Appendix to “Long-Run Risk, the Wealth-Consumption Ratio, and the Temporal Pricing of Risk”

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In this appendix, we first derive four risk premia: the expected excess returns on a consumption claim, on equity and on real and nominal bonds. We then obtain the Alvarez and Jermann (2005) decomposition of the SDF in the long-run risk model. We present the parameter values used in our calibration and our simulation results. Finally, we report robustness checks and empirical variance ratios.

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I. Wealth-Consumption Ratio and Consumption Risk Premium

We start from the aggregate budget constraint:

$$W_{t+1} = R^c_{t+1}(W_t - C_t). \tag{1}$$

The beginning-of-period (or cum-dividend) total wealth W_t that is not spent on aggregate consumption C_t earns a gross return R^c_{t+1} and leads to beginning-of-next-period total wealth W_{t+1}. The return on a claim to aggregate consumption, the *total wealth return*, can be written as

$$R^c_{t+1} = \frac{W_{t+1}}{W_t - C_t} = \frac{C_{t+1}}{C_t}\frac{WC_{t+1}}{WC_t - 1}.$$

We use the Campbell (1991) approximation of the log total wealth return $r^c_t = \log(R^c_t)$ around the long-run average log wealth-consumption ratio $\mu_{wc} \equiv E[w_t - c_t]$:

$$r^c_{t+1} = \kappa^c_0 + \Delta c_{t+1} + wc_{t+1} - \kappa^c_1 wc_t,$$

where the linearization constants κ^c_0 and κ^c_1 are non-linear functions of the unconditional mean log wealth-consumption ratio μ_{wc}:

$$\kappa^c_1 = \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1} > 1 \;\text{ and }\; \kappa^c_0 = -\log\left(e^{\mu_{wc}} - 1\right) + \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1}\mu_{wc}.$$

Throughout the paper, we use lower letters to denote logs.

The Euler equation for any asset i with lognormal return R^i implies:

$$0 = \mathbb{E}_t\left[sdf_{t+1}\right] + \mathbb{E}_t\left[r^i_{t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left[sdf_{t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left[r^i_{t+1}\right] + \mathrm{Cov}_t\left[sdf_{t+1}, r^i_{t+1}\right] \tag{2}$$

We conjecture that the wealth-consumption ratio is linear in the state variables x_t, σ^2_{gt} and σ^2_{xt}:

$$wc_t = \mu_{wc} + W_x x_t + W_{gs}\left(\sigma^2_{gt} - \sigma^2_g\right) + W_{xs}\left(\sigma^2_{xt} - \sigma^2_x\right)$$

We first compute the different components of equation 2:

$$\begin{aligned}
r^c_{t+1} &= r^c_0 + \left[1 + W_x\left(\rho - \kappa^c_1\right)\right]x_t + W_{gs}\left(\nu_g - \kappa^c_1\right)\left(\sigma^2_{gt} - \sigma^2_g\right) + W_{xs}\left(\nu_x - \kappa^c_1\right)\left(\sigma^2_{xt} - \sigma^2_x\right) \\
&\quad + \sigma_{gt}\eta_{t+1} + W_x\sigma_{xt}e_{t+1} + W_{gs}\sigma_{gw}w_{g,t+1} + W_{xs}\sigma_{xw}w_{x,t+1} \\
\mathbb{E}_t\left[r^c_{t+1}\right] &= r_0 + \left[1 + W_x\left(\rho - \kappa^c_1\right)\right]x_t + W_{gs}\left(\nu_g - \kappa^c_1\right)\left(\sigma^2_{gt} - \sigma^2_g\right) + W_{xs}\left(\nu_x - \kappa^c_1\right)\left(\sigma^2_{xt} - \sigma^2_x\right) \\
r^c_{t+1} - \mathbb{E}_t\left[r^c_{t+1}\right] &= \sigma_{gt}\eta_{t+1} + W_x\sigma_{xt}e_{t+1} + W_{gs}\sigma_{gw}w_{g,t+1} + W_{xs}\sigma_{xw}w_{x,t+1} \\
\mathbb{V}_t\left[r^c_{t+1}\right] &= \sigma^2_{gt} + W^2_x\sigma^2_{xt} + W^2_{gs}\sigma^2_{gw} + W^2_{xs}\sigma^2_{xw} \\
r^c_0 &= \kappa^c_0 + \mu_g + \left(1 - \kappa^c_1\right)\mu_{wc}
\end{aligned}$$

Epstein and Zin (1989) show that the log real stochastic discount factor is

$$
\begin{aligned}
sdf_{t+1} &= \theta \log \delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1)\, r^c_{t+1} \\
&= \mu_s + \left\{ -\frac{\theta}{\psi} + (\theta-1)\left[1 + W_x\left(\rho - \kappa^c_1\right)\right] \right\} x_t \\
&\quad + \left\{ W_{gs}\left(\nu_g - \kappa^c_1\right)(\theta-1)\right\}\left(\sigma^2_{gt} - \sigma^2_g\right) + \left\{ W_{xs}\left(\nu_x - \kappa^c_1\right)(\theta-1)\right\}\left(\sigma^2_{xt} - \sigma^2_x\right) \\
&\quad + \left\{ \theta\left(1 - \frac{1}{\psi}\right) - 1 \right\}\sigma_{gt}\eta_{t+1} + (\theta-1)\left\{ W_x \sigma_{xt} e_{t+1} + W_{gs}\sigma_{gw} w_{g,t+1} + W_{xs}\sigma_{xw} w_{x,t+1} \right\} \\
sdf_{t+1} - \mathbb{E}_t\left[sdf_{t+1}\right] &= \left\{ \theta\left(1 - \frac{1}{\psi}\right) - 1 \right\}\sigma_{gt}\eta_{t+1} + (\theta-1)\left\{ W_x \sigma_{xt} e_{t+1} + W_{gs}\sigma_{gw} w_{g,t+1} + W_{xs}\sigma_{xw} w_{x,t+1} \right\} \\
\mathbb{E}_t\left[sdf_{t+1}\right] &= \mu_s + \left\{ -\frac{\theta}{\psi} + (\theta-1)\left[1 + W_x\left(\rho - \kappa^c_1\right)\right] \right\} x_t \\
&\quad + \left\{ W_{gs}\left(\nu_g - \kappa^c_1\right)(\theta-1)\right\}\left(\sigma^2_{gt} - \sigma^2_g\right) + \left\{ W_{xs}\left(\nu_x - \kappa^c_1\right)(\theta-1)\right\}\left(\sigma^2_{xt} - \sigma^2_x\right) \\
\mathbb{V}_t\left[sdf_{t+1}\right] &= \left\{ \theta\left(1 - \frac{1}{\psi}\right) - 1 \right\}^2 \sigma^2_{gt} + (\theta-1)^2\left\{ W^2_x\sigma^2_{xt} + W^2_{gs}\sigma^2_{gw} + W^2_{xs}\sigma^2_{xw} \right\} \\
\mu_s &= \theta \log \delta - \frac{\theta}{\psi}\mu_g + (\theta-1)\, r^c_0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}_t\left[r^c_{t+1}, sdf_{t+1}\right] &= \mathbb{E}_t\left[(r^c_{t+1} - \mathbb{E}_t\left[r^c_{t+1}\right])(sdf_{t+1} - \mathbb{E}_t\left[sdf_{t+1}\right])\right] \\
&= \left\{ \theta\left(1 - \frac{1}{\psi}\right) - 1 \right\}\sigma^2_{gt} + W^2_x(\theta-1)\,\sigma^2_{xt} + W^2_{gs}(\theta-1)\,\sigma^2_{gw} + W^2_{xs}(\theta-1)\,\sigma^2_{xw}
\end{aligned}
$$

Plugging these different components into equation (2) evaluated at $i = c$ yields:

$$0 = r^c_0 + \mu_s + \frac{\theta^2}{2}\left\{ \left(1 - \frac{1}{\psi}\right)^2 \sigma^2_g + W^2_x\sigma^2_x + W^2_{gs}\sigma^2_{gw} + W^2_{xs}\sigma^2_{xw} \right\} \tag{3}$$

$$+\theta\left\{ -\frac{1}{\psi} + \left[1 + W_x\left(\rho - \kappa^c_1\right)\right] \right\} x_t \tag{4}$$

$$+\frac{\theta}{2}\left\{ 2W_{gs}\left(\nu_g - \kappa^c_1\right) + \theta\left(1 - \frac{1}{\psi}\right)^2 \right\}\left(\sigma^2_{gt} - \sigma^2_g\right) \tag{5}$$

$$+\frac{\theta}{2}\left\{ 2W_{xs}\left(\nu_x - \kappa^c_1\right) + \theta W^2_x \right\}\left(\sigma^2_{xt} - \sigma^2_x\right) \tag{6}$$

Then setting all coefficients equal to zero we obtain:

$$(4) \quad \Longrightarrow W_x = \frac{1-\frac{1}{\psi}}{\kappa_1^c - \rho}$$

$$(5) \quad \Longrightarrow W_{gs} = \frac{\theta\left(1-\frac{1}{\psi}\right)^2}{2\left(\kappa_1^c - \nu_g\right)}$$

$$(6) \quad \Longrightarrow W_{xs} = \frac{\theta}{2\left(\kappa_1^c - \nu_x\right)}\left(1 - \frac{\frac{1}{\psi}}{\kappa_1^c - \rho}\right)^2$$

If the IES exceeds 1, then $W_x > 0$, $W_{gs} < 0$, and $W_{xs} < 0$.
Plugging these coefficients back into equation (3) implicitly defines a nonlinear equation in one unknown (μ_{wc}), which can be solved for numerically, characterizing the average wealth-consumption ratio.
According to (2), the risk premium (expected excess real return corrected for a Jensen term) on the consumption claim is given by[1]:

$$\begin{aligned}
\mathbb{E}_t\left[r_{t+1}^{c,e}\right] &= -\mathrm{Cov}_t\left[r_{t+1}^c, sdf_{t+1}\right] \\
&= \left\{1-\theta\left(1-\frac{1}{\psi}\right)\right\}\sigma_{gt}^2 + W_x^2\left(1-\theta\right)\sigma_{xt}^2 + W_{gs}^2\left(1-\theta\right)\sigma_{gw}^2 + W_{xs}^2\left(1-\theta\right)\sigma_{xw}^2 \\
&= \lambda_\eta\sigma_{gt}^2 + W_x\lambda_e\sigma_{xt}^2 + W_{gs}\lambda_{gw}\sigma_{gw}^2 + W_{xs}\lambda_{xw}\sigma_{xw}^2
\end{aligned}$$

with the market price of risk vector $\Lambda = [\lambda_\eta, \lambda_e, \lambda_{gw}, \lambda_{xw}]$ given by:

$$\begin{aligned}
\lambda_\eta &= -\left\{\theta\left(1-\frac{1}{\psi}\right)-1\right\} = \gamma > 0 \\
\lambda_e &= \left(1-\theta\right)W_x = \frac{\gamma-\frac{1}{\psi}}{\kappa_1^c-\rho} \\
\lambda_{gw} &= \left(1-\theta\right)W_{gs} = -\frac{\left(\gamma-1\right)\left(\gamma-\frac{1}{\psi}\right)}{2\left(\kappa_1^c-\nu_g\right)} \\
\lambda_{xw} &= \left(1-\theta\right)W_{xs} = -\frac{\left(\gamma-1\right)\left(\gamma-\frac{1}{\psi}\right)}{2\left(\kappa_1^c-\nu_x\right)\left(\kappa_1^c-\rho\right)^2}
\end{aligned}$$

If the IES is sufficiently large ($\gamma > 1/\psi$), then $\lambda_e > 0$, $\lambda_{gw} < 0$, and $\lambda_{xw} < 0$.

II. Equity Risk Premium

We log-linearize return on portfolio: $r_{t+1} = \kappa_0 + \Delta d_{t+1} + pd_{t+1} - \kappa_1 pd_t$, and conjecture that the price-dividend ratio is linear in the state variables: $pd_t = \mu_{pd} + D_x x_t + D_{gs}\left(\sigma_{gt}^2 - \sigma_g^2\right) + D_{xs}\left(\sigma_{xt}^2 - \sigma_x^2\right)$
As we did for the return on the consumption claim, we compute innovations in the

[1]Recall that the log riskfree rate is $y_t(1) = -\mathbb{E}_t\left[sdf_{t+1}\right] - \frac{1}{2}\mathrm{Var}_t\left[sdf_{t+1}\right]$.

dividend claim return, and its conditional mean and variance:

$$
\begin{aligned}
r_{t+1} &= r_0 + \{\phi_x + D_x(\rho - \kappa_1)\} x_t + D_{gs}(\nu_g - \kappa_1)\left(\sigma_{gt}^2 - \sigma_g^2\right) + D_{xs}(\nu_x - \kappa_1)\left(\sigma_{xt}^2 - \sigma_x^2\right) \\
&\quad + \varphi_d \sigma_{gt} \eta_{d,t+1} + D_x \sigma_{xt} e_{t+1} + D_{gs} \sigma_{gw} w_{g,t+1} + D_{xs} \sigma_{xw} w_{x,t+1} \\
r_{t+1} - \mathbb{E}_t[r_{t+1}] &= \varphi_d \sigma_{gt} \eta_{d,t+1} + D_x \sigma_{xt} e_{t+1} + D_{gs} \sigma_{gw} w_{g,t+1} + D_{xs} \sigma_{xw} w_{x,t+1} \\
\mathbb{E}_t[r_{t+1}] &= r_0 + \{\phi_x + D_x(\rho - \kappa_1)\} x_t + D_{gs}(\nu_g - \kappa_1)\left(\sigma_{gt}^2 - \sigma_g^2\right) \\
&\quad + D_{xs}(\nu_x - \kappa_1)\left(\sigma_{xt}^2 - \sigma_x^2\right) \\
\mathrm{Var}_t[r_{t+1}] &= \varphi_d^2 \sigma_{gt}^2 + D_x^2 \sigma_{xt}^2 + D_{gs}^2 \sigma_{gw}^2 + D_{xs}^2 \sigma_{xw}^2 \\
r_0 &= \kappa_0 + \mu_{pd}(1 - \kappa_1) + \mu_d
\end{aligned}
$$

$$
\mathrm{Cov}_t[r_{t+1}, sdf_{t+1}] = (\theta - 1)\left[W_{gs} D_{gs} \sigma_{gw}^2 + W_{xs} D_{xs} \sigma_{xw}^2\right] - \gamma \varphi_d \tau_{gd} \sigma_{gt}^2 + (\theta - 1) W_x D_x \sigma_{xt}^2
$$

Plug these different components into equation (2):

$$
\begin{aligned}
0 \;=\; & \mu_s + r_0 + \frac{1}{2}\left[\gamma^2 - 2\gamma\varphi_d\tau_{dg} + \varphi_d^2\right]\sigma_g^2 + \frac{1}{2}\left[W_x(\theta - 1) + D_x\right]^2 \sigma_x^2 + \frac{1}{2}\left[W_{gs}(\theta - 1) + D_{gs}\right]^2 \sigma_{gw}^2 \\
& + \frac{1}{2}\left[W_{xs}(\theta - 1) + D_{xs}\right]^2 \sigma_{xw}^2 \quad (7) \\
& + \left\{ (8) \; \frac{1}{\psi} + \left[\phi_x + D_x(\rho - \kappa_1)\right] \right\} x_t \\
& + \left\{ (9) \; \frac{1}{2}\left[\gamma^2 - 2\gamma\varphi_d\tau_{dg} + \varphi_d^2\right] + W_{gs}(\kappa_1^c - \nu_g)(1 - \theta) + D_{gs}(\nu_g - \kappa_1) \right\} \left(\sigma_{gt}^2 - \sigma_g^2\right) \\
& + \left\{ (10) \; \frac{1}{2}\left[W_x(\theta - 1) + D_x\right]^2 + W_{xs}(\kappa_1^c - \nu_x)(1 - \theta) + D_{xs}(\nu_x - \kappa_1) \right\} \left(\sigma_{xt}^2 - \sigma_x^2\right)
\end{aligned}
$$

Then setting all coefficients equal to zero we get:

$$
\begin{aligned}
(8) \quad & \Longrightarrow D_x = \frac{\phi_x - \frac{1}{\psi}}{\kappa_1 - \rho} \\
(9) \quad & \Longrightarrow D_{gs} = \frac{\frac{1}{2}\left[\gamma^2 - 2\gamma\varphi_d\tau_{dg} + \varphi_d^2\right] - \frac{1}{2}\left(\gamma - \frac{1}{\psi}\right)(\gamma - 1)}{\kappa_1 - \nu_g} \\
(10) \quad & \Longrightarrow D_{xs} = \frac{\frac{1}{2}\left[\frac{\phi_x - \frac{1}{\psi}}{\kappa_1 - \rho} - \frac{\gamma - \frac{1}{\psi}}{\kappa_1^c - \rho}\right]^2 - \frac{1}{2}\frac{(\gamma - 1)\left(\gamma - \frac{1}{\psi}\right)}{\left(\kappa_1^c - \rho\right)^2}}{\kappa_1 - \nu_x}
\end{aligned}
$$

Plugging these into (7) implicitly defines a nonlinear equation in one unknown (i.e., μ_{pd}), which can be solved for numerically, characterizing the mean price-dividend ratio.

The D coefficients are the betas of the equity market portfolio with respect to the four fundamental consumption growth shocks.

The equity risk premium is equal to:

$$
\begin{aligned}
\mathbb{E}_t\left[r_{t+1}^e\right] &= -\mathrm{Cov}_t\left[r_{t+1}, sdf_{t+1}\right] \\
&= \left(\varphi_d \tau_{gd}\right)\lambda_\eta \sigma_{gt}^2 + D_x \lambda_e \sigma_{xt}^2 + D_{gs}\lambda_{gw}\sigma_{gw}^2 + D_{xs}\lambda_{xw}\sigma_{xw}^2 \\
&= \left[G_0 + G_{gs}\sigma_g^2 + G_{xs}\sigma_x^2\right] + G_{gs}\left(\sigma_{gt}^2 - \sigma_g^2\right) + G_{xs}\left(\sigma_{xt}^2 - \sigma_x^2\right) \\
G_0 &= D_{gs}\lambda_{gw}\sigma_{gw}^2 + D_{xs}\lambda_{xw}\sigma_{xw}^2 \\
G_{gs} &= \varphi_d \tau_{gd}\gamma \\
G_{xs} &= D_x \lambda_e
\end{aligned}
$$

III. Real Bond Returns and Risk Premium

We start off the expression for the real stochastic discount factor derived in the first sub-section above. Let define the following three parameters: $s_x \equiv -\frac{1}{\psi}$, $s_{gs} \equiv -\frac{1}{2}(\gamma - 1)(\gamma - \frac{1}{\psi})$, and $s_{xs} \equiv -\frac{1}{2}(\gamma-1)(\gamma-\frac{1}{\psi})\frac{1}{(\kappa_1^c-\rho)^2}$. Using notation defined above and in the previous sub-sections, the real stochastic discount factor is:

$$
\begin{aligned}
sdf_{t+1} &= \mu_s + s_x x_t + s_{gs}\left(\sigma_{gt}^2 - \sigma_g^2\right) + s_{xs}\left(\sigma_{xt}^2 - \sigma_x^2\right) \\
&\quad -\lambda_\eta \sigma_{gt}\eta_{t+1} - \lambda_e \sigma_{xt} e_{t+1} - \lambda_{gw}\sigma_{gw} w_{g,t+1} - \lambda_{xw}\sigma_{xw} w_{x,t+1}
\end{aligned}
$$

Let $p_t^b(n) = \log\left(P_t^b(n)\right)$ be the log price and $y_t^b(n) = -\frac{1}{n}p_t^b(n)$ the yield of an n-period real bond.
We conjecture that the log prices of real bonds are linear in the state variables: $p_t(n) = -B_0(n) - B_x(n)x_t - B_{gs}(n)\left(\sigma_{gt}^2 - \sigma_g^2\right) - B_{xs}\left(\sigma_{xt}^2 - \sigma_x^2\right)$

The coefficients are initialized at zero and satisfy the following recursions:

$$
\begin{aligned}
B_0(n) &= B_0(n-1) - \mu_s - \frac{1}{2}\left[\lambda_{gw} + B_{gs}(n-1)\right]^2 \sigma_{gw}^2 \\
&\quad -\frac{1}{2}\left\{\left[\lambda_{xw} + B_{xs}(n-1)\right]^2 \sigma_{xw}^2 + \lambda_\eta^2 \sigma_g^2\right\} \\
&\quad -\frac{1}{2}\left[\lambda_e + B_x(n-1)\right]^2 \sigma_x^2 \\
B_x(n) &= \rho B_x(n-1) + \frac{1}{\psi} \\
B_{gs}(n) &= \nu_g B_{gs}(n-1) + \frac{1}{2}(\gamma-1)(\gamma - \frac{1}{\psi}) - \frac{1}{2}\gamma^2 \\
B_{xs}(n) &= \nu_x B_{xs}(n-1) + \frac{1}{2}(\gamma-1)\frac{(\gamma-\frac{1}{\psi})}{(\kappa_1^c-\rho)^2} - \frac{1}{2}\left[\frac{\gamma-\frac{1}{\psi}}{\kappa_1^c-\rho} + B_x(n-1)\right]^2.
\end{aligned}
$$

These recursions imply the following limit values:

$$
\begin{aligned}
B_x(\infty) &= \frac{1}{\psi(1-\rho)} \\
B_{gs}(\infty) &= \frac{\frac{1}{2}(\gamma-1)(\gamma-\frac{1}{\psi})-\frac{1}{2}\gamma^2}{1-\nu_g} \\
B_{xs}(\infty) &= \frac{\frac{1}{2}(\gamma-1)\frac{(\gamma-\frac{1}{\psi})}{(\kappa_1^c-\rho)^2}-\frac{1}{2}\left[\frac{\gamma-\frac{1}{\psi}}{\kappa_1^c-\rho}+B_x(\infty)\right]^2}{1-\nu_x}.
\end{aligned}
$$

We define $B(\infty) \equiv [B_x(\infty), B_{gs}(\infty), B_{xs}(\infty)]'$.
The real bond risk premium on monthly holding period returns is equal to:

$$
\begin{aligned}
r^b_{t+1}(n) &\equiv ny^b_t(n)-(n-1)y^b_{t+1}(n-1) \\
r^b_{t+1}(n)-\mathbb{E}_t\left[r^b_{t+1}(n)\right] &= -B_x(n-1)\sigma_{xt}e_{t+1}-B_{gs}(n-1)\sigma_{gw}w_{g,t+1} \\
&\quad -B_{xs}(n-1)\sigma_{xw}w_{x,t+1} \\
\mathbb{E}_t\left[r^{b,e}_{t+1}(n)\right] &= -\mathrm{Cov}_t\left[r^b_{t+1}, sdf_{t+1}\right] \\
&= \left[F_0(n)+F_{gs}(n)\sigma_g^2+F_{xs}(n)\sigma_x^2\right]+F_{gs}(n)\left(\sigma^2_{gt}-\sigma^2_g\right)+F_{xs}(n)\left(\sigma^2_{xt}-\sigma^2_x\right) \\
F_0(n) &= -B_{gs}(n-1)\lambda_{gw}\sigma^2_{gw}-B_{xs}(n-1)\lambda_{xw}\sigma^2_{xw}, \\
F_{gs}(n) &= 0, \\
F_{xs}(n) &= -B_x(n-1)\lambda_e.
\end{aligned}
$$

We now define some vectors and matrices to present results in a more compact way. Let the vector X_t summarize all real state variables: $X_t \equiv [x_t,\ \sigma^2_{gt}-\sigma^2_g,\ \sigma^2_{xt}-\sigma^2_x]'$. Let ε_{t+1} denote the corresponding gaussian, i.i.d shocks: $\varepsilon_{t+1} \equiv [e_{t+1},\ w_{g,t+1},\ w_{x,t+1}]'$. We define $\Sigma_t \equiv diag[\sigma^2_{xt}, \sigma^2_{gw}, \sigma^2_{xw}]$. The law of motion of the state vector X_t is $X_{t+1} = \Gamma X_t + \Sigma_t^{\frac{1}{2}}\varepsilon_{t+1}$, where Γ is a 3 by 3 diagonal matrix with ρ, φ_{zg}, and φ_{zx} on the diagonal. Let $B(n)$ denote all the n-period real bond parameters: $B(n) \equiv [B_x(n),\ B_{gs}(n),\ B_{xs}(n)]'$. Using this notation, we can rewrite the real bond risk premium as:

$$
\mathbb{E}_t\left[r^{b,e}_{t+1}(n)\right] = -B(n-1)'\Sigma_t\Lambda.
$$

IV. Nominal Bond Returns and Risk Premium

We start off the expression for the real stochastic discount factor derived above. We use a $^\$$ superscript to denote nominal variables. The nominal stochastic discount factor is then:

$$
\begin{aligned}
sdf^\$_{t+1} &\equiv sdf_{t+1}-\pi_{t+1} \\
&= \mu_s-\mu_\pi+s_x x_t+s_{gs}\left(\sigma^2_{gt}-\sigma^2_g\right)+s_{xs}\left(\sigma^2_{xt}-\sigma^2_x\right)-(\bar{\pi}_t-\mu_\pi) \\
&\quad -\left(\lambda_\eta+\varphi_{\pi g}\right)\sigma_{gt}\eta_{t+1}-\left(\lambda_e+\varphi_{\pi x}\right)\sigma_{xt}e_{t+1}-\lambda_{gw}\sigma_{gw}w_{g,t+1}-\lambda_{xw}\sigma_{xw}w_{x,t+1}-\sigma_\pi\xi_{t+1}
\end{aligned}
$$

Let $p^\$_t(n) = \log\left(P^\$_t(n)\right)$ be the log price and $y^\$_t(n) = -\frac{1}{n}p^\$_t(n)$ the yield of an n-period nominal bond.

We conjecture that the log prices of nominal bonds are linear in the state variables: $p^{\$}_t(n) = -B^{\$}_0(n) - B^{\$}_x(n)x_t - B^{\$}_{gs}(n)\left(\sigma^2_{gt} - \sigma^2_g\right) - B^{\$}_{xs}\left(\sigma^2_{xt} - \sigma^2_x\right) - B^{\$}_\pi(n)\left(\bar{\pi}_t - \mu_\pi\right)$

The coefficients are initialized at zero and satisfy the following recursions:

$$
\begin{aligned}
B^{\$}_0(n) &= B^{\$}_0(n-1) - \mu_s + \mu_\pi - \frac{1}{2}\left\{\left[\sigma_\pi + B^{\$}_\pi(n-1)\sigma_z\right]^2 + \left[\lambda_{gw} + B^{\$}_{gs}(n-1)\right]^2 \sigma^2_{gw}\right\} \\
&\quad -\frac{1}{2}\left\{\left[\lambda_{xw} + B^{\$}_{xs}(n-1)\right]^2 \sigma^2_{xw} + \left[\varphi_{\pi g} + \lambda_\eta + \varphi_{zg}B^{\$}_\pi(n-1)\right]^2 \sigma^2_g\right\} \\
&\quad -\frac{1}{2}\left[\varphi_{\pi x} + \lambda_e + B^{\$}_x(n-1) + \varphi_{zx}B^{\$}_\pi(n-1)\right]^2 \sigma^2_x \\
B^{\$}_x(n) &= \rho B^{\$}_x(n-1) + \alpha_x B^{\$}_\pi(n-1) - s_x \\
B^{\$}_{gs}(n) &= \nu_g B^{\$}_{gs}(n-1) - s_{gs} - \frac{1}{2}\left[\lambda_\eta + \varphi_{\pi g} + \varphi_{zg}B^{\$}_\pi(n-1)\right]^2 \\
B^{\$}_{xs}(n) &= \nu_x B^{\$}_{xs}(n-1) - s_{xs} - \frac{1}{2}\left[\lambda_e + \varphi_{\pi x} + B^{\$}_x(n-1) + \varphi_{zx}B^{\$}_\pi(n-1)\right]^2 \\
B^{\$}_\pi(n) &= \alpha_\pi B^{\$}_\pi(n-1) + 1.
\end{aligned}
$$

These recursions imply the following limit values:

$$
\begin{aligned}
B^{\$}_x(\infty) &= \frac{\alpha_x B^{\$}_\pi(\infty) - s_x}{1-\rho} \\
B^{\$}_{gs}(\infty) &= \frac{-s_{gs} - \frac{1}{2}\left[\lambda_\eta + \varphi_{\pi g} + \varphi_{zg}B^{\$}_\pi(\infty)\right]^2}{1-\nu_g} \\
B^{\$}_{xs}(\infty) &= \frac{-s_{xs} - \frac{1}{2}\left[\lambda_e + \varphi_{\pi x} + B^{\$}_x(\infty) + \varphi_{zx}B^{\$}_\pi(\infty)\right]^2}{1-\nu_x} \\
B^{\$}_\pi(\infty) &= \frac{1}{1-\alpha_\pi}.
\end{aligned}
$$

We define $B^{\$}(\infty) \equiv [B^{\$}_x(\infty), B^{\$}_{gs}(\infty), B^{\$}_{xs}(\infty), B^{\$}_\pi(\infty)]'$.

The nominal bond risk premium on monthly holding period returns is equal to:

$$
\begin{aligned}
r^{b,\$}_{t+1}(n) &\equiv n y^{\$}_t(n) - (n-1)y^{\$}_{t+1}(n-1) \\
r^{b,\$}_{t+1}(n) - \mathbb{E}_t\left[r^{b,\$}_{t+1}(n)\right] &= -\left(B^{\$}_x(n-1) + B^{\$}_\pi(n-1)\varphi_{zx}\right)\sigma_{xt}e_{t+1} - B^{\$}_{gs}(n-1)\sigma_{gw}w_{g,t+1} \\
&\quad -B^{\$}_{xs}(n-1)\sigma_{xw}w_{x,t+1} - B^{\$}_\pi(n-1)\left(\varphi_{zg}\sigma_{gt}\eta_{t+1} + \sigma_z\xi_{t+1}\right) \\
\mathbb{E}_t\left[r^{b,\$,e}_{t+1}(n)\right] &= -\mathrm{Cov}_t\left[r^{b,\$}_{t+1}, sdf^{\$}_{t+1}\right] \\
&= \left[F^{\$}_0(n) + F^{\$}_{gs}(n)\sigma^2_g + F^{\$}_{xs}(n)\sigma^2_x\right] + F^{\$}_{gs}(n)\left(\sigma^2_{gt} - \sigma^2_g\right) + F^{\$}_{xs}(n)\left(\sigma^2_{xt} - \sigma^2_x\right) \\
F^{\$}_0(n) &= -\left\{\lambda_{gw}B^{\$}_{gs}(n-1)\sigma^2_{gw} + \lambda_{xw}B^{\$}_{xs}(n-1)\sigma^2_{xw} + \sigma_\pi\sigma_z B^{\$}_\pi(n-1)\right\} \\
F^{\$}_{gs}(n) &= -\left(\lambda_\eta + \varphi_{\pi g}\right)\varphi_{zg}B^{\$}_\pi(n-1) \\
F^{\$}_{xs}(n) &= -\left(\lambda_e + \varphi_{\pi x}\right)\left(B^{\$}_x(n-1) + B^{\$}_\pi(n-1)\varphi_{zx}\right)
\end{aligned}
$$

Define the following vector and matrix objects:

$$\begin{aligned}
\widehat{\Lambda}^{\$} &\equiv [\lambda_\eta + \varphi_{\pi g}, \lambda_e + \varphi_{\pi x}, \lambda_{gw}, \lambda_{xw}, \sigma_\pi], \\
\widehat{B}^{\$}(n) &\equiv [B^{\$}_\pi(n)\varphi_{zg}, B^{\$}_x(n) + B^{\$}_\pi(n)\varphi_{zx}, B^{\$}_{gs}(n), B^{\$}_{xs}(n), B^{\$}_\pi(n)\sigma_z], \\
\widehat{\Sigma}_t &\equiv diag[\sigma^2_{gt}, \sigma^2_{xt}, \sigma^2_{gw}, \sigma^2_{xw}, 1], \\
\widehat{\varepsilon}_{t+1} &\equiv [\eta_{t+1}, e_{t+1}, w_{g,t+1}, w_{x,t+1}, \xi_{t+1}]
\end{aligned}$$

Then we can write the nominal bond risk premium compactly as:

$$\mathbb{E}_t\left[r^{b,\$,e}_{t+1}(n)\right] = -\widehat{B}^{\$\prime}(n-1)\widehat{\Sigma}_t\widehat{\Lambda}^{\$}.$$

V. Decomposition of the Real SDF

The following proposition shows how to decompose the SDF of the long-run risk model into a martingale component and the dominant pricing component.

Proposition 1. *The stochastic discount factor of the long-run risk model can be decomposed into a martingale component and the dominant pricing component:*

$$\begin{aligned}
\frac{M^T_{t+1}}{M^T_t} &= \beta\exp\left(-B'_\infty\left(I-\Gamma\right)X_t + B'_\infty\Sigma^{\frac{1}{2}}_t\varepsilon_{t+1}\right), \\
\frac{M^P_{t+1}}{M^P_t} &= \beta^{-1}\exp\left(\mu_s + \left[S' + B'_\infty\left(I-\Gamma\right)\right]X_t - \left(\Lambda' + B'_\infty\right)\Sigma^{\frac{1}{2}}_t\varepsilon_{t+1} - \lambda_\eta\sigma_{gt}\eta_{t+1}\right).
\end{aligned}$$

To show this, we start from the definition of the dominant pricing component of the pricing kernel:

$$M^T_t = \lim_{n\to\infty}\frac{\beta^{t+n}}{P^b_t\left(n\right)},$$

Recall that log real bond prices are affine in the state vector:

$$\begin{aligned}
p^b_t(n) &= -B_0(n) - B_x(n)x_t - B_{gs}(n)\left(\sigma^2_{gt}-\sigma^2_g\right) - B_{xs}\left(\sigma^2_{xt}-\sigma^2_x\right) \\
&= -B_0(n) - B(n)'X_t.
\end{aligned}$$

We can then write the dominant pricing component of the SDF as:

$$M^T_t = \lim_{n\to\infty}\beta^{t+n}\exp\left(B_0(n) + B(n)'X_t\right).$$

The constant β is chosen in order to satisfy Assumption 1 in Alvarez and Jermann (2005):

$$0 < \lim_{n\to\infty}\frac{P^b_t(n)}{\beta^n} < \infty.$$

Recall that $B_0(n)$ is defined recursively:

$$\begin{aligned}
B_0(n) &= B_0(n-1) - \mu_s - \frac{1}{2}\left\{\left[\lambda_{gw} + B_{gs}(n-1)\right]^2\sigma^2_{gw}\right\} \\
&- \frac{1}{2}\left\{\left[\lambda_{xw} + B_{xs}(n-1)\right]^2\sigma^2_{xw} + \lambda^2_\eta\sigma^2_g + \left[\lambda_e + B_x(n-1)\right]^2\sigma^2_x\right\}
\end{aligned}$$

Because of the affine term structure of the model and the stationarity of the state vector X, the limit $\lim_{n\to\infty} B(n) = B(\infty)$ is finite. Taking limits on both sides of the equation above leads to:

$$\begin{aligned}\lim_{n\to\infty} B_0(n) - B_0(n-1) &= -\mu_s - \frac{1}{2}\left\{[\lambda_{gw} + B_{gs}(\infty)]^2 \sigma_{gw}^2\right\} \\ &\quad -\frac{1}{2}\left\{[\lambda_{xw} + B_{xs}(\infty)]^2 \sigma_{xw}^2 + \lambda_\eta^2\sigma_g^2 + [\lambda_e + B_x(\infty)]^2 \sigma_x^2\right\}\end{aligned}$$

The limit of $B_0(n) - B_0(n-1)$ is finite, so that $B_0(n)$ grows at a linear rate in the limit. We choose the constant β to offset the growth in $B_0(n)$ as n becomes very large. Setting

$$\beta = \exp\left(\mu_s + \frac{1}{2}\left\{[\lambda_{gw} + B_{gs}(\infty)]^2 \sigma_{gw}^2 + [\lambda_{xw} + B_{xs}(\infty)]^2 \sigma_{xw}^2 + \lambda_\eta^2\sigma_g^2 + [\lambda_e + B_x(\infty)]^2 \sigma_x^2\right\}\right)$$

guarantees that Assumption 1 in Alvarez and Jermann (2005) is satisfied.

We can now write the dominant pricing component of the SDF as:

$$\frac{M_{t+1}^T}{M_t^T} = \beta \exp\left(-B_\infty' (I-\Gamma) X_t + B_\infty' \Sigma_t^{\frac{1}{2}} \varepsilon_{t+1}\right),$$

To derive the martingale component of the SDF, let us go back to the SDF itself. Let S and Λ denote the parameters of the real SDF: $S \equiv [s_x,\ s_{gs},\ s_{xs}]'$, $\Lambda \equiv [\lambda_e,\ \lambda_{gw},\ \lambda_{xw}]'$. Then the real SDF is:

$$SDF_{t+1} = \frac{M_{t+1}}{M_t} = \exp\left(\mu_s + S'X_t - \Lambda'\Sigma_t^{\frac{1}{2}}\varepsilon_{t+1} - \lambda_\eta \sigma_{gt}\eta_{t+1}\right).$$

As a result, the martingale component of the SDF is:

$$\begin{aligned}\frac{M_{t+1}^P}{M_t^P} &= \frac{M_{t+1}}{M_t}\left(\frac{M_{t+1}^T}{M_t^T}\right)^{-1} \\ &= \beta^{-1}\exp\left(\mu_s + [S' + B_\infty'(I-\Gamma)]X_t - (\Lambda' + B_\infty')\Sigma_t^{\frac{1}{2}}\varepsilon_{t+1} - \lambda_\eta\sigma_{gt}\eta_{t+1}\right).\end{aligned}$$

We need to verify that the martingale component is a martingale, i.e that $E_t[M_{t+1}^P/M_t^P] = 1$.

To do this, recall that the bond parameters evolve as:

$$\begin{aligned}B_x(n) &= \rho B_x(n-1) - s_x \\ B_{gs}(n) &= \nu_g B_{gs}(n-1) - s_{gs} - \frac{1}{2}\lambda_\eta^2 \\ B_{xs}(n) &= \nu_x B_{xs}(n-1) - s_{xs} - \frac{1}{2}[\lambda_e + B_x(n-1)]^2.\end{aligned}$$

Taking limits as $n \to \infty$ leads to:

$$B(\infty)'(I-\Gamma) = -S' + [0,\ -\frac{1}{2}\lambda_\eta^2,\ -\frac{1}{2}[\lambda_e + B_x(\infty)]^2]'.$$

To check the martingale condition, plug the definition of β in the following expression:

$$\mathbb{E}_t\left[\frac{M^P_{t+1}}{M^P_t}\right] = \beta^{-1}\exp\left(\mu_s + \left[S' + B'_\infty\left(I-\Gamma\right)\right]X_t + \frac{1}{2}\left(\Lambda' + B'_\infty\right)\Sigma_t\left(\Lambda + B_\infty\right) + \frac{1}{2}\lambda^2_\eta\sigma^2_{gt}\right).$$

The term in front of X_t is equal to $[0,\ -\frac{1}{2}\lambda^2_\eta,\ -\frac{1}{2}\left[\lambda_e + B_x(\infty)\right]^2]'$. Terms in σ^2_{gt} and σ^2_{xt} cancel out. We next plug in the expression for β and check that $E_t[\frac{M^P_{t+1}}{M^P_t}] = 1$.

We now turn to the conditional variances of the log SDF and its dominant pricing and martingale components, $\mathrm{Var}_t[sdf_{t+1}]$, $\mathrm{Var}_t[sdf^T_{t+1}]$ and $\mathrm{Var}_t[sdf^P_{t+1}]$.

$$\begin{aligned}
\mathrm{Var}_t[sdf_{t+1}] &= \Lambda'\Sigma_t\Lambda + \lambda^2_\eta\sigma^2_{gt} \\
\mathrm{Var}_t[sdf^T_{t+1}] &= B'_\infty\Sigma_t B_\infty \\
\mathrm{Var}_t[sdf^P_{t+1}] &= (\Lambda' + B'_\infty)\Sigma_t(\Lambda + B_\infty) + \lambda^2_\eta\sigma^2_{gt}.
\end{aligned}$$

The conditional variance ratio $\mathrm{Var}_t[sdf^P_{t+1}]/V_t[sdf_{t+1}]$ equals

$$\frac{V_t[sdf^P_{t+1}]}{V_t[sdf_{t+1}]} = 1 - \frac{-B'_\infty\Sigma_t\Lambda - \frac{1}{2}B'_\infty\Sigma_t B_\infty}{\frac{1}{2}\Lambda'\Sigma_t\Lambda + \frac{1}{2}\lambda^2_\eta\sigma^2_{gt}}$$

The first term in the numerator corresponds to the bond risk premium $(-B'_\infty\Sigma_t\Lambda)$. It includes the Jensen term $(\frac{1}{2}B'_\infty\Sigma_t B_\infty)$. As a result, the numerator corresponds to the bond risk premium without the Jensen term. The denominator corresponds to the maximum risk premium (also without the Jensen term).

Note that the maximal Sharpe ratio in the model is:

$$\begin{aligned}
MaxSR_t &= \sigma_t(\log SDF_{t+1}) \\
&= \sqrt{\lambda^2_e\sigma^2_{xt} + \lambda^2_{gw}\sigma^2_{gw} + \lambda^2_{xw}\sigma^2_{xw} + \lambda^2_\eta\sigma^2_{gt}} \\
&= \left(\Lambda'\Sigma_t\Lambda + \lambda^2_\eta\sigma^2_{gt}\right)^{\frac{1}{2}}
\end{aligned}$$

VI. Decomposition of the Nominal SDF

The following proposition shows how to decompose the nominal SDF of the long-run risk model into a martingale and a dominant pricing component. To avoid confusion, we use MN to denote the nominal pricing kernel.

Proposition 2. *The stochastic discount factor of the long-run risk model can be decomposed into a martingale component and the dominant pricing component:*

$$\begin{aligned}
\frac{MN^T_{t+1}}{MN^T_t} &= \tilde{\beta}\exp\left(-B^{\$\prime}_\infty(I-\tilde{\Gamma})\tilde{X}_t + \widehat{B}^{\$\prime}_\infty\widehat{\Sigma}^{\frac{1}{2}}_t\widehat{\varepsilon}_{t+1}\right), \\
\frac{MN^P_{t+1}}{MN^P_t} &= \tilde{\beta}^{-1}\exp\left(\mu_s - \mu_\pi + \left[\tilde{S}' + B^{\$\prime}_\infty\left(I-\tilde{\Gamma}\right)\right]\tilde{X}_t - (\widehat{\Lambda}^\$ + \widehat{B}^\$_\infty)'\widehat{\Sigma}^{\frac{1}{2}}_t\widehat{\varepsilon}_{t+1}\right).
\end{aligned}$$

To show this, we start from the definition of the dominant pricing component of the pricing kernel:

$$MN_t^T = \lim_{n\to\infty} \frac{\tilde{\beta}^{t+n}}{P_t^{\$b}(n)},$$

Recall that log real bond prices are affine in the state vector:

$$\begin{aligned} p_t^{\$b}(n) &= -B_0^{\$}(n) - B_x^{\$}(n)x_t - B_{gs}^{\$}(n)\left(\sigma_{gt}^2 - \sigma_g^2\right) - B_{xs}^{\$}\left(\sigma_{xt}^2 - \sigma_x^2\right) - B_\pi^{\$}\left(\bar{\pi}_t - \mu_\pi\right) \\ &= -B_0^{\$}(n) - B^{\$}(n)'\tilde{X}_t, \end{aligned}$$

where we define $\tilde{X}_t = [x_t, \sigma_{gt}^2 - \sigma_g^2, \sigma_{xt}^2 - \sigma_x^2, \bar{\pi}_t - \mu_\pi]$.

We can then write the dominant pricing component of the SDF as:

$$MN_t^T = \lim_{n\to\infty} \tilde{\beta}^{t+n} \exp\left(B_0^{\$}(n) + B^{\$}(n)'\tilde{X}_t\right).$$

The constant $\tilde{\beta}$ is chosen in order to satisfy Assumption 1 in Alvarez and Jermann (2005):

$$0 < \lim_{n\to\infty} \frac{P_t^{\$b}(n)}{\beta^n} < \infty.$$

Recall that $B_0^{\$}(n)$ is defined recursively:

$$\begin{aligned} B_0^{\$}(n) &= B_0^{\$}(n-1) - \mu_s + \mu_\pi - \frac{1}{2}\left\{\left[\sigma_\pi + B_\pi^{\$}(n-1)\sigma_z\right]^2 + \left[\lambda_{gw} + B_{gs}^{\$}(n-1)\right]^2 \sigma_{gw}^2\right\} \\ &\quad -\frac{1}{2}\left\{\left[\lambda_{xw} + B_{xs}^{\$}(n-1)\right]^2 \sigma_{xw}^2 + \left[\varphi_{\pi g} + \lambda_\eta + \varphi_{zg} B_\pi^{\$}(n-1)\right]^2 \sigma_g^2\right\} \\ &\quad -\frac{1}{2}\left[\varphi_{\pi x} + \lambda_e + B_x^{\$}(n-1) + \varphi_{zx} B_\pi^{\$}(n-1)\right]^2 \sigma_x^2 \end{aligned}$$

Because of the affine term structure of the model and the stationarity of the state vector $\tilde{X}$, the limit $\lim_{n\to\infty} B^{\$}(n) \equiv B^{\$}(\infty)$ is finite. Taking limits on both sides of the equation above leads to:

$$\begin{aligned} \lim_{n\to\infty} B_0^{\$}(n) - B_0^{\$}(n-1) &= -\mu_s + \mu_\pi - \frac{1}{2}\left\{\left[\sigma_\pi + B_\pi^{\$}(\infty)\sigma_z\right]^2 + \left[\lambda_{gw} + B_{gs}^{\$}(\infty)\right]^2 \sigma_{gw}^2\right\} \\ &\quad -\frac{1}{2}\left\{\left[\lambda_{xw} + B_{xs}^{\$}(\infty)\right]^2 \sigma_{xw}^2 + \left[\varphi_{\pi g} + \lambda_\eta + \varphi_{zg} B_\pi^{\$}(\infty)\right]^2 \sigma_g^2\right\} \\ &\quad -\frac{1}{2}\left[\varphi_{\pi x} + \lambda_e + B_x^{\$}(\infty) + \varphi_{zx} B_\pi^{\$}(\infty)\right]^2 \sigma_x^2 \end{aligned}$$

The limit of $B_0^{\$}(n) - B_0^{\$}(n-1)$ is finite, so that $B_0^{\$}(n)$ grows at a linear rate in the limit. We choose the constant $\tilde{\beta}$ to offset the growth in $B_0^{\$}(n)$ as n becomes very large.

Setting

$$\tilde{\beta} = \exp\left(\mu_s - \mu_\pi + \frac{1}{2}\left\{\left[\sigma_\pi + B^{\$}_\pi(\infty)\sigma_z\right]^2 + \left[\lambda_{gw} + B^{\$}_{gs}(\infty)\right]^2 \sigma^2_{gw}\right\}\right.$$
$$+\frac{1}{2}\left\{\left[\lambda_{xw} + B^{\$}_{xs}(\infty)\right]^2 \sigma^2_{xw} + \left[\varphi_{\pi g} + \lambda_\eta + \varphi_{zg}B^{\$}_\pi(\infty)\right]^2 \sigma^2_g\right\}$$
$$\left.+\frac{1}{2}\left[\varphi_{\pi x} + \lambda_e + B^{\$}_x(\infty) + \varphi_{zx}B^{\$}_\pi(\infty)\right]^2 \sigma^2_x\right)$$

guarantees that Assumption 1 in Alvarez and Jermann (2005) is satisfied.

We can now write the dominant pricing component of the SDF as:

$$\frac{MN^T_{t+1}}{MN^T_t} = \tilde{\beta}\exp\left(-B^{\$\prime}_\infty(I-\tilde{\Gamma})\tilde{X}_t + B^{\$}_\pi(\infty)\varphi_{zg}\sigma_{gt}\eta_{t+1} + B^{\$}_\pi(\infty)\sigma_z\xi_{t+1}\right.$$
$$\left.+[B^{\$}_x(\infty) + B^{\$}_\pi(\infty)\varphi_{zx}]\sigma_{xt}e_{t+1} + B^{\$}_{gs}(\infty)\sigma_{gw}w_{g,t+1} + B^{\$}_{xs}(\infty)\sigma_{xw}w_{x,t+1}\right)$$
$$= \tilde{\beta}\exp\left(-B^{\$\prime}_\infty(I-\tilde{\Gamma})\tilde{X}_t + \widehat{B}^{\$\prime}_\infty\widehat{\Sigma}^{.5}_t\widehat{\varepsilon}_{t+1}\right),$$

where

$$\tilde{\Gamma} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \nu_g & 0 & 0 \\ 0 & 0 & \nu_x & 0 \\ \alpha_x & 0 & 0 & \alpha_\pi \end{bmatrix}$$

To derive the martingale component of the SDF, let us go back to the SDF itself. Let $\tilde{S} \equiv [s_x,\ s_{gs},\ s_{xs}, -1]'$. Then the nominal SDF is:

$$\frac{MN_{t+1}}{MN_t} = \exp\left(\mu_s - \mu_\pi + \tilde{S}'\tilde{X}_t - (\lambda_\eta + \varphi_{\pi g})\,\sigma_{gt}\eta_{t+1}\right.$$
$$\left.- (\lambda_e + \varphi_{\pi x})\,\sigma_{xt}e_{t+1} - \lambda_{gw}\sigma_{gw}w_{g,t+1} - \lambda_{xw}\sigma_{xw}w_{x,t+1} - \sigma_\pi\xi_{t+1}\right)$$
$$= \exp\left(\mu_s - \mu_\pi + \tilde{S}'\tilde{X}_t - \widehat{\Lambda}^{\$\prime}\Sigma^{.5}_t\widehat{\varepsilon}_{t+1}\right)$$

As a result, the martingale component of the SDF is:

$$\frac{MN^P_{t+1}}{MN^P_t} = \frac{MN_{t+1}}{MN_t}\left(\frac{MN^T_{t+1}}{MN^T_t}\right)^{-1}$$
$$= \tilde{\beta}^{-1}\exp\left(\mu_s - \mu_\pi + \left[\tilde{S}' + B^{\$\prime}_\infty\left(I-\tilde{\Gamma}\right)\right]\tilde{X}_t - [\lambda_\eta + \varphi_{\pi g} + B^{\$}_\pi(\infty)\varphi_{zg}]\sigma_{gt}\eta_{t+1}\right.$$
$$-[\lambda_e + \varphi_{\pi x} + B^{\$}_x(\infty) + B^{\$}_\pi(\infty)\varphi_{zx}]\sigma_{xt}e_{t+1}$$
$$\left.-[\lambda_{gw} + B^{\$}_{gs}(\infty)]\sigma_{gw}w_{g,t+1} - [\lambda_{xw} + B^{\$}_{xs}(\infty)]\sigma_{xw}w_{x,t+1} - [\sigma_\pi + B^{\$}_\pi(\infty)\sigma_z]\xi_{t+1}\right)$$
$$= \tilde{\beta}^{-1}\exp\left(\mu_s - \mu_\pi + \left[\tilde{S}' + B^{\$\prime}_\infty\left(I-\tilde{\Gamma}\right)\right]\tilde{X}_t - (\widehat{\Lambda}^{\$} + \widehat{B}^{\$}_\infty)'\widehat{\Sigma}^{.5}_t\widehat{\varepsilon}_{t+1}\right).$$

We need to verify that the martingale component is a martingale, i.e that $E_t[M^P_{t+1}/M^P_t] =$

1. To do so, recall that the bond parameters evolve as:

$$\begin{aligned}
B_x^{\$}(n) &= \rho B_x^{\$}(n-1) + \alpha_x B_\pi^{\$}(n-1) - s_x \\
B_{gs}^{\$}(n) &= \nu_g B_{gs}^{\$}(n-1) - s_{gs} - \frac{1}{2}\left[\lambda_\eta + \varphi_{\pi g} + \varphi_{zg} B_\pi^{\$}(n-1)\right]^2 \\
B_{xs}^{\$}(n) &= \nu_x B_{xs}^{\$}(n-1) - s_{xs} - \frac{1}{2}\left[\lambda_e + \varphi_{\pi x} + B_x^{\$}(n-1) + \varphi_{zx} B_\pi^{\$}(n-1)\right]^2 \\
B_\pi^{\$}(n) &= \alpha_\pi B_\pi^{\$}(n-1) + 1.
\end{aligned}$$

Taking limits as $n \to \infty$ leads to:

$$B(\infty)^{\$\prime}(I-\tilde{\Gamma})+\tilde{S}' = \left[0,\ -\frac{1}{2}\left[\lambda_\eta + \varphi_{\pi g} + \varphi_{zg} B_\pi^{\$}(\infty)\right]^2,\ -\frac{1}{2}\left[\lambda_e + \varphi_{\pi x} + B_x^{\$}(\infty) + \varphi_{zx} B_\pi^{\$}(\infty)\right]^2, 0\right]'.$$

To check the martingale condition, we plug in the definition of $\tilde{\beta}$ in the expression for the martingale component of the nominal SDF, and use the above equation for $B(\infty)^{\$\prime}(I-\tilde{\Gamma}) + \tilde{S}'$. After some algebra, we indeed find that

$$\mathbb{E}_t\left[\frac{MN_{t+1}^P}{MN_t^P}\right] = 1.$$

We now turn to the conditional variances of the log SDF and its dominant pricing and martingale components, $\text{Var}_t[sdf_{t+1}^{\$}]$, $\text{Var}_t[sdf_{t+1}^{\$,T}]$ and $\text{Var}_t[sdf_{t+1}^{\$,P}]$.

$$\begin{aligned}
\text{Var}_t[sdf_{t+1}^{\$}] &= (\lambda_\eta + \varphi_{\pi g})^2 \sigma_{gt}^2 + (\lambda_e + \varphi_{\pi x})^2 \sigma_{xt}^2 + \lambda_{gw}^2 \sigma_{gw}^2 + \lambda_{xw}^2 \sigma_{xw}^2 + \sigma_\pi^2 \\
&= \widehat{\Lambda}_\infty^{\$\prime} \widehat{\Sigma}_t \widehat{\Lambda}_\infty^{\$} \\
\text{Var}_t[sdf_{t+1}^{\$,T}] &= B_\pi^{\$}(\infty)^2 \varphi_{zg}^2 \sigma_{gt}^2 + [B_x^{\$}(\infty) + B_\pi^{\$}(\infty)\varphi_{zg}]^2 \sigma_{xt}^2 \\
&\quad + B_{gs}^{\$}(\infty)^2 \sigma_{gw}^2 + B_{xs}^{\$}(\infty)^2 \sigma_{xw}^2 + B_\pi^{\$}(\infty)^2 \sigma_z^2 \\
&= \widehat{B}_\infty^{\$\prime} \widehat{\Sigma}_t \widehat{B}_\infty^{\$} \\
\text{Var}_t[sdf_{t+1}^{\$,P}] &= [\lambda_\eta + \varphi_{\pi g} + B_\pi^{\$}(\infty)\varphi_{zg}]^2 \sigma_{gt}^2 + [\lambda_e + \varphi_{\pi x} + B_x^{\$}(\infty) + B_\pi^{\$}(\infty)\varphi_{zx}]^2 \sigma_{xt}^2 \\
&\quad + [\lambda_{gw} + B_{gs}^{\$}(\infty)]^2 \sigma_{gw}^2 + [\lambda_{xw} + B_{xs}^{\$}(\infty)]^2 \sigma_{xw}^2 + [\sigma_\pi + B_\pi^{\$}(\infty)\sigma_z]^2 \\
&= (\widehat{\Lambda}^{\$} + \widehat{B}_\infty^{\$})' \widehat{\Sigma}_t (\widehat{\Lambda}^{\$} + \widehat{B}_\infty^{\$})
\end{aligned}$$

The conditional variance ratio $\text{Var}_t[sdf_{t+1}^P]/V_t[sdf_{t+1}]$ equals

$$\frac{V_t[sdf_{t+1}^{\$,P}]}{V_t[sdf_{t+1}^{\$}]} = 1 - \frac{-\widehat{B}_\infty^{\$\prime} \widehat{\Sigma}_t \widehat{\Lambda}^{\$} - \frac{1}{2}\widehat{B}_\infty^{\$\prime} \widehat{\Sigma}_t \widehat{B}_\infty^{\$}}{\frac{1}{2}\widehat{\Lambda}_\infty^{\$\prime} \widehat{\Sigma}_t \widehat{\Lambda}_\infty^{\$}}$$

The first term in the numerator corresponds to the nominal bond risk premium of an infinite horizon bond, which includes a Jensen term. The second term in the numerator is that Jensen term. As a result, the numerator corresponds to the nominal bond risk premium without the Jensen term. The denominator corresponds to the maximum nominal risk premium, also without the Jensen term.

VII. Calibration

Table 1 reports the model parameter values we use; they are the ones proposed in Bansal and Shaliastovich (2007). Table 2 reports the model loadings on state variables.

The model is simulated for 60,000 months and aggregated up to quarterly frequency for comparison with our quarterly data. In the simulation, negative values for $\sigma^2_{g,t+1}$ and $\sigma^2_{x,t+1}$ are replaced by very small positive values in simulation.

Table 4 reports the mean, standard deviation and autocorrelation of the stochastic discount factor (SDF), its martingale (SDF^P) and dominant pricing (SDF^T) components, the conditional variance ratio ω, the maximum risk premium without Jensen adjustment ($Max\ RP$) and the risk premium of an infinite maturity bond without Jensen adjustment ($BRP(\infty)$). Table 3 reports the mean and standard deviations of the real and nominal yields and bond risk premia in the model and compare them to the same moments in the actual nominal data. Table 5 reports moments of quarterly inflation in the model and in the data. Quarterly inflation is obtained as the sum of three consecutive monthly inflation rates.

The Bansal and Shaliastovich (2007) calibration generates an annual consumption growth rate of 2.12 percent with a standard deviation of 3.52 percent. It generates an annual inflation rate of 3.52 percent with a standard deviation of 2.49 percent.

VIII. Robustness Checks

As robustness checks, we considered both changes on the real and on the nominal side of the economy.

On the real side, we conduct two experiments. First, we find that a slight decrease in the persistence of the long-run component in consumption growth ρ_x could decrease the long-horizon consumption variance ratios and the real variance ratio significantly, and increase the long term real yield from negative to positive values. As a result, the model would need to rely less on a large inflation risk premium in order to match the nominal yield curve, thus lowering the variation of M_t^T in the nominal pricing kernel. However, if all the other parameters are maintained at their previous values, the model would then imply too much volatility of the wealth-consumption ratio and an equity risk premium that is much too low. Second, we shut down the heteroscedasticity in consumption growth by calibrating σ_{xw} and σ_{gw} to very low values. We keep all the other parameters at their previous values. In this case, the real and nominal conditional variance ratios are respectively 1.20 and 0.63 (see Table 6). They are closer to 1, but equity and bond risk premia are constant.

On the nominal side, we first check the robustness of our results to a slightly different calibration of the inflation dynamics. First, we vary each inflation parameter independently in either direction. We report in Table 7 the mean maximum risk premium (MRP), the mean bond risk premium BRP_J (including the Jensen term) and the mean variance ratio ω for different values of the inflation parameters. We simulate the model for a low and a high value of each parameter (25 percent above and below the benchmark value reported in Table 1). The only exception is the parameter α_π, which we cannot increase by 25 percent without running into stationarity issues. The high value is a 10 percent increase for that parameter. We find that ω_t only changes noticeably with α_x and α_π.

To further investigate the sensitivity to these two parameters, Figure 1 in the appendix plots ω_t (left axis) and the five-year nominal bond risk premium (right axis) against α_x (horizontal axis). As we vary α_x away from its benchmark value of -0.35, we simultaneously vary α_π to match the observed persistence of quarterly inflation. We also choose μ_π and σ_π to keep the mean and volatility of inflation at their benchmark values. The figure shows that ω_t is essentially unchanged over a wide range of values for α_x and never comes close to the desired value of one.

Next, we consider a calibration that matches the observed mean, variance, and persistence of inflation, the 5-1-year yield spread, and the persistence of the 5-year nominal bond risk premium. This calibration delivers a nominal variance ratio ω_t that is much too high.

Finally, we ask whether we can find inflation parameters that deliver a nominal variance ratio of 1. We find that we can, while matching the mean inflation, the slope of the nominal term structure, and the persistence of the nominal BRP, but inflation ends up being 2.5 times too volatile and not persistent enough.

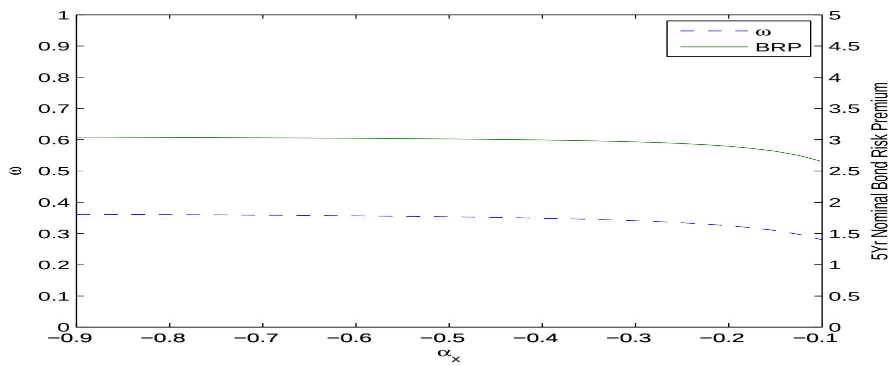

Figure 1. Variance Ratio and Nominal Bond Risk Premium: Sensitivity Analysis

The figure plots the conditional variance ratio ω_t (against the left axis) and the five-year nominal bond risk premium (against the right axis) for different values of the parameter α_x (on the horizontal axis). As we vary α_x away from its benchmark value of -0.35, we simultaneously vary α_π to match the observed persistence of quarterly inflation. We also choose μ_π and σ_π to keep the mean and volatility of inflation at their benchmark values.

IX. Empirical Variance Ratios

Alvarez and Jermann (2005) show that – assuming that the process X_t satisfies the same regularity conditions as above and that X_{t+1}/X_t is strictly stationary and $\lim_{k\to\infty} \frac{1}{k}\mathrm{Var}(\mathbb{E}_{t+k}[X_t]) = 0$ – then

$$\mathrm{Var}\left(\frac{X^P_{t+1}}{X^P_t}\right) = \lim_{k\to\infty} \frac{1}{k}\mathrm{Var}\left(\frac{X_{t+k}}{X_t}\right),$$

Note that the entropy measure used by Alvarez and Jermann (2005) collapses to the half-variance since all variables are conditionally normal. This result implies that long-horizon variance ratios are informative about the variance of the martingale component. We now turn to the empirical variance ratios of the two components of the SDF, e.g consumption growth and the wealth consumption ratio.

If changes in log consumption or changes in the log wealth-consumption ratio are i.i.d, then the variance of long-horizon changes in each variable should grow with the horizon. We compute variance ratios at horizon h as $VR(h) = \mathrm{Var}[\sum_{j=0}^{h} \Delta x_{t+j}]/[h\mathrm{Var}(\Delta x_t)]$, for $x = c$ and $x = wc$. We simulate the model at monthly frequency. Table 1 in the appendix reports the model parameters. We start from the parameter values in Bansal and Shaliastovich (2007).

Figure 2 reports these variance ratios for consumption growth, the change in the wealth-consumption ratio, and inflation. The left panel corresponds to actual data; the right panel uses simulated series. Let us first focus on actual data. The variance ratio of the wealth-consumption ratio clearly decreases with the horizon. It is below 0.6 within five years. Consumption growth exhibits a very different pattern: its variance ratio first increases for horizons up to 5 years; it then decreases, but even after 15 years, the variance ratio is still above one. As a result, there is strong evidence of persistence and mean-reversion in the wealth-consumption ratio, but not in consumption growth.

Let us now turn to simulated data. The variance ratios of the wealth-consumption ratio are in line with the data. They decrease linearly with the horizon, from 1 to approximately 0.5 at the 30-year horizon. In the data, the variance ratio decreases from 1 to 0.6. Consumption growth, however, exhibits a very different pattern. At long horizons, it displays more persistence in the model than in the data. The bottom panel shows that the inflation persistence is similar in model and data, with a slight divergence maybe at longer horizons.

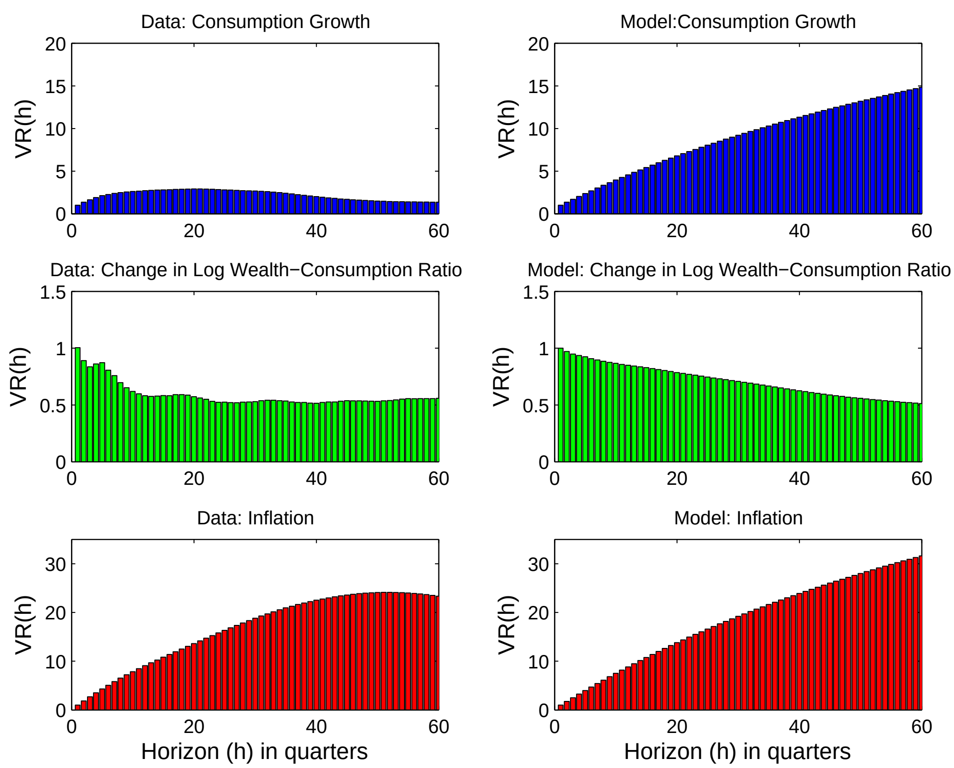

Figure 2. Variance Ratios for Consumption Growth, the Change in the Log Wealth Consumption Ratio and Inflation in the Data and in the Model.

The variance ratio of Δx_t is equal to $VR(h) = Var[\sum_{j=0}^{h} \Delta x_{t+j}]/[hVar(\Delta x_t)]$. The left panel corresponds to actual data. The right panel corresponds to simulated data. Data are quarterly. Actual data come from Lustig et al. (2009). The sample is 1952:II-2008:IV.

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TABLE 1—MODEL PARAMETER VALUES

Parameter		$BS(2007)$
Preference Parameters:		
Subjective discount factor	δ	0.9987
Intertemporal elasticity of substitution	ψ	1.5
Risk aversion coefficient	γ	8
Consumption Growth Parameters:		
Mean of consumption growth	μ_g	0.0016
Long-run risk persistence	ρ	0.991
News volatility level	σ_g	0.004
News volatility persistence	ν_g	0.85
News volatility of volatility	σ_{gw}	$1.15e-6$
Long run-risk volatility level	σ_x	$0.004\sigma_g$
Long run-risk volatility persistence	ν_x	0.996
Long run-risk volatility of volatility	σ_{xw}	$0.06^2\sigma_{gw}$
Dividend Growth Parameters:		
Mean of dividend growth	μ_d	0.0015
Dividend leverage	ϕ_x	1.5
Dividend loading on news volatility	ϕ_{gs}	0
Dividend loading on long-run risk volatility	ϕ_{xs}	0
Volatility loading of dividend growth	φ_d	6.0
Correlation of consumption and dividend news	τ_{gd}	0.1
Inflation Parameters:		
Mean of inflation rate	μ_π	0.0032
Inflation leverage on news	$\varphi_{\pi g}$	0
Inflation leverage on long-run news	$\varphi_{\pi x}$	-2.0
Inflation shock volatility	σ_π	0.0035
Expected inflation AR coefficient	α_π	0.83
Expected inflation loading on long-run risk	α_x	-0.35
Expected inflation leverage on news	φ_{zg}	0
Expected inflation leverage on long-run news	φ_{zx}	-1.0
Expected inflation shock volatility	σ_z	$4.0e-6$

This table reports the calibrated parameters values for our simulation. We take them from Table IV and Table C.I in Bansal and Shaliastovich (2007).

Table 2—Model Loadings on State Variables

	constant	x	$\sigma_{gt}^2-\sigma_g^2$	$\sigma_{xt}^2-\sigma_x^2$
wc	μ_{wc}	$W_x=\frac{1-\frac{1}{\psi}}{\kappa_1^c-\rho}$	$W_{gs}=\frac{\theta\left(\frac{1}{\psi}-1\right)^2}{2\left(\kappa_1^c-\nu_g\right)}$	$W_{xs}=\frac{\theta}{2\left(\nu_x-\kappa_1^c\right)}\left(\frac{\frac{1}{\psi}-1}{\kappa_1^c-\rho}\right)^2$
	6.4	31	-7.7	-1.8×10^6
pd	μ_{pd}	$D_x=\frac{\phi_x-\frac{1}{\psi}}{\kappa_1-\rho}$	$D_{gs}=\frac{\frac{1}{2}\left[\gamma^2-2\gamma\varphi_d\tau_{dg}+\varphi_d^2\right]}{\kappa_1-\nu_g}+\frac{\frac{1}{2}\left[\left(\gamma-\frac{1}{\psi}\right)(1-\gamma)+\phi_{gs}\right]}{\kappa_1-\nu_g}$	$D_{xs}=\frac{\frac{1}{2}\left\{\left[\frac{\gamma-\frac{1}{\psi}}{\rho-\kappa_1^c}+\frac{\frac{1}{\psi}-\phi_x}{\rho-\kappa_1}\right]^2+\frac{(1-\gamma)\left(\gamma-\frac{1}{\psi}\right)}{\left(\rho-\kappa_1^c\right)^2}\right\}}{\kappa_1-\nu_x}+\frac{\phi_{xs}}{\kappa_1-\nu_x}$
	5.6	66	1.3×10^2	-4.3×10^6
ERP	$(1-\theta)\,W_{gs}D_{gs}\sigma_{gw}^2$ $+(1-\theta)\,W_{xs}D_{xs}\sigma_{xw}^2$ $+\varphi_d\tau_{gd}\gamma\sigma_g^2$ $+D_x\,(\theta-1)\,W_x\sigma_x^2$		$G_{gs}=\gamma\varphi_d\tau_{gd}$	$G_{xs}=(1-\theta)\,W_xD_x$
	0.003	0	4.8	4.6×10^4
BRP (Real)	$(\theta-1)\,W_{gs}B_{gs}(n-1)\sigma_{gw}^2$ $+(\theta-1)\,W_{xs}B_{xs}(n-1)\sigma_{xw}^2$ $+(\theta-1)\,W_xB_x(n-1)\sigma_x^2$	$F_x(n)=0$	$F_{gs}(n)=0$	$F_{xs}(n)=(\theta-1)\,W_xB_x(n-1)$
	-0.0014	0	0	-2.1×10^4
BRP (Nominal)	$(\theta-1)\,W_{gs}B_{gs}^{\$}(n-1)\sigma_{gw}^2$ $+(\theta-1)\,W_{xs}B_{xs}^{\$}(n-1)\sigma_{xw}^2$ $+\sigma_\pi\sigma_zB_\pi^{\$}(n-1)$ $-(\gamma+\varphi_{\pi g})\,\varphi_{zg}B_\pi^{\$}(n-1)\sigma_g^2$ $+((\theta-1)\,W_x-\varphi_{\pi x})\left(B_x^{\$}(n-1)+B_\pi^{\$}(n-1)\varphi_{zx}\right)\sigma_x^2$	$F_x^{\$}(n)=0$	$F_{gs}^{\$}(n)=-(\gamma+\varphi_{\pi g})$ $\times\varphi_{zg}B_\pi^{\$}(n-1)$	$F_{xs}^{\$}(n)=[(\theta-1)\,W_x-\varphi_{\pi x}]$ $\times\left(B_x^{\$}(n-1)+B_\pi^{\$}(n-1)\varphi_{zx}\right)$
	0.0015	0	-0	4.3×10^4

This table reports the model loadings on a constant and the state variables. We consider the log wealth-consumption ratio (wc), the log price-dividend ratio (pd), the equity risk premium (ERP), the real and nominal bond risk premia (BRP) at the n-year horizon.

Table 3—Real and Nominal Yield Curves

Maturity	1	2	3	4	5	30	200
			Nominal Bonds - Data				
Mean Yields	5.33	5.52	5.69	5.80	5.89		
Std	2.81	2.77	2.70	2.69	2.65		
			Nominal Bonds - Model				
Mean Yields	5.19	5.46	5.75	6.06	6.38	12.82	20.02
Std	2.92	2.79	2.65	2.53	2.43	1.60	0.36
Mean BRP	0.33	0.93	1.59	2.27	2.97	16.81	24.43
Std	0.07	0.18	0.28	0.38	0.46	1.13	1.18
			Real Bonds - Model				
Mean Yields	1.26	1.05	0.83	0.61	0.39	−4.71	−13.63
Std	1.39	1.35	1.32	1.30	1.29	1.10	0.25
Mean BRP	−0.39	−0.83	−1.28	−1.73	−2.19	−11.14	−16.21
Std	0.05	0.10	0.15	0.19	0.23	0.52	0.55

The top panel reports the mean and standard deviation of nominal bond yields in the Fama-Bliss data. The data are for 1952 until 2008, and only bond yields of maturities one through five years are available. The maturity is in years. The yields and returns are annualized and reported in percentage points. The middle panel does the same for nominal bond yields for a 60,000 month simulation of the LRR model. It also reports the mean and standard deviation of the nominal bond risk premia. The bottom panel reports the same model-implied moments for real bonds.

Table 4—Conditional Variance Ratio

	Mean	*Std*	*AR*(1)
	Nominal SDF		
$SDF^{\$}$	0.99	0.23	−0.01
$SDF^{\$,P}$	1.00	0.14	−0.01
$SDF^{\$,T}$	0.98	0.10	−0.01
$\omega_t^{\$}$	**0.37**	0.06	0.98
Max RP	30.62	2.52	0.99
$BRP(\infty)$	18.72	1.04	0.99
	Real SDF		
SDF	1.00	0.23	−0.01
SDF^{P}	1.00	0.30	−0.01
SDF^{T}	1.02	0.07	−0.01
ω_t	**1.65**	0.11	0.98
Max RP	30.69	2.54	0.99
$BRP(\infty)$	−19.05	0.58	0.99

This table reports the mean, standard deviation and autocorrelation of the stochastic discount factor (SDF), its martingale (SDF^P) and dominant pricing (SDF^T) components, the conditional variance ratio ω, the maximum risk premium without Jensen adjustment (*Max RP*) and the risk premium of an infinite maturity bond without Jensen adjustment ($BRP(\infty)$). The table reports the autocorrelation of each monthly variable in logs. The top (bottom) panel focuses on the nominal (real) stochastic discount factor. The numbers are computed from a 60,000 month simulation.

Table 5—Inflation: Model vs Data

	Data			Model		
	Mean	*Std*	*AR*(1)	*Mean*	*Std*	*AR*(1)
π_t	0.85	0.62	0.86	0.88	1.25	0.76

This table reports the mean, standard deviation and autocorrelation of the quarterly inflation rate. The left panel corresponds to actual data, from Lustig, Van Nieuwerburgh and Verdelhan (2009). The right panel corresponds to simulated data, from the model. The mean and standard deviation are in percentage.

Table 6—Conditional Variance Ratio: No Heteroscedasticity

	Mean	*Std*	*AR*(1)
	Nominal SDF		
$SDF^{\$}$	1.00	0.12	−0.01
$SDF^{\$,P}$	1.00	0.13	−0.01
$SDF^{\$,T}$	1.00	0.01	−0.01
$\omega_t^{\$}$	1.20	0.00	1.00
$Max\ RP$	8.74	0.00	1.00
$BRP(\infty)$	−1.74	0.00	1.00
	Real SDF		
SDF	0.99	0.12	−0.01
SDF^P	1.00	0.10	−0.01
SDF^T	0.99	0.03	−0.01
ω_t	0.63	0.00	1.00
$Max\ RP$	8.70	0.00	1.00
$BRP(\infty)$	3.18	0.00	1.00

This table reports the mean, standard deviation and autocorrelation of the stochastic discount factor (SDF), its martingale (SDF^P) and dominant pricing (SDF^T) components, the conditional variance ratio ω, the maximum risk premium without Jensen adjustment ($Max\ RP$) and the risk premium of an infinite maturity bond without Jensen adjustment ($BRP(\infty)$). The table reports the autocorrelation of each monthly variable in logs. The top (bottom) panel focuses on the nominal (real) stochastic discount factor. The numbers are computed from a 60,000 month simulation.

TABLE 7—SENSITIVITY TO INFLATION SPECIFICATION

	Max RP		$BRP(\infty)$		ω	
	Low	*High*	*Low*	*High*	*Low*	*High*
μ_p	30.62	30.62	18.72	18.72	0.37	0.37
$\varphi_{\pi g}$	30.62	30.62	18.72	18.72	0.37	0.37
$\varphi_{\pi x}$	30.64	30.60	18.70	18.74	0.37	0.37
σ_π	30.62	30.62	18.72	18.72	0.37	0.37
α_π	30.62	30.62	5.61	26.42	0.81	0.10
α_x	30.62	30.62	14.54	21.84	0.51	0.27
φ_{zg}	30.62	30.62	18.72	18.72	0.37	0.37
φ_{zx}	30.62	30.62	18.63	18.81	0.38	0.37
σ_z	30.62	30.62	18.72	18.72	0.37	0.37

This table reports the mean maximum risk premium (*Max RP*) , the mean bond risk premium $BRP(\infty)$ (including the Jensen term) and the mean variance ratio ω. We vary one parameter at a time, and simulate the model for a low and a high value of each parameter (25 percent above and below the benchmark value reported in the first column of Table 1). The only exception is the parameter α_π, which we cannot increase by 25 percent without running into stationarity issues. The high value is a 10 percent increase for that parameter.