

# Financial Innovation and Asset Price Volatility

## Online Technical Appendix

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In this technical appendix we derive all numbered equations displayed in the paper.

### Equations (1)

For the two models in the paper, the first set of numbered equations,

$$c_t^{y,h} = \frac{1}{3}e_t^h, \quad c_t^{m,h} = \frac{1}{2}\kappa_t^{m,h}, \quad c_t^{o,h} = \kappa_t^{o,h} \quad (1)$$

with  $h = 1, 2$ , follows already from Huffman (1987) and Kubler and Schmedders (2011). Here we derive this result again in some more generality for economies with  $I \geq 1$  long-lived securities,  $J$  short-lived securities and  $S$  states.

For the purpose of this derivation only, we need to introduce some additional notation. It suffices to examine a single individual's utility optimization problem in the OLG economy and so we omit the type superscript  $h$  and the time subscript  $t$ . We need to explicitly refer to the dependence of the decision variables on the exogenous shocks and denote consumption at middle age by  $c_{s_2}^m$  and consumption at old age by  $c_{s_2, s_3}^o$  for  $(s_2, s_3) \in \mathcal{S} \times \mathcal{S}$ . The agent can invest in  $I$  different stocks,  $i \in \mathcal{I} = \{1, 2, \dots, I\}$ , and  $J$  different one-period securities,  $j \in \mathcal{J} = \{1, 2, \dots, J\}$ . We denote the stock dividends by  $d_s^i$  and the payoffs of the one-period securities by  $f_s^j$  for  $s \in \mathcal{S}$ . The corresponding prices are denoted by  $p_s^i$  and  $q_s^j$ , respectively.

We consider an individual's utility optimization problem in the OLG economy. The decision variables are the consumption choices,  $c^y$ ,  $c_{s_2}^m$  for  $s_2 \in \mathcal{S}$ ,  $c_{s_2, s_3}^o$  for  $(s_2, s_3) \in \mathcal{S} \times \mathcal{S}$ , the portfolio choices of long-lived securities,  $\phi^{y,i}$  for  $i \in \mathcal{I}$  and  $\phi_{s_2}^{m,i}$  for  $i \in \mathcal{I}$  and  $s_2 \in \mathcal{S}$ , and the positions of short-lived securities  $\theta^{y,j}$  for  $j \in \mathcal{J}$  and  $\theta_{s_2}^{m,j}$  for  $j \in \mathcal{J}$  and  $s_2 \in \mathcal{S}$ .

$$\max \quad \ln(c^y) + \sum_{s_2=1}^S \pi_{s_2} \ln(c_{s_2}^m) + \sum_{s_2=1}^S \sum_{s_3=1}^S \pi_{s_2, s_3} \ln(c_{s_2, s_3}^o)$$

$$c^y + \sum_{i=1}^I p^i \phi^{y,i} + \sum_{j=1}^J q^j \theta^{y,j} - e = 0 \quad (a)$$

$$c_{s_2}^m + \sum_{i=1}^I p_{s_2}^i \phi_{s_2}^{m,i} + \sum_{j=1}^J q_{s_2}^j \theta_{s_2}^{m,j} - \left( \sum_{i=1}^I (p_{s_2}^i + d_{s_2}^i) \phi_{s_2}^{y,i} + \sum_{j=1}^J f_{s_2}^j \theta_{s_2}^{y,j} \right) = 0 \quad \forall s_2 \in \mathcal{S} \quad (b)$$

$$c_{s_2, s_3}^o - \left( \sum_{i=1}^I (p_{s_2, s_3}^i + d_{s_3}^i) \phi_{s_2}^{m,i} + \sum_{j=1}^J f_{s_3}^j \theta_{s_2}^{m,j} \right) = 0 \quad \forall (s_2, s_3) \in \mathcal{S} \times \mathcal{S}. \quad (c)$$

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We denote the Lagrange multipliers for the budget constraints (a), (b), and (c) by  $\lambda$ ,  $\mu_{s_2}$  for all  $s_2 \in \mathcal{S}$ , and  $\nu_{s_2, s_3}$  for all  $(s_2, s_3) \in \mathcal{S} \times \mathcal{S}$ , respectively. Then the necessary and sufficient first-order conditions for this optimization problem are as follows.

$$\frac{1}{c^y} - \lambda = 0 \quad (\text{d})$$

$$\frac{\pi_{s_2}}{c_{s_2}^m} - \mu_{s_2} = 0 \quad \forall s_2 \in \mathcal{S} \quad (\text{e})$$

$$\frac{\pi_{s_2, s_3}}{c_{s_2, s_3}^o} - \nu_{s_2, s_3} = 0 \quad \forall (s_2, s_3) \in \mathcal{S} \times \mathcal{S} \quad (\text{f})$$

$$\lambda p^i - \sum_{s_2 \in \mathcal{S}} \mu_{s_2} (p_{s_2}^i + d_{s_2}^i) = 0 \quad \forall i \in \mathcal{I} \quad (\text{g})$$

$$\lambda q^j - \sum_{s_2 \in \mathcal{S}} \mu_{s_2} f_{s_2}^j = 0 \quad \forall j \in \mathcal{J} \quad (\text{h})$$

$$\mu_{s_2} p_{s_2}^i - \sum_{s_3 \in \mathcal{S}} \nu_{s_2, s_3} (p_{s_2, s_3}^i + d_{s_3}^i) = 0 \quad \forall i \in \mathcal{I}, \forall s_2 \in \mathcal{S} \quad (\text{i})$$

$$\mu_{s_2} q_{s_2}^j - \sum_{s_3 \in \mathcal{S}} \nu_{s_2, s_3} f_{s_3}^j = 0 \quad \forall j \in \mathcal{J}, \forall s_2 \in \mathcal{S}. \quad (\text{j})$$

Multiplying (g) by  $\phi^{y,i}$  and (i) by  $\phi_{s_2}^{m,i}$  for all  $i \in \mathcal{I}$  and then summing over all  $i \in \mathcal{I}$  and  $s_2 \in \mathcal{S}$  leads to

$$\lambda \sum_{i=1}^I p^i \phi^{y,i} - \sum_{i=1}^I \sum_{s_2 \in \mathcal{S}} \mu_{s_2} (p_{s_2}^i + d_{s_2}^i) \phi^{y,i} + \sum_{i=1}^I \sum_{s_2 \in \mathcal{S}} \mu_{s_2} p_{s_2}^i \phi_{s_2}^{m,i} - \sum_{i=1}^I \sum_{s_2 \in \mathcal{S}} \sum_{s_3 \in \mathcal{S}} \nu_{s_2, s_3} (p_{s_2, s_3}^i + d_{s_3}^i) \phi_{s_2}^{m,i} = 0.$$

Similarly, multiplying (h) by  $\theta^{y,j}$  and (j) by  $\theta_{s_2}^{m,j}$  for all  $j \in \mathcal{J}$  and then summing over all  $j \in \mathcal{J}$  and  $s_2 \in \mathcal{S}$  leads to

$$\lambda \sum_{j=1}^J q^j \theta^{y,j} - \sum_{j=1}^J \sum_{s_2 \in \mathcal{S}} \mu_{s_2} f_{s_2}^j \theta^{y,j} + \sum_{j=1}^J \sum_{s_2 \in \mathcal{S}} \mu_{s_2} q_{s_2}^j \theta_{s_2}^{m,j} - \sum_{j=1}^J \sum_{s_2 \in \mathcal{S}} \sum_{s_3 \in \mathcal{S}} \nu_{s_2, s_3} f_{s_3}^j \theta_{s_2}^{m,j} = 0.$$

Adding the last two equations and replacing the Lagrange multipliers  $\lambda$ ,  $\mu_{s_2}$ , and  $\nu_{s_2, s_3}$  by the expressions from (d), (e), and (f), respectively, leads to

$$\begin{aligned} \frac{1}{c^y} \left( \sum_{i=1}^I p^i \phi^{y,i} + \sum_{j=1}^J q^j \theta^{y,j} \right) &= \sum_{s_2 \in \mathcal{S}} \frac{\pi_{s_2}}{c_{s_2}^m} \left( \sum_{i=1}^I (p_{s_2}^i + d_{s_2}^i) \phi^{y,i} + \sum_{j=1}^J f_{s_2}^j \theta^{y,j} - \sum_{i=1}^I p_{s_2}^i \phi_{s_2}^{m,i} - \sum_{j=1}^J q_{s_2}^j \theta_{s_2}^{m,j} \right) \\ &\quad + \sum_{s_2 \in \mathcal{S}} \sum_{s_3 \in \mathcal{S}} \frac{\pi_{s_2, s_3}}{c_{s_2, s_3}^o} \left( \sum_{i=1}^I (p_{s_2, s_3}^i + d_{s_3}^i) \phi_{s_2}^{m,i} + \sum_{j=1}^J f_{s_3}^j \theta_{s_2}^{m,j} \right). \end{aligned}$$

Combining the last equation with the budget equations (b) and (c) leads to

$$\frac{1}{c^y} \left( \sum_{i=1}^I p^i \phi^{y,i} + \sum_{j=1}^J q^j \theta^{y,j} \right) = \sum_{s_2 \in \mathcal{S}} \pi_{s_2} + \sum_{s_2 \in \mathcal{S}} \sum_{s_3 \in \mathcal{S}} \pi_{s_2, s_3} = 2.$$

The budget equations (a) for the first period now imply  $c^y = \frac{1}{3}e$  which is the first equation in condition (1).

Multiplying (i) by  $\phi_{s_2}^{m,i}$  for all  $i \in \mathcal{I}$  and then summing over all  $i \in \mathcal{I}$  leads to

$$\sum_{i=1}^I \mu_{s_2} p_{s_2}^i \phi_{s_2}^{m,i} - \sum_{i=1}^I \sum_{s_3 \in \mathcal{S}} \nu_{s_2, s_3} (p_{s_2, s_3}^i + d_{s_3}^i) \phi_{s_2}^{m,i} = 0 \quad \forall s_2 \in \mathcal{S}.$$

Similarly for (j),

$$\sum_{j=1}^J \mu_{s_2} q_{s_2}^j \theta_{s_2}^{m,j} - \sum_{j=1}^J \sum_{s_3 \in \mathcal{S}} \nu_{s_2, s_3} f_{s_3}^j \theta_{s_2}^{m,j} = 0 \quad \forall s_2 \in \mathcal{S}.$$

Adding the last two equations and replacing the Lagrange multipliers  $\mu_{s_2}$  and  $\nu_{s_2,s_3}$  by the expressions from (e) and (f), respectively, leads to

$$\begin{aligned} \frac{\pi_{s_2}}{c_{s_2}^m} \left( \sum_{i=1}^I p_{s_2}^i \phi_{s_2}^{m,i} + \sum_{j=1}^J q_{s_2}^j \theta_{s_2}^{m,j} \right) &= \sum_{s_3 \in \mathcal{S}} \frac{\pi_{s_2,s_3}}{c_{s_2,s_3}^o} \left( \sum_{i=1}^I (p_{s_2,s_3}^i + d_{s_3}^i) \phi_{s_2}^{m,i} + \sum_{j=1}^J \sum_{s_3 \in \mathcal{S}} f_{s_3}^j \theta_{s_2}^{m,j} \right) \\ &= \sum_{s_3 \in \mathcal{S}} \pi_{s_2,s_3} \end{aligned}$$

and so, since  $\pi_{s_2} = \sum_{s_3 \in \mathcal{S}} \pi_{s_2,s_3}$ , it follows that

$$c_{s_2}^m = \sum_{i=1}^I p_{s_2}^i \phi_{s_2}^{m,i} + \sum_{j=1}^J q_{s_2}^j \theta_{s_2}^{m,j} = \frac{1}{2} \left( \sum_{i=1}^I (p_{s_2}^i + d_{s_2}^i) \phi_{s_2}^{y,i} + \sum_{j=1}^J f_{s_2}^j \theta_{s_2}^{y,j} \right)$$

which is the second set of equations in condition (1). The budget equations (c) are the third and last equations in condition (1). This completes the proof of equations (1).

In the remainder of this technical appendix we employ the same notation as in the paper.

### Equations (2)

In a competitive equilibrium the market-clearing condition for the consumption good requires

$$\sum_{h=1}^2 \left( c_t^{y,h} + c_t^{m,h} + c_t^{o,h} \right) = e_t + d_t = \frac{1}{3} e_t + \frac{1}{2} \kappa_t^m + \kappa_t^o.$$

Combining this condition with the market-clearing condition for the stock market,  $\kappa_t^m + \kappa_t^o = p_t + d_t$ , we obtain

$$e_t + d_t + \frac{1}{2} \kappa_t^m = \frac{1}{3} e_t + p_t + d_t.$$

Solving for the stock price then yields the expressions

$$\begin{aligned} p_t &= \frac{2}{3} e_t + \frac{1}{2} \kappa_t^m \\ &= \frac{4}{3} e_t + d_t - \kappa_t^o. \end{aligned} \tag{2}$$

### Equation (3)

In the economy with incomplete markets, the stock is the only security available for trade. The stock holdings of the young agents satisfy

$$p_t \left( \phi_t^{y,1} + \phi_t^{y,2} \right) = \frac{2}{3} e_t.$$

Combining this equation with expression (2) for the stock price yields the aggregate stock holdings of the young agents,

$$\phi_t^{y,1} + \phi_t^{y,2} = \frac{2e_t}{2e_t + \frac{3}{2}\kappa_t^m}.$$

The aggregated cash-at-hand positions of the middle-aged agents satisfy

$$\kappa_{t+1}^m = \left( \phi_t^{y,1} + \phi_t^{y,2} \right) (p_{t+1} + d_{t+1}).$$

Combining the last two equations and replacing  $p_{t+1}$  according to (2) leads to

$$\kappa_{t+1}^m = \frac{2e_t}{2e_t + \frac{3}{2}\kappa_t^m} \left( \frac{2}{3} e_{t+1} + \frac{1}{2} \kappa_{t+1}^m + d_{t+1} \right).$$

Solving for  $\kappa_{t+1}^m$  yields the rule for the evolution of the aggregated cash-at-hand positions of the middle-aged agents,

$$\kappa_{t+1}^m = \frac{1}{\frac{1}{2} + \frac{3}{4} \frac{\kappa_t^m}{e_t}} \left( \frac{2}{3} e_{t+1} + d_{t+1} \right). \tag{3}$$

**Equation (4)**

In the economy with complete markets, the prices  $q_{1,t}$  and  $q_{2,t}$  of the two Arrow securities satisfy the first-order conditions of all agents trading in period  $t$ . The Euler equations of the young agents state

$$\pi_{s_{t+1}}^h c_t^{y,h} = q_{s_{t+1},t} c_{t+1}^{m,h}$$

for  $h = 1, 2$  and  $s_{t+1} \in \{1, 2\}$ . Similarly, the Euler equations of the middle-aged agents state

$$\pi_{s_{t+1}}^h c_t^{m,h} = q_{s_{t+1},t} c_{t+1}^{o,h}.$$

Combining the two sets of Euler equations leads to

$$q_{s_{t+1},t} = \frac{\sum_{h=1}^2 \left( \pi_{s_{t+1}}^h c_t^{y,h} + \pi_{s_{t+1}}^h c_t^{m,h} \right)}{\sum_{h=1}^2 \left( c_{t+1}^{m,h} + c_{t+1}^{o,h} \right)}.$$

Equation (1) implies  $c_t^y = \frac{1}{3}e_t$  and so aggregate consumption of the middle-aged and old agents satisfies

$$\sum_{h=1}^2 \left( c_t^{m,h} + c_t^{o,h} \right) = \frac{2}{3}e_t + d_t.$$

With  $\kappa_t^{m,h} = 2c_t^{m,h}$  from equation (1), we can express the prices of the Arrow securities by

$$q_{s_{t+1},t} = \frac{\sum_{h=1}^2 \left( \pi_{s_{t+1}}^h \frac{1}{3}e_t^h + \pi_{s_{t+1}}^h \frac{1}{2}\kappa_t^{m,h} \right)}{\frac{2}{3}e_{t+1} + d_{t+1}}.$$

Once again using the Euler equations of the young agents we obtain

$$\kappa_{t+1}^{m,h} = 2c_{t+1}^{m,h} = 2 \frac{\pi_{s_{t+1}}^h c_t^{y,h}}{q_{s_{t+1},t}}.$$

Combining the last two equations yields the rule for the evolution of the cash-at-hand positions of the middle-aged agents of type  $h$  in the complete-markets economy,

$$\kappa_{t+1}^{m,h} = \frac{\pi_{s_{t+1}}^h e_t^h \left( \frac{4}{3}e_{t+1} + 2d_{t+1} \right)}{\pi_{s_{t+1}}^1 e_t^1 + \pi_{s_{t+1}}^2 e_t^2 + \frac{3}{2} \left( \pi_{s_{t+1}}^1 \kappa_t^{m,1} + \pi_{s_{t+1}}^2 \kappa_t^{m,2} \right)}. \quad (4)$$

**Equation (5)**

Substituting  $d_{t+1} = \frac{1}{3}e_{t+1}$  into equation (3) leads to

$$\kappa_{t+1}^m = \frac{1}{\frac{1}{2} + \frac{3}{4} \frac{\kappa_t^m}{e_t}} e_{t+1}.$$

The definition of the normalized aggregated cash-at-hand position  $\hat{\kappa}_t^m = \frac{\kappa_t^m}{e_t}$  of the middle-aged agents then yields

$$\hat{\kappa}_{t+1}^m = \frac{1}{\frac{1}{2} + \frac{3}{4} \hat{\kappa}_t^m}. \quad (5)$$

**Equation (6)**

Individual endowments of the young agents satisfy  $e^1(s_t) = e^2(s_t) = \frac{1}{2}e(s_t)$ . Agents of type 1 have beliefs  $(\eta, 1 - \eta)$  and agents of type 2 hold the beliefs  $(1 - \eta, \eta)$  for the two possible exogenous states with  $\eta \geq \frac{1}{2}$ . The Euler equations of the young agents imply

$$c_{t+1}^{m,1}(1) = \eta \frac{\frac{1}{6}e_t}{q_{1,t}}, \quad c_{t+1}^{m,2}(1) = (1 - \eta) \frac{\frac{1}{6}e_t}{q_{1,t}} \quad \text{and} \quad c_{t+1}^{m,1}(2) = (1 - \eta) \frac{\frac{1}{6}e_t}{q_{2,t}}, \quad c_{t+1}^{m,2}(2) = \eta \frac{\frac{1}{6}e_t}{q_{2,t}}.$$

These equations imply

$$\frac{c_t^{m,1}(1)}{c_t^{m,2}(1)} = \frac{\kappa_t^{m,1}(1)}{\kappa_t^{m,2}(1)} = \frac{\eta}{1-\eta} \quad \text{and} \quad \frac{c_t^{m,1}(2)}{c_t^{m,2}(2)} = \frac{\kappa_t^{m,1}(2)}{\kappa_t^{m,2}(2)} = \frac{1-\eta}{\eta}$$

and thus

$$\kappa_t^{m,1}(1) = \eta \kappa_t^m(1), \quad \kappa_t^{m,2}(1) = (1-\eta) \kappa_t^m(1) \quad \text{and} \quad \kappa_t^{m,1}(2) = (1-\eta) \kappa_t^m(2), \quad \kappa_t^{m,2}(2) = \eta \kappa_t^m(2).$$

Since  $\pi_{s_{t+1}}^1 + \pi_{s_{t+1}}^2 = 1$  and  $e^1(s_t) = e^2(s_t) = \frac{1}{2}e(s_t)$  equation (4) leads to

$$\kappa_{t+1}^m = \kappa_{t+1}^{m,1} + \kappa_{t+1}^{m,2} = \frac{e_t e_{t+1}}{\frac{1}{2}e_t + \frac{3}{2} \left( \pi_{s_{t+1}}^1 \kappa_t^{m,1} + \pi_{s_{t+1}}^2 \kappa_t^{m,2} \right)}.$$

For  $s_{t+1} = s_t$  the second term in the denominator is

$$\frac{3}{2} \left( \pi_{s_{t+1}}^1 \kappa_t^{m,1} + \pi_{s_{t+1}}^2 \kappa_t^{m,2} \right) = \frac{3}{2} (\eta^2 + (1-\eta)^2) \kappa_t^m$$

while for  $s_{t+1} \neq s_t$  this term is

$$\frac{3}{2} \left( \pi_{s_{t+1}}^1 \kappa_t^{m,1} + \pi_{s_{t+1}}^2 \kappa_t^{m,2} \right) = \frac{3}{2} (2\eta(1-\eta)) \kappa_t^m.$$

And so the evolution of the normalized aggregated cash-at-hand position of the middle-aged agents in the complete-markets economy satisfies

$$\hat{\kappa}_{t+1}^m = \frac{1}{\frac{1}{2} + \frac{3}{4}\Pi_{t+1}\hat{\kappa}_t^m} \tag{6}$$

with

$$\Pi_{t+1} = \begin{cases} 2(\eta^2 + (1-\eta)^2) & \text{if } s_{t+1} = s_t, \\ 4\eta(1-\eta) & \text{if } s_{t+1} \neq s_t. \end{cases}$$