costly participation can lead to a similar labor supply and consumption decision rule. We assume that each 'household' is composed of a continuum of family members. Labor is indivisible: each member of the household decides whether to work a fixed amount of hours or not to participate in the labor market. Participating in the labor market entails a cost. We assume perfect risk sharing within the household. The maximization problem for the household becomes

$$E_t\sum_{T=t}^{\infty}\beta^{T-t}u\left(C_T,L_T\right)$$

subject to

$$\begin{aligned}C_t+K_{t+1}+X_t\Phi\left(e_t\right) &= R_t^K\left(u_tK_t\right)+W_tH_t+\left(1-\delta\left(U_t\right)\right)K_t\\ L_t &= 1-H_t,\end{aligned}$$

where e_t denotes the fraction of the household members that are working and where household consumption is defined as

$$C_t=e_tC_t^e+\left(1-e_t\right)C_t^u$$

where C_t^e denotes consumption of employed members and C_t^u is consumption of the unemployed. The utility function is defined as

$$u(C_t, L_t) = e_t \frac{(C_t^e)^{1-\sigma} \nu(h)}{1-\sigma} + (1 - e_t) \frac{(C_t^u)^{1-\sigma} \nu(0)}{1-\sigma},$$

where

$$L_t = 1 - H_t = 1 - e_t h.$$

Finally, the cost function $\Phi(e_t)$ has the following properties

$$\begin{aligned} \Phi_e(e_t) &> 0, \\ \Phi_{ee}(\bar{e}) &> 0. \end{aligned}$$

The first-order conditions

$$(C_t^e)^{-\sigma} \nu(h) = \Lambda_t \tag{7}$$

$$(C_t^u)^{-\sigma} \nu(0) = \Lambda_t$$

imply the following risk-sharing condition

$$\frac{C_t^e}{C_t^u} = \left[\frac{\nu(h)}{\nu(0)}\right]^{\frac{1}{\sigma}}$$

so that employed members enjoy more consumption in order to be compensated for work effort. The first-order conditions with respect to the participation rate give

$$\frac{1}{1-\sigma}\left[-(C_t^e)^{1-\sigma} \nu(h) + (C_t^u)^{1-\sigma} \nu(0)\right] = \Lambda_t \left[W_t h - C_t^e + C_t^u - \Phi_{e,t}\right]$$

which, rearranging, becomes

$$\frac{\sigma}{\sigma-1}(C_t^e - C_t^u) = W_t h - \Phi_{e,t}.$$

Re-expressing the variables in stationary levels and after log-linearization we are left with the following equations:

$$\begin{aligned} \hat{c}_t^e &= \hat{c}_t^u \\ \hat{c}_t &= \frac{\bar{e}(\bar{c}^e - \bar{c}^u)}{\bar{c}} \hat{H}_t + \hat{c}_t^e \\ \frac{\sigma}{\sigma-1} \frac{e(\bar{c}^e - \bar{c}^u)}{\bar{c}} \hat{c}_t^e &= \psi \hat{w}_t - \epsilon_\phi \bar{\phi} \hat{H}_t, \end{aligned} \tag{8}$$

where

$$\psi = \frac{\bar{H}\bar{w}}{\bar{c}}, \epsilon_\phi = \frac{\Phi_{ee}\bar{e}}{\Phi_e}, \bar{\phi} = \frac{\Phi_e \bar{e}}{\bar{c}}.$$

Next, notice that in steady state the following holds

$$\frac{\bar{e}(\bar{c}^e - \bar{c}^u)}{\bar{c}} = \frac{\sigma-1}{\sigma}\left(\psi - \bar{\phi}\right) > 0.$$

Substituting for $\hat{c}_t^e$ and using the above steady-state relation we get

$$\left(\psi - \bar{\phi}\right)\left[\hat{c}_t - \frac{\sigma-1}{\sigma}\left(\psi - \bar{\phi}\right)\hat{H}_t\right] = \psi \hat{w}_t - \epsilon_\phi \bar{\phi} \hat{H}_t,$$

which gives the following constant-consumption labor supply

$$\left[\frac{\epsilon_\phi \bar{\phi}}{\left(\psi-\bar{\phi}\right)}-\frac{\sigma-1}{\sigma}\left(\psi-\bar{\phi}\right)\right]\hat{H}_t=-\hat{c}_t+\frac{\psi}{\left(\psi-\bar{\phi}\right)}\hat{w}_t.$$

Let us consider the optimal decision rule. Again, the flow budget constraint can be expressed in terms of stationary variables

$$c_t+k_{t+1}+\Phi\left(e_t\right)=\left(\gamma_t\right)^{-1}R_t^K\left(U_t k_t\right)+w_t H_t+\left(1-\delta\left(U_t\right)\right)\left(\gamma_t\right)^{-1}k_t.$$

Log-linearization gives

$$\bar{\gamma}^{\sigma-1}\beta^{-1}\hat{k}_t=\left\{\begin{array}{c}\frac{\Phi_e \bar{e}}{\bar{c}}\frac{c}{k}\hat{H}_t+\frac{\bar{c}}{k}\hat{c}_t+\hat{k}_{t+1}- \\ \bar{R}\bar{\gamma}^{\sigma-1}\left(\hat{R}_t^K+\hat{U}_t+\frac{1-\alpha}{\alpha}\hat{w}_t+\frac{1-\alpha}{\alpha}\hat{H}_t-\hat{\gamma}_t\right)- \\ \frac{(1-\delta)}{\bar{\gamma}}\left[-\hat{\gamma}_t-\frac{\delta}{(1-\delta)}\theta\hat{U}_t\right]\end{array}\right\},$$

which becomes

$$\hat{k}_t=\tilde{\beta}\left[\frac{\bar{c}}{\bar{k}}\hat{c}_t+\bar{\phi}\frac{\bar{c}}{\bar{k}}\hat{H}_t+\hat{k}_{t+1}+\tilde{\beta}^{-1}\hat{\gamma}_t-\tilde{R}\left(\hat{R}_t^K+\frac{1-\alpha}{\alpha}\hat{w}_t+\frac{1-\alpha}{\alpha}\hat{H}_t\right)\right].$$

Using the for the constant-consumption labor supply we get

$$\hat{k}_t=\tilde{\beta}\left(\epsilon_c^\phi\hat{c}_t+\tilde{\beta}^{-1}\hat{\gamma}_t+\hat{k}_{t+1}-\epsilon_w^\phi\hat{w}_t-\tilde{R}\hat{R}_t^K\right)$$

where

$$\begin{aligned}\epsilon_c^\phi &= \frac{\bar{c}}{\bar{k}}-\bar{\phi}\frac{\bar{c}}{\bar{k}}\left[\frac{\epsilon_\phi\bar{\phi}}{\left(\psi-\bar{\phi}\right)}-\frac{\sigma-1}{\sigma}\left(\psi-\bar{\phi}\right)\right]^{-1}+\left[\frac{\epsilon_\phi\bar{\phi}}{\left(\psi-\bar{\phi}\right)}-\frac{\sigma-1}{\sigma}\left(\psi-\bar{\phi}\right)\right]^{-1}\tilde{R}\frac{1-\alpha}{\alpha} \\ &= \frac{\bar{c}}{\bar{k}}+\left[\frac{\epsilon_\phi\bar{\phi}}{\left(\psi-\bar{\phi}\right)}-\frac{\sigma-1}{\sigma}\left(\psi-\bar{\phi}\right)\right]^{-1}\left[\tilde{R}\frac{1-\alpha}{\alpha}-\bar{\phi}\frac{\bar{c}}{\bar{k}}\right]\end{aligned}$$

and

$$\epsilon_w^\phi=\left(1+\frac{\psi}{\left(\psi-\bar{\phi}\right)}\left[\frac{\epsilon_\phi\bar{\phi}}{\left(\psi-\bar{\phi}\right)}-\frac{\sigma-1}{\sigma}\left(\psi-\bar{\phi}\right)\right]^{-1}\right)\tilde{R}\frac{1-\alpha}{\alpha}$$

By iterating forward and taking expectations we get

$$\epsilon_c^\phi\hat{E}_t\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\hat{c}_T=\tilde{\beta}^{-1}\hat{k}_t+\hat{E}_t\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\left(\epsilon_w^\phi\hat{w}_T+\tilde{R}\hat{R}_T^K-\tilde{\beta}^{-1}\hat{\gamma}_T\right).$$

The expression above defines household expected intertemporal budget constraint, as of time t. Recall the Euler equation is

$$\beta\hat{E}_t\left[\beta^{-1}\left(\hat{\lambda}_{t+1}-\hat{\lambda}_t-\sigma\hat{\gamma}_{t+1}\right)+\left(\beta^{-1}-\frac{(1-\delta)}{\gamma^\sigma}\right)\hat{R}_{t+1}^K\right]=0.$$

Combining (7) and (8) we obtain

$$\hat{c}_t=\frac{\sigma-1}{\sigma}\left(\psi-\bar{\phi}\right)\hat{H}_t-\sigma^{-1}\hat{\lambda}_t$$

which, combined with the Euler equation, gives

$$\hat{Q}_t = -\hat{E}_t \beta \bar{R} \left(\hat{R}^K_{t+1} \right) + \hat{E}_t \hat{Q}_{t+1} + \sigma \hat{E}_t \hat{\gamma}_{t+1}$$

where

$$\hat{Q}_t = \sigma \hat{c}_t + \left(\psi - \bar{\phi} \right) (1 - \sigma) \hat{H}_t.$$

By solving backward from time T we get

$$\begin{aligned} \hat{E}_t Q_T &= \hat{Q}_t + \hat{E}_t \left[\sum_{T=t}^{T-1} \left(\tilde{\beta} \tilde{R} \hat{R}^K_{T+1} - \sigma \hat{\gamma}_{T+1} \right) \right] \\ \hat{E}_t \left(\sigma \hat{c}_T + \left(\psi - \bar{\phi} \right) (1 - \sigma) \hat{H}_T \right) &= \sigma \hat{c}_t + \left(\psi - \bar{\phi} \right) (1 - \sigma) \hat{H}_t + \hat{E}_t \left[\sum_{T=t}^{T-1} \left(\tilde{\beta} \tilde{R} \hat{R}^K_{T+1} - \sigma \hat{\gamma}_{T+1} \right) \right]. \end{aligned}$$

Substituting for the constant-consumption labor supply

$$\hat{E}_t \left(\sigma \hat{c}_T + \left(\psi - \bar{\phi} \right) (1 - \sigma) \left[\frac{\epsilon_\phi \bar{\phi}}{\left(\psi - \bar{\phi} \right)} - \frac{\sigma - 1}{\sigma} \left(\psi - \bar{\phi} \right) \right]^{-1} \left(-\hat{c}_T + \frac{\psi}{\left(\psi - \bar{\phi} \right)} \hat{w}_T \right) \right)$$

$$= \sigma \hat{c}_t + \left(\psi - \bar{\phi} \right) (1 - \sigma) \hat{H}_t + \hat{E}_t \left[\sum_{T=t}^{T-1} \left(\tilde{\beta} \tilde{R} \hat{R}^K_{T+1} - \sigma \hat{\gamma}_{T+1} \right) \right]$$

which can be simplified to

$$\hat{E}_t \left(\left[(1 - \chi) \sigma \hat{c}_T + \chi \sigma \hat{w}_T \right] \right) = \sigma \hat{c}_t + \psi (1 - \sigma) \hat{H}_t + \hat{E}_t \left[\sum_{T=t}^{T-1} \left(\tilde{\beta} \tilde{R} \hat{R}^K_{T+1} - \sigma \hat{\gamma}_{T+1} \right) \right]$$

by using

$$\begin{aligned} \left(\psi - \bar{\phi} \right) (1 - \sigma) \left[\frac{\epsilon_\phi \bar{\phi}}{\left(\psi - \bar{\phi} \right)} - \frac{\sigma - 1}{\sigma} \left(\psi - \bar{\phi} \right) \right]^{-1} &= \left[\frac{\epsilon_\phi \bar{\phi}}{\left(\psi - \bar{\phi} \right)^2 (1 - \sigma)} + \sigma^{-1} \right]^{-1} \\ &= \sigma \left[\frac{\sigma \epsilon_\phi \bar{\phi}}{\left(\psi - \bar{\phi} \right)^2 (1 - \sigma)} + 1 \right]^{-1} \\ &= \sigma \left[\frac{\left(\psi - \bar{\phi} \right)^2 (1 - \sigma)}{\sigma \epsilon_\phi \bar{\phi} + \left(\psi - \bar{\phi} \right)^2 (1 - \sigma)} \right] = \sigma \chi. \end{aligned}$$

Rearranging in terms of expected consumption

$$\hat{E}_t \hat{c}_T = \frac{1}{1 - \chi} \left\{ \hat{c}_t + \sigma^{-1} \left(\psi - \bar{\phi} \right) (1 - \sigma) \hat{H}_t + \hat{E}_t \left[\sum_{T=t}^{T-1} \left(\sigma^{-1} \tilde{\beta} \tilde{R} \hat{R}^K_{T+1} - \hat{\gamma}_{T+1} \right) \right] - \chi \hat{w}_T \right\}$$

and substituting into the intertemporal budget constraint we get

$$\begin{aligned} & \epsilon_c^\phi \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t} \left[\frac{1}{1 - \chi} \left\{ \hat{c}_t + \sigma^{-1} \left(\psi - \bar{\phi} \right) (1 - \sigma) \hat{H}_t + \hat{E}_t \left[\sum_{T=t}^{T-1} \left(\sigma^{-1} \tilde{\beta} \tilde{R} \hat{R}^K_{T+1} - \hat{\gamma}_{T+1} \right) \right] - \chi \hat{w}_T \right\} \right] \\ = \quad & \tilde{\beta}^{-1} \hat{k}_t + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t} \left(\epsilon_w^\phi \hat{w}_T + \tilde{R} \hat{R}^K_T - \tilde{\beta}^{-1} \hat{\gamma}_T \right). \end{aligned}$$

Further simplification leads to

$$\hat{c}_t + \sigma^{-1}\left(\psi - \bar{\phi}\right)(1-\sigma)\hat{H}_t =$$

$$\frac{(1-\chi)\left(1-\tilde{\beta}\right)}{\epsilon_c^{\phi}\tilde{\beta}}\hat{k}_t +$$

$$\hat{E}_t\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\left[\frac{(1-\chi)\left(1-\tilde{\beta}\right)}{\epsilon_c^{\phi}}\left(\left(\epsilon_w^{\phi}+\epsilon_c^{\phi}\frac{\chi}{1-\chi}\right)\hat{w}_T + \tilde{R}\hat{R}_T^K - \tilde{\beta}^{-1}\hat{\gamma}_T\right) - \tilde{\beta}\left(\sigma^{-1}\tilde{\beta}\tilde{R}\hat{R}_{T+1}^K - \hat{\gamma}_{T+1}\right)\right].$$

Finally, we obtain the consumption decision rule, depending only on forecast of prices that are beyond the control of the household,

$$\begin{aligned}
\hat{c}_t + \sigma^{-1}\left(\psi - \bar{\phi}\right)(1-\sigma)\hat{H}_t &= \frac{(1-\chi)\left(1-\tilde{\beta}\right)}{\epsilon_c^{\phi}}\left[\tilde{\beta}^{-1}\hat{k}_t + \tilde{R}\hat{R}_t^K - \tilde{\beta}^{-1}\hat{\gamma}_t + \left(\epsilon_w^{\phi}+\epsilon_c^{\phi}\frac{\chi}{1-\chi}\right)\hat{w}_t\right] \\
&+\hat{E}_t\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\left[\tilde{\beta} - \frac{(1-\chi)\left(1-\tilde{\beta}\right)}{\epsilon_c^{\phi}}\right]\hat{\gamma}_{T+1} \\
&+\hat{E}_t\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\left[\frac{(1-\chi)\left(1-\tilde{\beta}\right)}{\epsilon_c^{\phi}} - \tilde{\beta}\sigma^{-1}\right]\tilde{\beta}\tilde{R}\hat{R}_{T+1}^K \\
&+\hat{E}_t\sum_{T=t}^{\infty}\tilde{\beta}^{T-t}\frac{(1-\chi)\left(1-\tilde{\beta}\right)}{\epsilon_c^{\phi}}\tilde{\beta}\left(\epsilon_w^{\phi}+\epsilon_c^{\phi}\frac{\chi}{1-\chi}\right)\hat{w}_{T+1}.
\end{aligned}$$

For low values of $\bar{\phi}$ (the cost of participating) the decision rule approximates our representative case model. Also notice that with $\sigma > 1$, by letting $\Phi_{ee}, \Phi_e \to 0$ ($\epsilon_\phi, \bar{\phi} \to 0$) the model becomes Rogerson's lottery model with non-separable preferences described in King and Rebelo (1999) and the decision rule corresponds to the case of non-separable preferences with infinitely elastic labor supply, where $\epsilon_c^{\phi} = \epsilon_c$ and $\epsilon_w^{\phi} = \epsilon_w$.

5 Stability under learning

Decreasing gain. Consider a decreasing gain consistent with Recursive Least Squares, $g_t = t^{-1}$.[2] The true data generating process becomes

$$z_t = T_1(\hat{\omega}_{t-1})q_{t-1} + T_2(\hat{\omega}_{t-1})\hat{\gamma}_t \tag{9}$$

$$\hat{\omega}_t = \hat{\omega}_{t-1} + t_t^{-1}R_t^{-1}q_{t-1}\left(\left[\left(T_1(\hat{\omega}_{t-1}) - \hat{\omega}_{t-1}'\right)q_{t-1} + T_2(\hat{\omega}_{t-1})\hat{\gamma}_t\right]\right)' \tag{10}$$

$$R_t = R_{t-1} + t_t^{-1}\left(q_{t-1}q_{t-1}' - R_{t-1}\right) \tag{11}$$

The system (9)-(11) is locally stable under learning if the estimated coefficients $\hat{\omega}_t$ converge to the rational expectations coefficients ω^*. Following Evans and Honkapohja (2001), local stability under

[2] Evans and Honkapohja (2001) discuss a wider class of decreasing gains that yield the same stability conditions.

learning of the system (9)-(11) can be studied by evaluating the local stability of the following ODE

$$\begin{aligned} \dot{\omega} &= RM_q^{-1}\left[T_1(\omega)' - \omega\right] \\ \dot{R} &= M_q - R \end{aligned} \tag{12}$$

at the rational expectations equilibrium where $T_1'(\omega^*) = \omega^*$. Stability is obtained obtained if the real part of the system's eigenvalues is negative. We verify numerically that stability obtains for the different parameter values discussed in the paper. We find that the system admits *real eigenvalues* so that agents' estimates converge monotonically to their rational expectations values. Figure 1 shows the evolution of the intercept for the rental rate and wage (in efficiency units) equations according to the ordinary differential equation

$$\dot{\omega} = \left[T_1(\omega)' - \omega\right]. \tag{13}$$

We consider a simple experiment where the initial values of the intercepts in the two equations are equal to 0.01 and −0.01 respectively, while other equations coefficients are kept at the rational expectations values. The plot shows the evolution of agents' estimates in continuos time, corresponding to the solution of (13). The estimated intercept initially increase, responding to higher observed returns to capital and then monotonically converges to zero, its rational expectation value. The opposite happens for the intercept in the efficiency wage equation. These dynamics in belief parameters are consistent with the optimism and pessimism of capital and labor returns discussion in the paper.

Constant gain. In the paper we consider a constant gain ($g > 0$) algorithm. In this case $\hat{\omega}_t$ cannot be expected to converge to a non-stochastic point (ω^*). As shown in Evans and Honkapohja (2001), stability of the ODE (12) implies that $\hat{\omega}_t$ converges to an invariant distribution centered around ω^*, for sufficiently small values of the constant gain. The distribution that corresponds to our benchmark calibration is shown in Figure 4 in the paper.

6 Constant gain learning and the Kalman filter

Agents update their beliefs using the following constant-gain algorithm

$$\hat{\omega}_t = \hat{\omega}_{t-1} + gR_t^{-1}q_{t-1}\left(\tilde{z}_t - \hat{\omega}_{t-1}'q_{t-1}\right) \tag{14}$$

$$R_t = R_{t-1} + g\left(q_{t-1}q_{t-1}' - R_{t-1}\right)$$

where we now assume for simplicity that $\tilde{z}_t = T(\hat{\omega}_{t-1})q_{t-1}$ is one-dimensional (for example k_{t+1}) and q_t is a *two*-dimensional vector. Following Evans and Honkapohja (2001) and Sargent and Williams (2005), the limiting behavior of the estimates are approximated by the following system

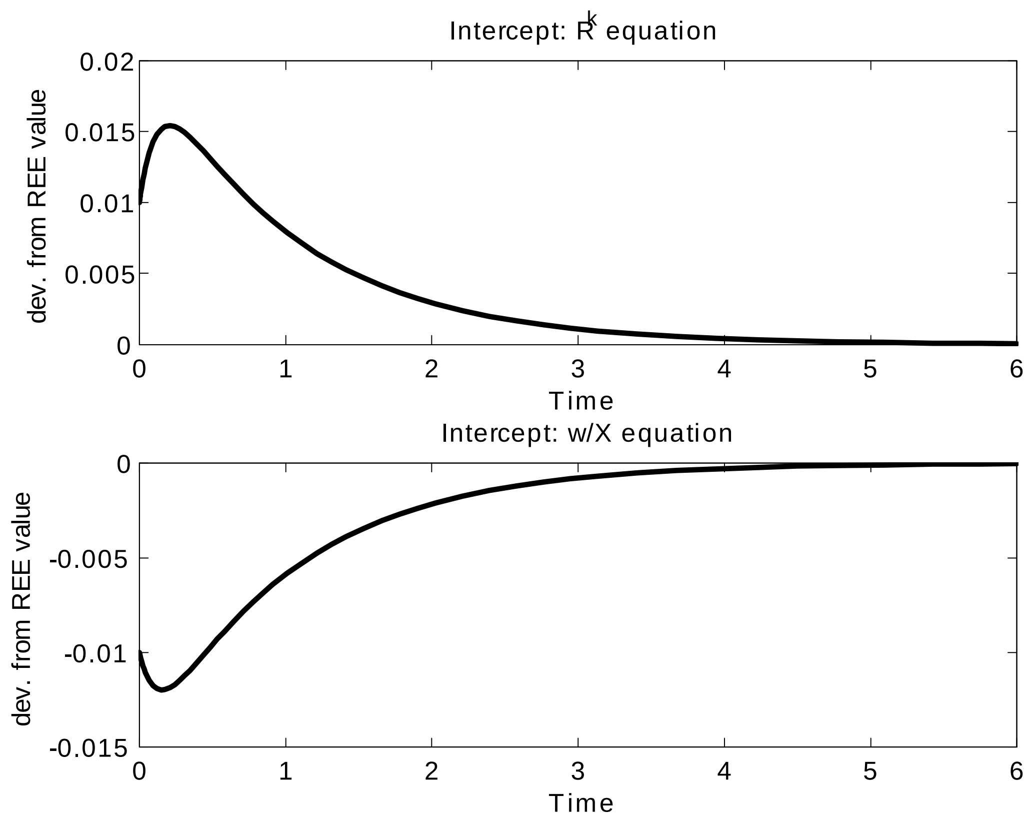

Figure 1: The figure shows the convergence of the estimated intercepts in the rental rate and wage equations. The top panel shows the estimate for the intercept of the equation for the rental rate of capital, while the bottom panel does the same for wages.

of ordinary differential equations[3]

$$\begin{aligned}\dot{\hat{\omega}} &= R^{-1}M_q(\hat{\omega})\left[T(\hat{\omega})-\hat{\omega}\right]\\ \dot{R} &= M_q(\hat{\omega})-R\end{aligned}$$

where[4] $M_q(\hat{\omega}) = E\left(q_{t-1}q'_{t-1}\right)$. Asymptotically R converges to $M_q(\hat{\omega})$.

As a way to justify and interpret the use of constant-gain algorithms it is common to relate them to the Kalman filter. Assume agents believe that the data generating process is the following random walk model of coefficient variation

$$\tilde{z}_t = \omega'_{t-1}q_{t-1} + \tilde{e}^z_t$$

$$\omega_t = \omega_{t-1} + \tilde{e}^\omega_t$$

where for simplicity we assume that $\tilde{z}_t$ is one-dimensional and q_t is a n-dimensional vector. The shock $\tilde{e}^z_t$ has standard deviation σ_z and variance-covariance matrix of $\tilde{e}^\omega_t$ is assumed to be $\Sigma^\omega << \sigma^2_z I$. The matrix Σ^ω defines agents' prior about the variance in the coefficients' drift. The model captures the possibility of permanent shocks to the structure of the economy which can change the long-term returns to capital and labor. The agents use a reduced-form model for forecasting purposes so that structural change could be due to a number of unspecified factors: for example, long-term shifts in the returns to aggregate capital or long-term shifts in the labor supply. At the rational expectations equilibrium beliefs are consistent with no structural changes: this corresponds to the case where $\sigma^2_z = 0$ and $\omega_t = \omega^*$. Let $\hat{\omega}_{t|t-1}$ the optimal estimate of ω_t conditional on information up to date $t-1$. This is obtained from the following Kalman filtering equations

$$\hat{\omega}_{t+1|t} = \hat{\omega}_{t|t-1} + \frac{P_t q_{t-1}}{1+q'_{t-1}P_t q_{t-1}}\left(\tilde{z}_t - \hat{\omega}'_{t|t-1}q_{t-1}\right) \quad (15)$$

$$P_{t+1} = P_t - \frac{P_t q_{t-1}q'_{t-1}P_t}{1+q'_{t-1}P_t q_{t-1}} + \frac{1}{\sigma^2_z}\Sigma^\omega \quad (16)$$

where we use

$$E\left[\left(\omega_t - \hat{\omega}_{t|t-1}\right)\left(\omega_t - \hat{\omega}_{t|t-1}\right)'\right] = \sigma^2_z P_t.$$

Sargent and Williams (2005) propose the following approximation to the filtering equations. For large t (16) can be approximated by

$$P_{t+1} = P_t - P_t M_q(\hat{\omega}) P_t + \frac{1}{\sigma^2_z}\Sigma^\omega.$$

Further assuming that $1/\left(1+q'_{t-1}P_t q_{t-1}\right) \approx 1$, the filtering equations can be re-written as

$$\begin{aligned}\hat{\omega}_{t+1|t} &= \hat{\omega}_{t|t-1} + P_t q_{t-1}\left(\tilde{z}_t - \hat{\omega}'_{t|t-1}q_{t-1}\right) \quad (17)\\ P_{t+1} &= P_t - P_t M_q(\hat{\omega}) P_t + \frac{1}{\sigma^2_z}\Sigma^\omega.\end{aligned}$$

[3] This ODE system is called the mean dynamics of the estimates. Sargent and Williams (2005) investigate a second ODE system which describe the escape dynamics, which are not the focus of this paper. In the simulations conducted with the calibrated model we did not observe escape dynamics.

[4] The unconditional expectation has finite value if the system is E-Stable. See also Evans and Honkapohja (2001).

The asymptotic behavior of (17) can be shown to be equivalent to the asymptotic behavior of constant-gain least squares, provided agents prior's on Σ^{ω} satisfy

$$\Sigma^{\omega} = g^2 \sigma_z^2 M_q \left(\hat{\omega}\right)^{-1}. \tag{18}$$

To show this, the matrix P converges asymptotically to a unique positive definite matrix which solves the Riccati equation

$$P M_q \left(\hat{\omega}\right) P = \frac{1}{\sigma_z^2} \Sigma^{\omega}$$

using (18) the solution becomes $P = g M_q \left(\hat{\omega}\right)^{-1}$. Hence, in large samples, P_t converges to P and R_t converges to $M_q \left(\hat{\omega}\right)$, implying that the constant gain algorithm and the Kalman filter have the same asymptotic behavior. As shown in Sargent and Williams (2005), the two algorithms share the same asymptotic behavior in large samples but their transitional dynamics displays differences in small samples. In this paper, we analyze the dynamics of agents' beliefs at their stationary distribution, and therefore evaluate the learning algorithm in large samples.

7 Model Dynamics: additional figures

Figure 2 shows the median impulse response for the rental rate of capital and the wage in efficiency units one period after a unit shock of technology. Impulse response are calculated as described in the main text of the paper.

In the period after the shock the rental rate of capital under learning (solid line) increases by more than under rational expectations. At the same time, efficiency wages decrease more than under rational expectations. The effects are reversed as time unfolds. By looking at the first period after the shock, *one-period-ahead* expectations of R_t^k and w_t are almost identical under both learning and rational expectations. Figure 3 shows the agents' forecasts of interest and wage rates for a *long time horizon*, in the period after the technology shock. These are not impulse response functions: they are the set of forecasts agents make about the future realization of prices in the period after the shock. The forecasts under learning (solid line) and under rational expectations (dashed line) differ significantly only at longer forecasting horizons (beyond 20 quarters).

Although the difference in the forecasted path appears to be small, the discounted sum of R_t^k and w_t has non-negligible impact on agents' decisions – see figure 2 in the main text. Figure 4 shows agents expectations at the time of the shock, but *before* their model's estimates have been updated. Here the forecasts under learning and rational expectations are aligned. (This because beliefs are centered around rational expectations. See discussion in the main text.)

Consider agents' decisions after their model's revision. On the one hand, higher expected returns to capital induce substitution effects toward future consumption. On the other hand, agents revise downward the expected path of wages, with respect to the previous period when the shock hit. This reduces the positive income effects generated by the technology shock. As a result, agents reduce

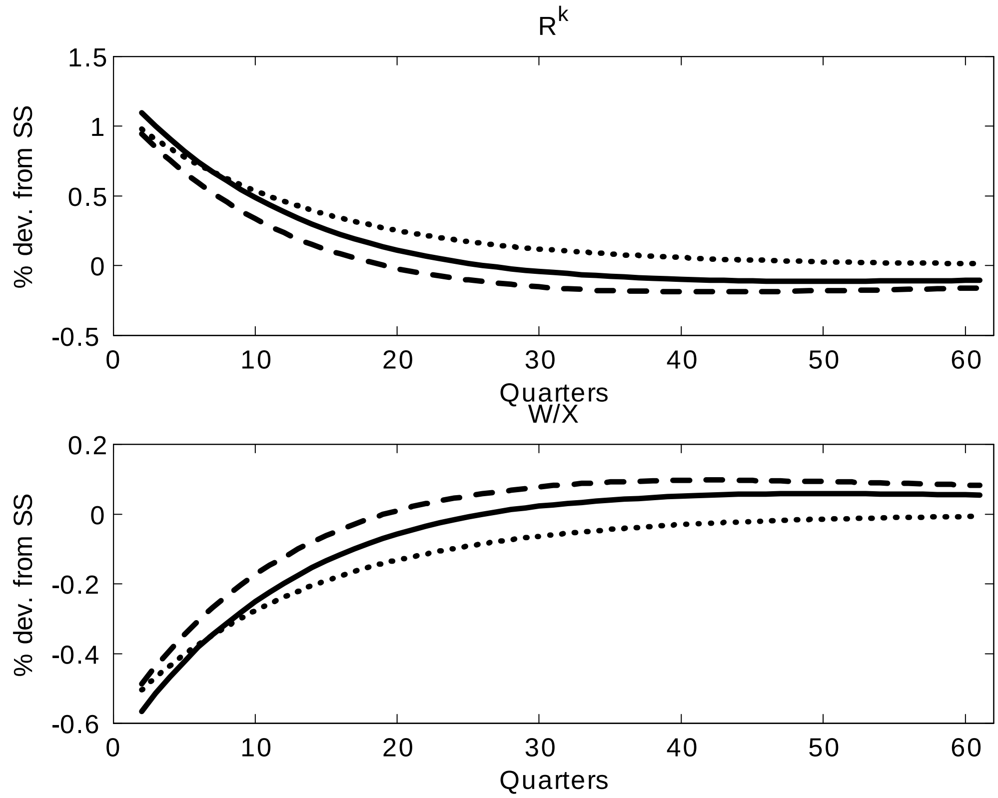

Figure 2: The figure shows the impulse response of factor prices, starting the period *after* a unit technoogy shock. On the top panel, the solid line shows the response of R_t^k and the dashed line shows $\hat{E}_t R_{t+1}^k$, generated from the learning model. The dotted line shows the path of the rental rate of capital in the model under rational expectations. The bottom panel does the same for the real wage in efficiency units.

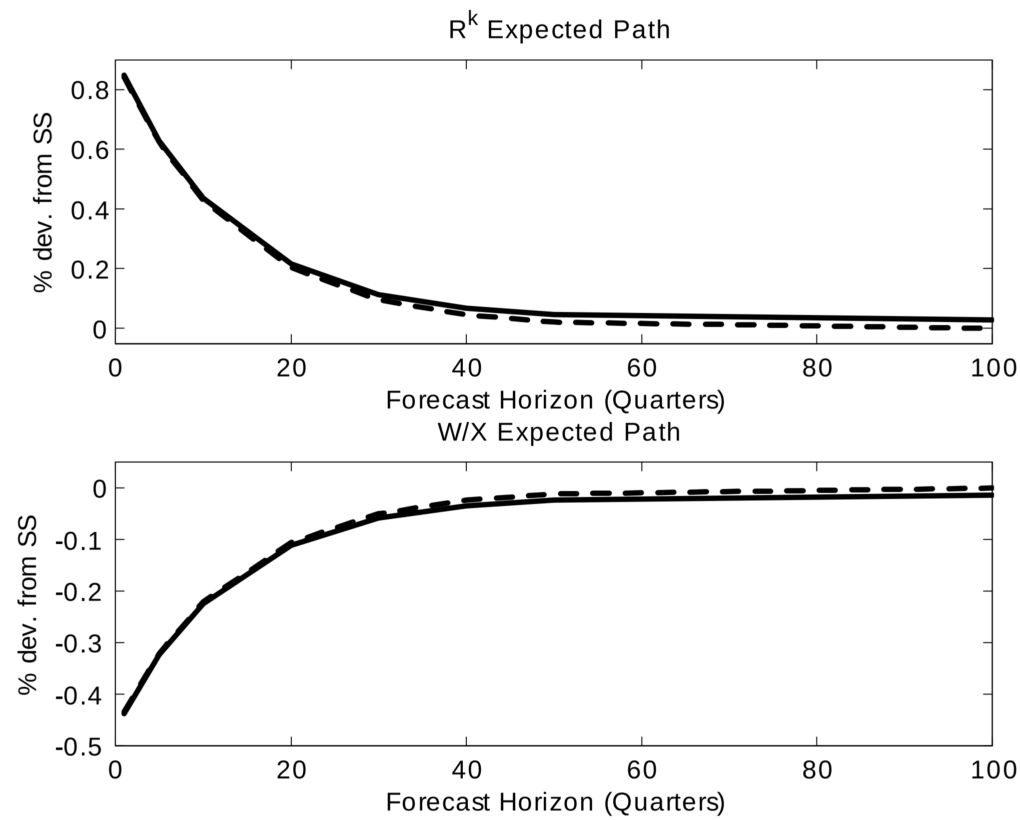

Figure 3: The figure shows the forecasted path of R^k and w at different forecasting horizons. This is agents' forecas on the period *after* a unit technology shock. The top panel show the forecast for R^k, under learning (solid line) and under rational expectations (dashed line).

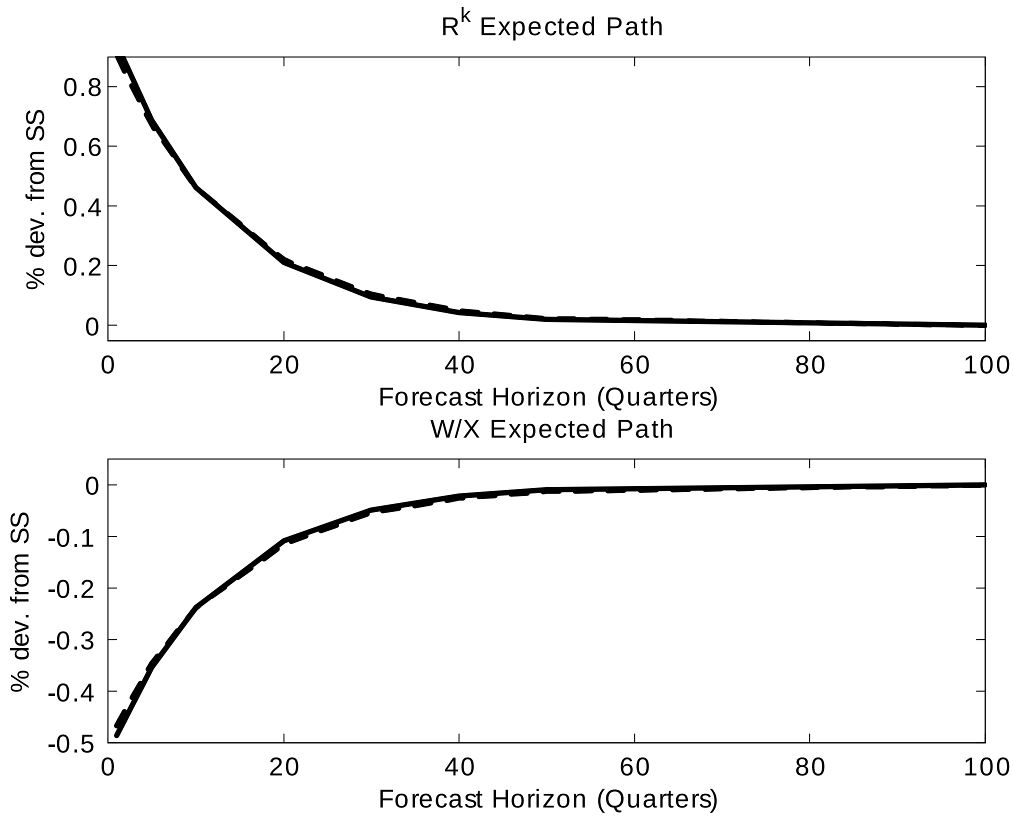

Figure 4: The figure shows the forecasted path of R^k and w at different forecasting horizons. This is agents' forecas on the period *before* updating their model's estimates. The top panel show the forecast for R^k, under learning (solid line) and under rational expectations (dashed line).

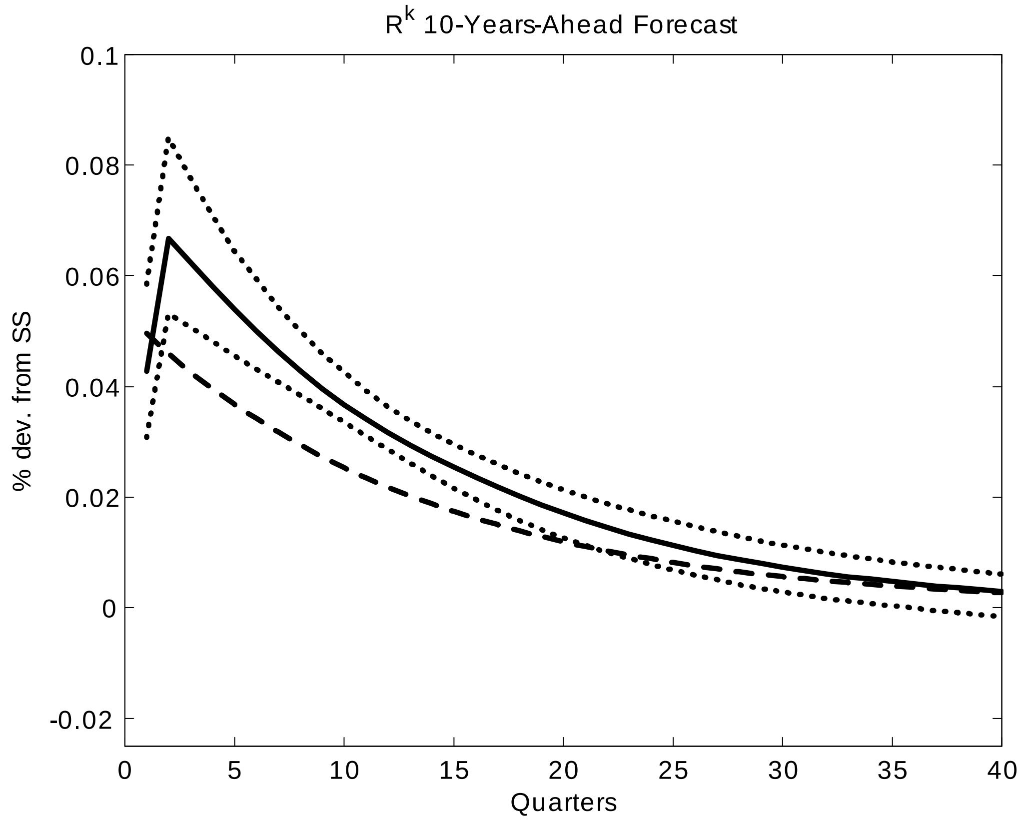

Figure 5: The figure shows the 10-years forecast for the retal rate of capital, after a unit technology shock. The solid line shows the response under learning with the dotted line representing the interquartile range. The dashed line represents the response under rational expectations.

consumption and increase their supply of labor; aggregate labor supply curve shifts outwards, as the marginal utility of income increases. The shift in expectations induces an increase in investment demand and labor supply through intertemporal substitution of consumption and leisure.

Finally, figure 5 shows the *time series* of the 10-year forecast for the rental rate of capital (the forecast for the wage is similar).

The model under learning induces a small departure from rational expectations. Notice also that for longer forecasting horizons the distribution of impulse responses under learning is fairly tight.

8 "Optimal Decisions" versus "Euler Equation" learning

Figure 3 offers the key to understanding the difference between the model dynamics under different decision rules. In the period after the shock, when the estimated model has been updated, short-term expectations in the learning model and in the model under rational expectations are virtually

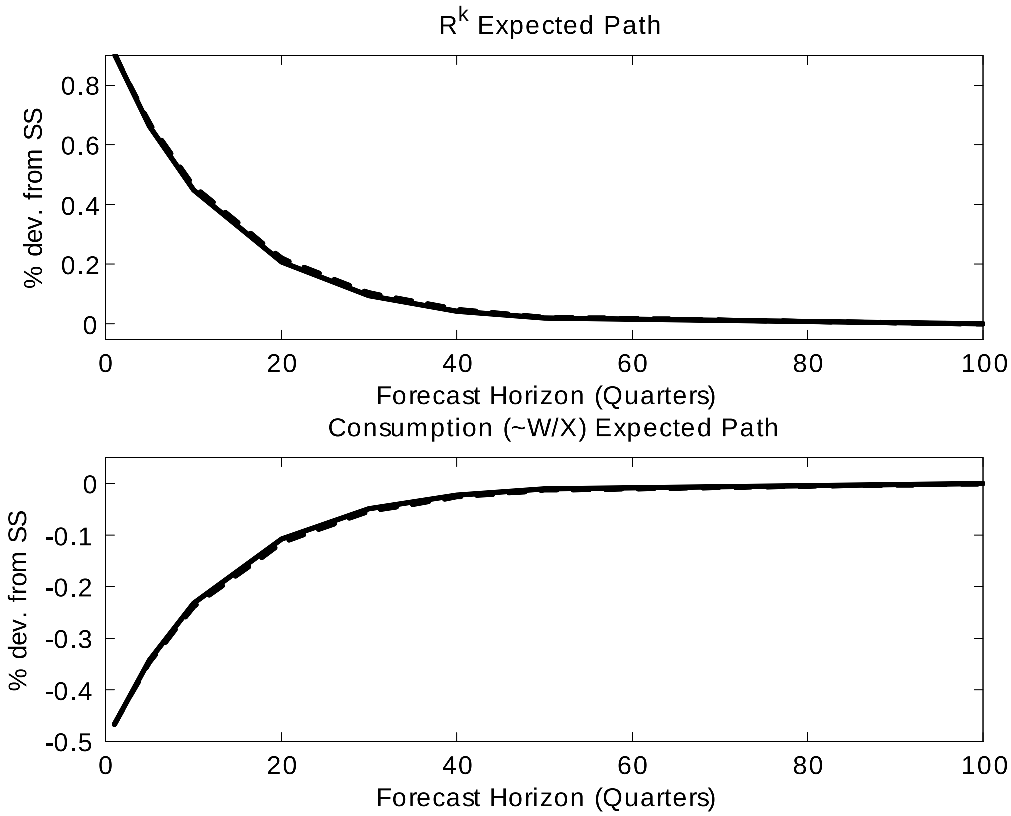

Figure 6: The figure shows the expected paths of R^k and w for different forecasting horizons, before agents forecasting update their model's coefficients. The forecasts are generated from the model under Euler learning.

identical. Because the agents' estimates under both Euler equation and optimal decision learning are centered around their rational expectation values and because under both approaches the learning algorithm used is the same, we should expect the forecasts under Euler learning to be very close to the forecasts produced under optimal decision learning. Figures 6 and 7 show exactly this.

The forecasted paths of R^k and c (the same as w under the benchmark calibration) obtained from the Euler learning model are very close to those obtained by the model with optimal decisions. The only potential source of difference is that they depend on capital, which follows a different path in the two models. As shown in the figures, the difference is not important.

It is then clear learning matters for the dynamics of our model because optimal decisions depend on the future path of factor prices. After the technology shock, the update in the forecasting model's coefficients feeds back into the investment and hours worked which then differ from the rational expectations model predictions. Thus, in successive periods, both agents' forecasts and the actual realizations of each variable will be different from rational expectations.

This does not happen under Euler equation learning because the consumption decision only

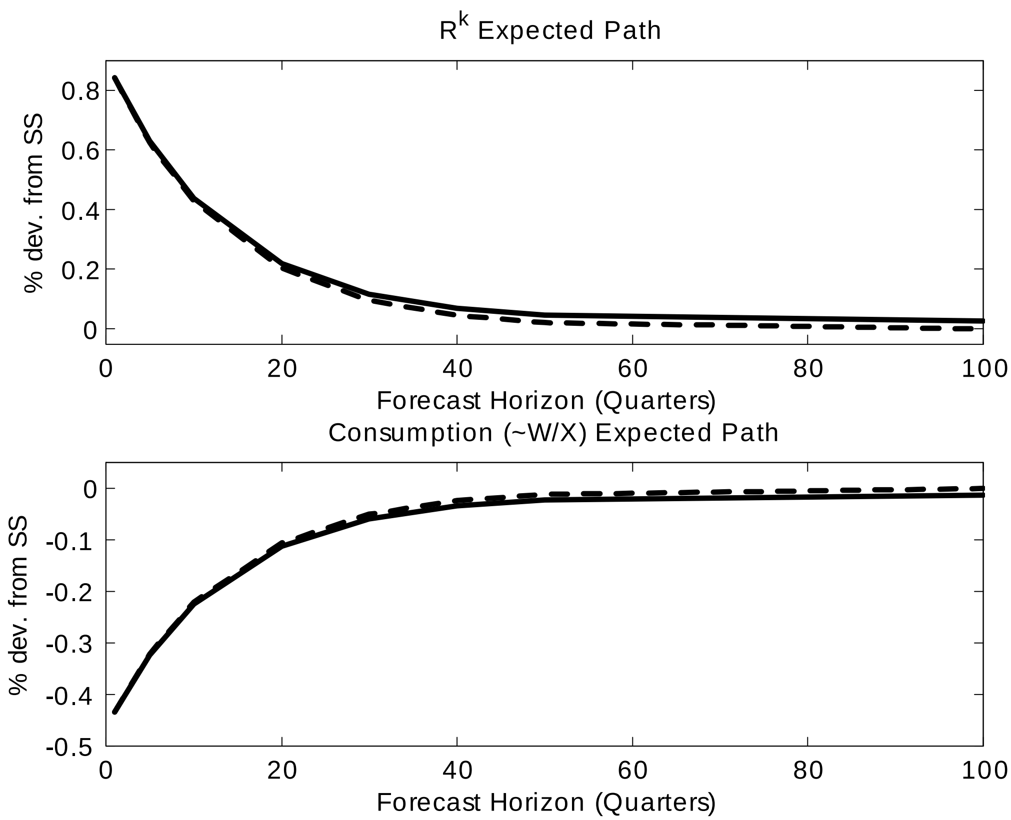

Figure 7: The figure shows the expected paths of R^k and w for different forecasting horizons, after agents forecasting update their model's coefficients. The forecasts are generated from the model under Euler learning.

depends on one-period-ahead forecasts. Finally, in both models, short-term expectations in the period after the shock are very close to rational expectations because most of their variation depends on the endogenous variable (in this case capital), not the forecasting model's coefficients. As the forecasting horizon increases, the role of the endogenous variable decreases because it is expected to revert back to its steady state value. The main determinants of the forecast at longer horizon are thus the changes in the forecasting model's coefficients!