

Online Appendix: Proof of Proposition 5

PROPOSITION 5: When reserve price increases, the total payment of every advertiser who remains (except the last) increase by an identical amount.

PROOF:

Setup. Advertisers are $1..n$, and slots are also $1..n$. Initial reserve price r^0 changes to some new reserve price r^1 . Suppose for now that the new reserve price r^1 remains sufficiently low that no advertiser is priced out of the market, *i.e.* $r^1 \leq s_n$.

Advertiser n . Advertiser n receives slot n and pays the reserve price r . So when the reserve price increases from r^0 to r^1 , advertiser n 's per-click payment increases by $\Delta r = r^1 - r^0$. Advertiser n receives α_n clicks, so advertiser n 's total payment increases by $\alpha_n \Delta r$.

Advertiser $n-1$. The per-click fee p_{n-1} paid by advertiser $n-1$ is determined by the per-click bid b_n of advertiser n . With reserve price r , advertiser n bids $b_n = s_n - \frac{\alpha_n}{\alpha_{n-1}}(s_n - r)$. (This is the base case of Theorem 2 of EOS.) So when the reserve price changes from r^0 to r^1 , advertiser $n-1$'s change in per-click payment equals advertiser n 's change in per-click bid, which is

$$\begin{aligned} \Delta p_{n-1} &= \Delta b_n = b_n^1 - b_n^0 \\ &= \left(s_n - \frac{\alpha_n}{\alpha_{n-1}}(s_n - r^1) \right) \\ &\quad - \left(s_n - \frac{\alpha_n}{\alpha_{n-1}}(s_n - r^0) \right) \\ &= \frac{\alpha_n}{\alpha_{n-1}} \Delta r \end{aligned}$$

Advertiser $n-1$ receives α_{n-1} clicks. So when the reserve price increases by Δr , advertiser $n-1$'s total payment increases by $\alpha_{n-1} \Delta p_{n-1} = \alpha_{n-1} \left(\frac{\alpha_n}{\alpha_{n-1}} \Delta r \right) = \alpha_n \Delta r$.

Advertiser $n-2$ and the general case. Advertiser $n-2$ pays a per-click fee p_{n-2} given by b_{n-1} . With reserve price r , advertiser $n-1$ bids $b_{n-1} = s_{n-1} - \frac{\alpha_{n-1}}{\alpha_{n-2}}(s_{n-1} - b_n)$. (This is the general case of Theorem 2 of EOS.) So when the reserve price changes from r^0 to r^1 , advertiser

$n-1$'s change in bid is

$$\begin{aligned} \Delta b_{n-1} &= b_{n-1}^1 - b_{n-1}^0 \\ &= \left(s_n - \frac{\alpha_{n-1}}{\alpha_{n-2}}(s_n - b_n^1) \right) \\ &\quad - \left(s_n - \frac{\alpha_{n-1}}{\alpha_{n-2}}(s_n - b_n^0) \right) \\ &= \frac{\alpha_{n-1}}{\alpha_{n-2}} (b_n^1 - b_n^0) \\ &= \frac{\alpha_{n-1}}{\alpha_{n-2}} (\Delta b_n) \\ &= \frac{\alpha_{n-1}}{\alpha_{n-2}} \left(\frac{\alpha_n}{\alpha_{n-1}} (r^1 - r^0) \right) \\ &= \frac{\alpha_n}{\alpha_{n-2}} \Delta r \end{aligned}$$

Advertiser $n-2$ receives α_{n-2} clicks. So advertiser $n-2$'s total payment increases by $\alpha_{n-2} \Delta b_{n-1} = \alpha_{n-2} \frac{\alpha_n}{\alpha_{n-2}} \Delta r = \alpha_n \Delta r$. Recursing upwards to advertiser 1 confirms that each advertiser's total payment increases by the same amount, $\alpha_n \Delta r$.

Reserve price that excludes one or more advertisers. We now allow an increase in reserve price such that one or more advertisers is priced out of the market (relaxing the assumption in the first paragraph of the proof).

Suppose the increased reserve r^1 exceeds advertiser n 's valuation s_n . Then advertiser n can never achieve a positive profit by buying ads; n would have to pay more than his valuation. So n exits. What about the other advertisers? Advertiser $n-1$ now takes on the role of n in the preceding analysis. $\Delta p_{n-1} = \Delta b_n = r^1 - b_n^0$ which is irregular and cannot be fully simplified. But advertisers $n-2$ and above follow the pattern of the preceding section, and for each of those advertiser, total payment increases by $\alpha_{n-1} \Delta r$.

More generally, if the increased reserve price leads j advertisers to drop out, then the total increase of advertisers 1 through $n-j-1$ is $\alpha_{n-j} \Delta r$.